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Bounds for neighbor connectivity of Cayley graphs generated by trees and unicyclic graphs

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Abstract

The neighbor connectivity refers to the minimum number of vertices whose removal, along with their neighbors, causes a previously connected graph to become disconnected. In this paper we focus on Cayley graphs constructed from the symmetric group S_n. We investigate the bounds of the neighbor connectivity for two cases: when the generating graph is a tree, and when it is a unicyclic graph with a unique cycle of length m, specifically considering cases where $m = 3$, $m = n - 1$, or $m = n$.

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1. This is a numbered first-level section head

An undirected graph $G = (V, E)$ is employed to represent an interconnection network, where V stands for the set of vertices, and E designates the set of edges. Within this framework, processors are aligned with vertices, and communication links are depicted by edges.

The notion of *graph connectivity* is a subject extensively explored in graph theory and network analysis. The *connectivity* of a graph G, denoted as $\kappa(G)$, signifies the smallest number of vertices in G that, upon removal, result in the formation of a disconnected or trivial graph. This measure serves as a straightforward gauge of the reliability and fault-tolerance of interconnection networks [3, 6, 18, 2].

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Gunter and Hartnell introduced the concept of neighbor connectivity in [8, 9, 10]. Their innovation involved extending the idea of connectivity by eliminating the closed neighborhood of a vertex rather than just removing the vertex itself. In network terms, this corresponds to a scenario where the failure of a vertex implies the failure of all its adjacent vertices. In the referenced paper [8], the authors opted for the term "subversion" in lieu of "failure." This choice was motivated by the graph's application in modeling an underground resistance movement, where vertices symbolize agents and edges represent communication lines among them. In the context of this model, when an agent is subverted, it results in the betrayal of all agents with whom they are in communication. Consistently, this paper adopts the same terminology as used in [8] to articulate the definition of neighbor connectivity. Let $G = (V, E)$ be a simple connected graph. The *neighborhood* of a vertex u of G is defined by $N(u) = \{v \in V;\ uv \in E\}$, and the *closed neighborhood* of u is defined by $N[u] = \{u\} \cup N(u)$. If H is a subgraph of G containing the vertex u, then $N_H(u) = \{v \in V(H);\ uv \in E(H)\}$. A vertex u of G is called *subverted* if the closed neighborhood $N[u]$ has been deleted from G. A set of vertices $U = \{u_1, \ldots, u_n\}$ is called a *subverted strategy* if each of the vertices $u_1, \ldots, u_n$ has been subverted. The *survival subgraph* of G for U, denoted by $G \ominus U$, is the subgraph of G induced by $V - N[U]$. The *neighbor connectivity* of G, denoted by $\kappa_{NB}(G)$, is the size of the minimum set U, such that $U \subseteq V$ and $G \ominus U$ is disconnected, complete, or empty. Such set U is called a *vertex-cut strategy*.

Consider a finite group A and a set Δ containing elements of A, excluding the group's identity, and satisfying the property that for any $u \in \Delta$, its inverse u^{-1} is also in Δ. The vertex set of the *Cayley graph* $Cay(A, \Delta)$ consists of all elements of A, with two vertices u and v being adjacent if and only if there exists an $s \in \Delta$ such that $u = vs$. Let S_n denote the symmetric group, representing permutations on $[n] = 1, 2, \ldots, n$, and let $\mathcal{T}$ be a set of transpositions. We define $G(\mathcal{T})$ as the *transposition generating graph*, where the vertex set of $G(\mathcal{T})$ is $[n]$, and its edge set is $\{(i, j);\ (i, j) \in \mathcal{T}\}$.

In this paper, the focus is on determining the neighbor connectivity of $Cay(S_n, \mathcal{T})$, where $G(\mathcal{T})$ represents either a tree with n vertices or a graph with n vertices containing a unique cycle of length $m = 3$ or $m = n - 1$. The examined graph families include well-known networks, such as the *star graph*, the *bubble-sort graph*, and the *modified bubble-sort graph* [15, 12, 7, 13, 20, 11]. The main results are provided in Theorem 3.2 and Theorem 4.2.

The paper's structure unfolds as follows: Section 2 presents various definitions. In Section 3, we explore the neighbor connectivity of Cayley graphs generated by trees, subsequently deducing the neighbor connectivity of the star graph and the bubble-sort graph. Moving on to Section 4, we determine the neighbor connectivity of Cayley graphs generated by unicyclic graphs and deduce the neighbor connectivity of the modified bubble-sort graph. Section 5 serves as the conclusion, where we summarize our findings and propose a conjecture regarding the neighbor connectivity of the Cayley graph when the generating graph is a graph with n vertices and contains a unique cycle of length m where $4 \leq m \leq n - 2$.

2. Preliminaries

We will follow usual graph terminology, which can be found in [16]. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *neighborhood* of $u \in V$, denoted

$N(u)$, is the set of vertices adjacent to u. The *closed neighborhood* of $u \in V$ is defined by $N[u] = N(u) \cup \{u\}$. If H is a subset of $V(G)$, we denote by $N[H] = \bigcup_{x \in H} N[x]$. The *degree of a vertex* v is the number of vertices of G adjacent to v. The minimum degree is denoted by $\delta(G)$ and the maximum degree by $\Delta(G)$. A set of edges are called *independent* if no two of them have a common endpoint. The graph G is k*-regular* if the degree of every vertex is k. A *vertex-cut* in G is a set X of vertices of G such that $G - X$ is disconnected. The *connectivity* of a graph G, denoted by $\kappa(G)$, is the least number of vertices of G whose removal results in a disconnected or trivial graph. We say that the graph G is *maximum connected* if $\kappa(G) = \delta(G)$; and G is *superconnected* if it is maximum connected and every minimum vertex-cut is composed of the neighborhood $N_G(u)$ of a vertex $u \in V$.

Let A be a finite group, and let Δ be a set of elements of A such that the identity of the group does not belong to Δ. The *Cayley graph* $Cay(A, \Delta)$ is the directed graph with vertex set consisting of the elements of G, and an arc is directed from u to v if and only if there is an $s \in \Delta$ such that $u = vs$. One of the main advantages of using Cayley graphs as models for interconnection networks is their vertex-transitivity, meaning that a graph viewed from any vertex looks the same; however, its vertex connectivity may be low. If whenever $u \in \Delta$, we also have its inverse $u^{-1} \in \Delta$, then for every arc, the reverse arc is also in the graph, hence we can treat this Cayley graph as an undirected graph by replacing each pair of arcs by an edge. In this paper, we use $[p_1p_2 \cdots p_n]$ to denote the permutation $\begin{bmatrix} 1 & 2 & \cdots & n \\ p_1 & p_2 & \cdots & p_n \end{bmatrix}$. For example, the permutation $\alpha = [31254]$ can be expressed in array form as $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{bmatrix}$ and its corresponding cycle notation is $\alpha = (1, 3, 2)(4, 5)$. A cycle (i, j) of length two is called *transposition*, and it swaps the numbers at positions i and j. For example, $[p_1p_2p_3 \cdots p_{n-1}p_n](2, n) = [p_1p_np_3 \cdots p_{n-1}p_2]$. Let S_n be the symmetric group, which is the set of permutations on $[n] = \{1, 2, \dots, n\}$, and let $\mathcal{T}$ be a set of transpositions. We call $G(\mathcal{T})$ the *transposition generating graph*, where the vertex set of $G(\mathcal{T})$ is $[n]$ and its edge set is $\{(i, j);\ (i, j) \in \mathcal{T}\}$. We call $G(\mathcal{T})$ a *transposition tree* if $G(\mathcal{T})$ is a tree.

3. Neighbor connectivity of Cayley graphs generated by trees

Let Γ_n denote the Cayley graph $Cay(S_n, \mathcal{T})$, where $G(\mathcal{T})$ represents a transposition tree with n vertices. As $G(\mathcal{T})$ has $n - 1$ edges, Γ_n is naturally $(n - 1)$-regular and consists of $n!$ vertices. This family of Cayley graphs encompasses well-known examples, such as the *star graph* when $G(\mathcal{T})$ is isomorphic to $K_{1,n-1}$, and the *bubble-sort graph* when $G(\mathcal{T})$ is isomorphic to P_n, a path with n vertices.

For a clearer understanding of Γ_n's structure, let's simplify by assuming, without loss of generality, that n is a leaf in $G(\mathcal{T})$. In the following proposition, we outline some fundamental characteristics of Γ_n.

Proposition 3.1. *[4, 17, 11] Let* $\Gamma_n = Cay(S_n, \mathcal{T})$*, where* $n \geq 4$ *and* n *is a leaf in* $G(\mathcal{T})$*.*

(I) Γ_n *consists of* n *vertex-disjoint subgraphs* $H_1, H_2, \dots, H_n$*, where* H_i *is the subgraph induced by the vertex set* $\{[p_1p_2 \cdots p_{n-1}i];\ p_j \in [n] - \{i\}$*, for* $j = 1, \dots, n - 1\}$*.*

(II) $H_i \cong \Gamma_{n-1}$*, where* $\Gamma_{n-1} = Cay(S_{n-1}, \mathcal{T}')$ *and* $G(\mathcal{T}')$ *is a transposition tree of* $n-1$ *vertices.*

(III) If n *is adjacent to* t *in* $G(\mathcal{T})$*, then* $(t,n) \in \mathcal{T}$ *and every vertex* $u \in V(H_i)$ *has exactly one neighbor,* $u' = u(t,n)$*, outside* H_i*. The edge* uu' *is called a* cross edge *and the vertex* u' *is called* outside neighbor *of* u*.*

(IV) Two distinct vertices in H_i *have different outside neighbors.*

(V) There are exactly $(n-2)!$ *cross edges between* H_i *and* H_j*, for* $1 \le i < j \le n$*.*

(VI) Γ_n *is bipartite.*

Lemma 3.1. *[17] Let* u *and* v *be two distinct vertices of* Γ_n*, then* u *and* v *have at most two common neighbors.*

Lemma 3.2. *Let* u *be a vertex in* H_k *for some* $k \in [n]$*. The maximum number of cross edges between* H_i *and* H_j *that are incident to* $N_{H_i}[u]$ *is* $n-1$*.*

Proof. Consider the graph $Cay(S_n, \mathcal{T})$, where $G(\mathcal{T})$ is a tree on n vertices. Let (j,n) be an edge of this tree, where n is a leaf. Without loss of generality, let $u = ()$ be in H_n, then the outside neighbor of u is in H_j. Since n is a leaf, then j is a vertex of the tree and it is adjacent to another vertex k, hence (j,k) is a vertex in $N_{H_n}[u]$, the closed neighborhood of u in H_n. Moreover, $(j,k)(j,n) = (j,n,k)$ which is a vertex in H_k. As a result, u and the vertex (j,k) have different outside neighbors. Since $N_{H_n}[u]$ contains n vertices, then the maximum number of cross edges between H_n and H_m that are incident to $N_{H_n}[u]$ is less than n, for $m \in [n] - \{m\}$. □

Theorem 3.1. *[5] Let* $n \ge 3$*, and let* G *be a Cayley graph obtained from a transposition generating graph* A *with* m *edges on* $\{1, 2, \ldots, n\}$*. Then* G *is maximally connected.*

Theorem 3.1 leads to the following useful lemma.

Lemma 3.3. *For* $n \ge 3$*,* $\kappa(\Gamma_n) = n-1$*.*

Lemma 3.4. *Let* $n \ge 4$ *and* $U \subseteq V(\Gamma_n)$*, such that* $1 \le |U| \le \lfloor \frac{n}{2} \rfloor - 1$*. Then* $\Gamma_n \ominus U$ *is* $(n-1-2|U|)$*-connected.*

Proof. For $n = 4$, we find that $|U| = 1$, and Γ_4 can be generated by either P_4 or $K_{1,3}$. We employed MathSage [1] to confirm that, in both scenarios, $\Gamma_4 - N[u]$ remains connected. Since Γ_4 is vertex transitive, we only had to remove the closed neighborhood of the vertex $() = [1234]$ and examine the resulting graph's connectivity. For instance, in the case of the generating tree being $K_{1,3}$, we utilized the following code:

```
G=SymmetricGroup(4)
S=[(1,2),(1,3),(1,4)]
C=G.cayley_graph(generators=S, simple=True)
U=C.to_undirected()
A=list(U.neighbor_iterator(G('()'), closed=True))
U.delete_vertices(A)
U.is_connected()
```

We proceed with mathematical induction on n. Assume that $\Gamma_{n-1} \ominus W$ is $(n-2-2|W|)$-connected for every set $W \subseteq V(\Gamma_{n-1})$, where $1 \leq |W| \leq \lfloor \frac{n-1}{2} \rfloor - 1$. Let $U \subseteq V(\Gamma_n)$, such that $1 \leq |U| \leq \lfloor \frac{n}{2} \rfloor - 1$, and let $F \subseteq V(\Gamma_n \ominus U)$, such that $|F| \leq n-2-2|U|$. Our aim is to demonstrate that $(\Gamma_n \ominus U) - F$ is connected. Define $U_i = U \cap V(H_i)$, $k_i = |N[U - U_i] \cap V(H_i)|$, and $F_i = F \cap V(H_i)$ for $i \in [n]$. We will consider cases based on the distribution of the vertices in U.

Case 1. $|U| = |U_i|$, *for some* $i \in [n]$. Without loss of generality, suppose that $i = 1$. Then all the vertices of U are in H_1 and $|U_i| = 0$ for $i \in [n] - \{1\}$. By Lemma 3.3, H_i is $(n-2)$-connected and the maximum number of vertices in $H_i \cap (N[U] \cup F)$ is $|U| + |F|$, for $i \in [n] - \{1\}$. Since $|F| \leq n-2-2|U|$ and $|U| \geq 1$, then $|U| + |F| \leq n-3$, then by Lemma 3.3 $H_i - N[U] - F$ is connected for $i \in [n] - \{1\}$. Let $i, j \in [n] - \{1\}$ such that $i \neq j$, then the number of cross edges between $H_i - F$ and $H_j - F$ is greater than $(n-2)! - (|U| + |F|) \geq (n-2)! - (n-3) \geq 1$, for $n \geq 5$. Therefore, there is a cross edge between $H_i - F$ and $H_j - F$, hence the subgraph C induced by $\bigcup_{i=2}^{n} (V(H_i) - N[U]) - F$ is connected. *If* $(H_1 - N[U]) - F$ *is connected*, then $(\Gamma_n - N[U]) - F$ is connected, since there are enough cross edges between $(H_1 - N[U]) - F$ and C. In fact, when $n \geq 5$, using the inequality $(n-2)! - |U| \geq (n-2)! - (\lfloor n/2 \rfloor - 1) \geq 5$ for $n \geq 5$, the number of these cross edges between $(H_1 - N[U]) - F$ and C is at least

$$
\begin{aligned}
(n-1)! - (n-1)|U| - |F| &\geq (n-1)[(n-2)! - |U|] - |F| \\
&\geq 5(n-1) - |F| \\
&\geq 5n - 5 + 2|U| + 2 - n \\
&\geq 4n - 3 + 2|U| \\
&\geq 4n - 1 \\
&\geq 19.
\end{aligned}
$$

Suppose that $(H_1 - N[U]) - F$ *is not connected.* Let C_1 be a connected component of $(H_1 - N[U]) - F$. We want to show that there is a cross edge between C and C_1.

Subcase 1.1. $|V(C_1)| = 1$. Let $V(C_1) = \{x\}$, then x is an isolated vertex in $(H_1 - N[U]) - F$. This can only happen if all the neighbors of x in H_1 are adjacent to vertices of $N[U_1] \cup F_1$. By Lemma 3.1, a vertex of U can share at most two common neighbors with x, then $deg_{H_1}(x) \leq 2|U| + |F_1|$, then $n-2 \leq 2|U| + |F_1|$, so $|F_1| \geq n-2-2|U|$. Then $|F| = |F_1|$ and all the elements of F are in H_1. By Proposition 3.1, the outside neighbor of x does not belong to $N[U]$, and since $|F_i| = 0$ for $i \in [n] - \{1\}$, then the outside neighbor of x is in C.

Subcase 1.2. $|V(C_1)| \geq 2$. Let x and y be two adjacent vertices of C_1. Since Γ_{n-1} is bipartite, then it contains no odd cycles, then $|N_{C_1}(x) \cap N_{C_1}(y)| = 0$. The maximum number of vertices in $N[x] \cup N[y]$ adjacent to $N[U]$ is $2|U|$; in fact, if $u \in U$, then u can be adjacent to at most two vertices of x, and u cannot be adjacent to a vertex in $N(x)$ and to a vertex in $N[y]$ because this would create an odd cycle and this is not possible because Γ_n is bipartite. The number of vertices in the subgraph induced by $(N[x] \cup N[y]) - N[U]$ is at least

$$\begin{aligned} 2 + 2(n-3) - 2|U| &= 2n - 4 - 2|U| \\ &= (n-2) + (n-2-2|U|) \\ &\geq (n-2) + |F| \end{aligned}$$

each of these $(n-2)+|F|$ vertices has an outside neighbor, then there are at least $n-2$ outside neighbors in C adjacent to vertices in $N_{H_1}[x]$ or $N_{H_1}[y]$. As a result, there is always an edge between C_1 and C, thus $(\Gamma_n \ominus U) - F$ is connected.

Case 2. $|U_i| \leq |U| - 1$, *for every* $i \in [n]$. By the induction hypothesis, the subgraph induced by $V(H_i) - N[U_i]$ is $(n-2-2|U_i|)$-connected. We claim that $(H_i - N[U]) - F$ is connected, because if not, then $k_i + |F_i| \geq n - 2 - 2|U_i|$, and since every vertex outside H_i may be adjacent to at most one vertex of H_i, then the maximum value of k_i is $|U| - |U_i|$, and the maximum value of $|F_i|$ is $n-2-2|U|$, then we have the inequality $n-2-2|U_i| \leq |U| - |U_i| + n - 2 - 2|U|$, and this implies that $|U| \leq |U_i|$, which is a contradiction. In addition, since $|U| \leq \lfloor \frac{n}{2} \rfloor - 1$, then there exists $j \in [n]$ such that $|U_j| = 0$. By Lemma 3.2, when $n \geq 5$, the number of cross edges between $(H_i - N[U]) - F$ and $(H_j - N[U]) - F$ is at least

$$\begin{aligned} (n-2)! - |F| - (n-1)|U_i| &\geq (n-2)! - (n-2-2|U|) - (n-1)(|U|-1) \\ &\geq (n-2)! - n + 2 - (n-3)|U| + (n-1) \\ &\geq (n-2)! + 1 - (n-3)|U| \\ &\geq 1 \end{aligned}$$

Therefore, there is always a cross edge between $(H_i - N[U]) - F$ and $(H_j - N[U]) - F$ for every $i \in [n] - \{j\}$. The maximum number of vertices of $N[U] \cup F$ removed from H_j is less than $n-2$; in fact $|U| + |F| \leq n - 2 - |U| \leq n - 3$. Then by Lemma 3.3, the subgraph induced by $(H_j - N[U]) - F$ is connected. Therefore $(\Gamma_n \ominus U) - F$ is connected. □

By the previous lemma, we conclude that $\kappa_{NB}(\Gamma_n) \geq \lfloor \frac{n}{2} \rfloor$. We now give an upper bound for $\kappa_{NB}(\Gamma_n)$.

Lemma 3.5. *Let* $n \geq 4$*, then* $\kappa_{NB}(\Gamma_n) \leq n - 1$.

Proof. Let $x \in V(\Gamma_n)$, and let $N(x) = \{x_1, x_2, \ldots, x_{n-1}\}$. Let $U = \{y_1, y_2, \ldots, y_{n-1}\} \subseteq V(\Gamma_n) - N[x]$ such that $x_i y_i \in E(\Gamma_n)$, and $y_i \neq y_j$, for $i, j \in [n-1]$ and $i \neq j$. Γ_n does not contain odd cycles because it is bipartite, therefore x is not adjacent to y_i for $i \in [n-1]$. Then $\Gamma_n - N[U]$ is disconnected because x is an isolated vertex in it. □

From the previous two lemmas, we deduce the following theorem.

Theorem 3.2. *Let* $n \geq 4$*, then* $\lfloor \frac{n}{2} \rfloor \leq \kappa_{NB}(\Gamma_n) \leq n - 1$. *Moreover, the bounds are tight.*

Proof. In [14], the authors proved that $\kappa_{NB}(S_n) = n - 1$, where S_n is the star graph. Consider the bubble-sort graph $B_n = Cay(S_n, P_n)$ where P_n is the path with vertex set $V(P_n) = [n]$ and edge set $E(P_n) = \{(i, i+1);\ i \in [n-1]\}$. Without loss of generality, let $u = ()$ be the identity permutation, then $N(u) = \{(i, i+1);\ i \in [n-1]\}$. If n is even, then the set of vertices $U = \{(i, i+1)(n-i, n-i+1);\ i = 1, \ldots, \frac{n}{2}-1\} \cup \{(\frac{n}{2}, \frac{n}{2}+1)(1, 2)\}$ is a vertex-cut strategy of size $\frac{n}{2}$. Then U is a vertex-cut strategy. If n is odd, let $U = \{(i, i+1)(\lfloor\frac{n}{2}\rfloor+i, \lfloor\frac{n}{2}\rfloor+i+1);\ i = 1, \ldots, \lfloor\frac{n}{2}\rfloor\}$ is a vertex-cut strategy of size $\lfloor\frac{n}{2}\rfloor$. Then $\kappa_{NB}(B_n) \leq \lfloor\frac{n}{2}\rfloor$, therefore $\kappa_{NB}(B_n) = \lfloor\frac{n}{2}\rfloor$. □

4. Neighbor connectivity of Cayley graphs generated by unicyclic graphs

In this section we consider Cayley graphs $UG_n = Cay(S_n, \mathcal{T})$, where $G(\mathcal{T})$ is a unicyclic graph with vertex set $[n]$. Let C_n be the cycle of n vertices, and let $H_{n,p}$ be the graph obtained by appending the cycle C_p to a pendant vertex of a path P_{n-p}. $H_{n,p}$ is called *lollipop* graph. The graph $H_{n,n-1}$ consists of the cycle C_{n-1} and one pendant vertex. When $G(\mathcal{T}) = C_n$, then UG_n becomes the *modified bubble-sort* graph MB_n, and when $G(\mathcal{T}) = H_{n,n-1}$, then we will denote such graph by LG_n.

4.1. Neighbor Connectivity of Modified Bubble-Sort Graph MB_n

Suppose that the generating graph of MB_n, $G(\mathcal{T})$ is $C_n = (1, 2, \ldots n, 1)$. Let $\mathcal{T}' = \mathcal{T} - \{(1, n), (n-1, n)\}$, then $G(\mathcal{T}')$ is a path of length $n-1$, and $Cay(S_{n-1}, \mathcal{T}')$ is the $(n-1)$-dimensional bubble-sort graph B_{n-1}. Let B^i_{n-1} be the subgraph of MB_n induced by the vertex set $\{[p_1 p_2 \ldots p_{n-1} i];\ p_k \in [n] - \{i\},$ for $k = 1, \ldots, n-1\}$, then $B^i_{n-1} \cong B_{n-1}$. Therefore, MB_n can be decomposed into n vertex disjoint subgraphs $B^1_{n-1}, \ldots, B^n_{n-1}$. The following proposition includes some useful topological properties of MB_n.

Proposition 4.1. *[19] Let MB_n be the n-dimensional modified bubble-sort graph, and let B^1_{n-1}, $\ldots, B^n_{n-1}$ be the subgraphs defined above.*

(I) MB_n is n-regular bipartite graph.

(II) If $u \in V(B^i_{n-1})$, then u has exactly two neighbors outside B^i_{n-1}, called the outside neighbors *of u.*

(III) The outside neighbors of B^i_{n-1} are all different.

(IV) The outside neighbors of a vertex are located in different B^i_{n-1} subgraphs.

(V) There are exactly $2(n-2)!$ independent edges between B^i_{n-1} and B^k_{n-1}, for $i, k \in [n]$ and $i \neq k$. Such edges are called cross edges.

Lemma 4.1. *Let $u \in V(B^i_{n-1})$ for some $i \in [n]$, and let u' and u'' be its outside neighbors. Then u' and u'' have no common neighbor in MB_n other than u.*

Proof. Without loss of generality, assume that $u = ()$, then $u \in V(B^n_{n-1})$. Let $u' = (1, n)$ and $u'' = (n-1, n)$ be the outside neighbors of u. If there is a common neighbor for u' and u'', then there exist two transpositions (a, b) and (c, d) such that $(1, n)(a, b) = (n-1, n)(c, d)$, equivalently $(1, n-1, n)(a, b) = (c, d)$. This situation occurs only if $a, b \in \{1, n-1, n\}$, which means when $(a, b) = (1, n)$ or $(a, b) = (n-1, n)$. If $(a, b) = (1, n)$, then $(c, d) = (n-1, n)$ and the common vertex will be $u = ()$. If $(a, b) = (n-1, n)$, then $(c, d) = (1, n-1)$, but this is not possible as $(1, n-1)$ is not in the set of generating transpositions. □

In the following lemma, we give an upper bound for $\kappa_{NB}(MB_n)$.

Lemma 4.2. *Let $n \geq 4$, then $\kappa_{NB}(MB_n) \leq \lceil \frac{n}{2} \rceil$.*

Proof. Suppose that n *is even*, then the set of vertices $U = \{(i, i+1)(n-i, n-i+1);\ i = 1, \ldots, \frac{n}{2}-1\} \cup \{(\frac{n}{2}, \frac{n}{2}+1)(1, n)\}$ is a vertex-cut strategy of size $\frac{n}{2}$ because the vertex corresponding to the identity permutation $()$ is isolated in $MB_n \ominus U$. Similarly, if n *is odd*, then $U = \{(i, i+1)(n-i, n-i+1);\ i = 1, \ldots, \lfloor \frac{n}{2} \rfloor - 1\} \cup \{(\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1)(1, n), (\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 2)(\lfloor \frac{n}{2} \rfloor + 2, \lfloor \frac{n}{2} \rfloor + 3)\}$ is a vertex-cut strategy of size $\lfloor \frac{n}{2} \rfloor + 1 = \lceil \frac{n}{2} \rceil$, because $()$ becomes an isolated vertex in $MB_n \ominus U$. Therefore, $\kappa_{NB}(MB_n) \leq \lceil \frac{n}{2} \rceil$. □

Lemma 4.3. *Let $n \geq 4$ and $u \in V(MB_n)$. Then $MB_n \ominus \{u\}$ is connected.*

Proof. If $u \in V(B_{n-1})$, then by Theorem 3.2 the graph $B_{n-1} \ominus \{u\}$ is connected, for $n \geq 4$. Now let $v \in V(MB_n)$, then $v \in V(B^i_{n-1})$ for some $i \in [n]$. Since $B^i_{n-1} \cong B_{n-1}$, then the graph induced by the vertices of $B^i_{n-1} - N[v]$ is connected. By Proposition 4.1, v has two outside neighbors v' and v'' in B^j_{n-1} and B^k_{n-1} respectively, where $j, k \in [n] - \{i\}$ and $j \neq k$. By Lemma 3.3, $B^j_{n-1} - \{v'\}$ and $B^k_{n-1} - \{v''\}$ are connected. Since there are $2(n-2)!$ cross edges between every pair of the B^i_{n-1}-subgraphs, then $MB_n \ominus \{u\}$ is connected. □

Lemma 4.4. *Let $u \in V(MB_n)$. Suppose $u \in V(B^i_{n-1})$ for some $i \in [n]$. If u has its outside neighbors u' and u'' in B^j_{n-1} and B^k_{n-1} for some j and k in $[n] - \{i\}$, then exactly $(n-3)$ vertices of $N_{MB^i_{n-1}}(u)$ have their outside neighbors in B^i_{n-1}.*

Proof. Since MB_n is vertex transitive, then without loss of generality assume that $u = () \in V(B^n_{n-1})$. Then the outside neighbors of u are $u' = (1, n) \in V(B^1_{n-1})$ and $u'' = (n-1, n) \in V(B^{n-1}_{n-1})$. The vertices corresponding to $(2, 3), \ldots, (n-2, n-1)$ are in $N_{B^n_{n-1}}(u)$ and they have their outside neighbors, $(2, 3)(1, n), (3, 4)(1, n), \ldots, (n-2, n-1)(1, n)$, in B^1_{n-1}. □

Lemma 4.5. *Let $n \geq 5$ and $U \subseteq V(MB_n)$, such that $2 \leq |U| \leq \lceil \frac{n}{2} \rceil - 1$. Then $MB_n \ominus U$ is $(n - 2|U|)$-connected.*

Proof. Let $F \subseteq V(MB_n)$, such that $|F| \leq n - 1 - 2|U|$. Our aim is to show that $(MB_n \ominus U) - F$ is connected. We consider cases depending on the distribution of the elements of U. Let $U_i = U \cap V(B^i_{n-1})$, $k_i = |N[U - U_i] \cap V(B^i_{n-1})|$, and $F_i = F \cap V(B^i_{n-1})$, for $i \in [n]$.

Case 1. $|U| = |U_1|$. For $i \in [n] - \{1\}$, $|F_i| + k_i \leq (n - 1 - 2|U|) + |U| \leq n - 3 < deg(B^i_{n-1})$, then by Lemma 3.3, the subgraph induced by the vertices of $(B^i_{n-1} - N[U]) - F$ is connected. The number of cross edges between the subgraphs induced by $(B^i_{n-1} - N[U]) - F$ and $(B^j_{n-1} - N[U]) -$

F, for $2 \leq i < j \leq n$, is greater than $2(n-2)! - [(n-1-2|U|) + 2|U|] \geq 2(n-2)! - n + 1 \geq 1$, hence there is always a cross edge between these two subgraphs. Then the subgraph C induced by the vertices of $\bigcup_{i=2}^{n} V(B_{n-1}^{i}) - (N[U] \cup F)$ is connected. If $(B_{n-1}^{1} - N[U]) - F$ is connected, then the graph $(MB_n \ominus U) - F$ becomes connected since there is at least one cross edge connecting a vertex from $(B_{n-1}^{1} - N[U]) - F$ and a vertex from C. Suppose that $(B_{n-1}^{1} - N[U]) - F$ is not connected. Let C_1 be a connected component in $(B_{n-1}^{1} - N[U]) - F$.

- If C_1 contains exactly one vertex u, then u must have an outside neighbor in C. In fact, the maximum number of vertices in $N[U] \cup F$ adjacent to u is $2|U| + |F| = n - 1$, and since $deg_{MB_n}(u) = n$, then u must have at least one outside neighbor in C.

- If $|C_1| \geq 2$, then let u and v be two adjacent vertices in C_1. Since MB_n contains no odd cycles, then $|N(u) \cap N(v)| = 0$. The subgraph induced by $(N_{B_{n-1}^{1}}[u] \cup N_{B_{n-1}^{1}}[v]) - (N[U] \cup F)$ contains at least $2n-4-2|U|-|F|$ vertices. Since $-2|U|-|F| \geq 1-n$, then this subgraph contains at least $n-3$ vertices. On the other hand, $|F| \leq n-1-2|U|$, so $|F| \leq n-5$. Therefore, there must be a cross edge between C_1 and C, and hence $(MB_n \ominus U) - F$ is connected.

Case 2. $|U_i| \leq |U| - 1$, *for every* $i \in [n]$.

Subcase 2.1. *Assume that* $(B_{n-1}^{i} - N[U]) - F$ *is connected for every* $i \in [n]$. Since $|U| \leq \lceil \frac{n}{2} \rceil - 1$, then at least half of the subgraphs B_{n-1}^{k}, for $k \in [n]$, contain no elements of U, and therefore each such subgraph contains at most $|U| + (n-1) - 2|U| \leq n-3$ vertices of $N[U] \cup F$, then by Theorem 3.1 these subgraphs are connected. Without loss of generality, suppose that B_{n-1}^{1} contains no elements of U, then $U_1 = \emptyset$. For every $i \in [n] - \{1\}$, we want to show that there is a cross edge between $(B_{n-1}^{1} - N[U]) - F$ and $(B_{n-1}^{i} - N[U]) - F$, and hence $(MB_n \ominus U) - F$ is connected. By Lemma 4.4, a vertex of U_i and its neighbors in B_{n-1}^{i} contribute to a maximum of $n-1$ cross edges between B_{n-1}^{1} and B_{n-1}^{i}. Then the maximum number of cross edges between B_{n-1}^{1} and B_{n-1}^{i} that are incident to elements of $N[U] \cup F$ is $|F| + (n-2)|U|$,

$$\begin{aligned}|F| + (n-2)|U| &\leq n-1-2|U| + (n-2)|U| \\ &\leq n-1-(n-4)|U| \\ &\leq n-1-(n-4)(\lceil n/2 \rceil - 1)\end{aligned}$$

Given that the total number of cross edges is $2(n-2)!$ and it is greater than $n-1-(n-4)(\lceil n/2 \rceil - 1)$ for $n \geq 5$, it follows that, for every i in $[n] - \{1\}$, there always exists a cross edge between $(B_{n-1}^{1} - N[U]) - F$ and $(B_{n-1}^{i} - N[U]) - F$. Consequently, $(MB_n \ominus U) - F$ is connected.

Subcase 2.2. *Assume that there exists* $i \in [n]$ *for which* $(B_{n-1}^{i} - N[U]) - F$ *is disconnected.* Without loss of generality, assume $i = 1$. We have $|U_1| \leq \lceil \frac{n}{2} \rceil - 2$, and since $\lceil \frac{n}{2} \rceil - 2 = \lfloor \frac{n-1}{2} \rfloor - 1$, then by Lemma 3.4, $B^1_{n-1} - N[U_1]$ is $(n-2-2|U_1|)$-connected, therefore

$$\begin{aligned}|F_1| + k_1 &\geq n-2-2|U_1|\\ |F_1| + |U| - |U_1| &\geq n-2-2|U_1|\\ |F_1| + |U| &\geq n-2-|U_1|\\ |F_1| + |U| &> n-2-|U|\\ |F_1| &> n-2-2|U|\\ |F_1| &> |F|\end{aligned}$$

this is a contradiction, therefore $(B^i_{n-1} - N[U]) - F$ is connected for every $i \in [n]$, hence by Subcase 2.1, $(MB_n \ominus U) - F$ is connected.

□

Theorem 4.1. *Let $n \geq 4$, then $\kappa_{NB}(MB_n) = \lceil \frac{n}{2} \rceil$.*

Proof. Lemma 4.3 and Lemma 4.5 imply that $\kappa_{NB}(MB_n)$ is greater than $\lceil \frac{n}{2} \rceil - 1$, then $\kappa_{NB}(MB_n) \geq \lceil \frac{n}{2} \rceil$. By Lemma 4.2, we have $\kappa_{NB}(MB_n) \leq \lceil \frac{n}{2} \rceil$, therefore $\kappa_{NB}(MB_n) = \lceil \frac{n}{2} \rceil$. □

4.2. Neighbor Connectivity of LG_n

Suppose that the generating graph of LG_n is $G(\mathcal{T}) = H_{n,n-1}$, which consists of the vertex set $[n]$ and edge set $\{(i, i+1), (1, n-1), (1, n);\ i = 1, \ldots, n-2\}$. Let $\mathcal{T}' = \mathcal{T} - \{(1, n)\}$, then $G(\mathcal{T}')$ is a cycle of length $n-1$, and $Cay(S_{n-1}, \mathcal{T}')$ is the $(n-1)$-dimensional modified bubble-sort graph MB_{n-1}. Let MB^i_{n-1} be the subgraph of LG_n induced by the vertex set $\{[p_1p_2 \ldots p_{n-1}i];\ p_k \in [n] - \{i\},$ for $k = 1, \ldots, n-1\}$, then $MB^i_{n-1} \cong MB_{n-1}$. Therefore, LG_n can be decomposed into n vertex disjoint subgraphs, $MB^1_{n-1}, \ldots, MB^n_{n-1}$, such that each one of them is isomorphic to MB_{n-1}.

Proposition 4.2. *[19] Let LG_n be the n-dimensional Cayley graph $Cay(S_n, H_{n,n-1})$.*

- *(I) LG_n is n-regular bipartite graph.*
- *(II) If $u \in V(MB^i_{n-1})$, then u has exactly one neighbor outside MB^i_{n-1}, called the* outside neighbor *of u.*
- *(III) The outside neighbors of MB^i_{n-1} are all different.*
- *(IV) There are exactly $(n-2)!$ independent edges between MB^i_{n-1} and MB^k_{n-1}, for $i, k \in [n]$ and $i \neq k$. Such edges are called cross edges.*

Lemma 4.6. *Let $u \in V(LG_n)$, for $n \geq 4$. Suppose $u \in V(MB^i_{n-1})$ for some $i \in [n]$. If u has its outside neighbor u' in MB^j_{n-1} for some $j \in [n] - \{i\}$, then exactly $(n-3)$ vertices of $N_{MB^i_{n-1}}(u)$ have their outside neighbors in MB^j_{n-1}.*

Proof. Since LG_n is vertex transitive, then without loss of generality assume that $u = ()$. Then the outside neighbor of u. $u' = (1, n)$, is in MB^1_{n-1}. The vertices corresponding to $(2, 3), \ldots, (n-2, n-1)$ are in $N_{MB^n_{n-1}}(u)$ and they have their outside neighbors in MB^1_{n-1}. □

Lemma 4.7. *Let $n \geq 4$ and $U \subseteq V(LG_n)$, such that $1 \leq |U| \leq \lceil \frac{n}{2} \rceil - 1$. Then $LG_n \ominus U$ is $(n-2|U|)$-connected.*

Proof. Let $F \subseteq V(LG_n)$, such that $|F| \leq n - 1 - 2|U|$. Our aim is to show that $(LG_n \ominus U) - F$ is connected. We consider cases depending on the distribution of the elements of U. Let $U_i = U \cap V(MB^i_{n-1})$, $k_i = |N[U - U_i] \cap V(B^i_{n-1})|$, and $F_i = F \cap V(MB^i_{n-1})$, for $i \in [n]$.

Case 1. $|U| = |U_1|$. For $i \in [n] - \{1\}$, $|F_i| + k_i \leq (n-1-2|U|) + |U| \leq n-2 < deg(MB^i_{n-1})$, then by Theorem 3.1, the subgraph induced by the vertices of $(MB^i_{n-1} - N[U]) - F$ is connected. The number of cross edges between the subgraphs induced by $(MB^i_{n-1} - N[U]) - F$ and $(MB^j_{n-1} - N[U]) - F$, for $2 \leq i < j \leq n$, is greater than $(n-2)! - [(n-1-2|U|)] \geq (n-2)! - n + 1 \geq 1$, for $n \geq 4$, hence there is always a cross edge between these two subgraphs. Then the subgraph C induced by the vertices of $\bigcup_{i=2}^{n} V(MB^i_{n-1}) - (N[U] \cup F)$ is connected. If $(MB^1_{n-1} - N[U]) - F$ is connected, then the graph $(LG_n \ominus U) - F$ becomes connected since there is at least one cross edge connecting a vertex from $(MB^1_{n-1} - N[U]) - F$ and a vertex from C. In fact, there are $(n-1)(n-2)!$ cross edges incident to vertices in MB^1_{n-1}, if the closed neighborhood of a vertex is removed from the graph, then this contributes to at most $(n-2)$ cross edges. Then the number of cross edges between $(MB^1_{n-1} - N[U]) - F$ and C is

$$
\begin{aligned}
(n-1)(n-2)! - (|F| + (n-2)|U|) &\geq (n-1)! - (n-1-2|U| + (n-2)|U|) \\
&\geq (n-1)! - n + 1 - (n-4)|U| \\
&\geq (n-1)! - n + 1 - (n-4)\frac{n}{2} \\
&\geq 3.
\end{aligned}
$$

Suppose that $(MB^1_{n-1} - N[U]) - F$ is not connected. Let C_1 be a connected component in $(MB^1_{n-1} - N[U]) - F$.

Subcase 1.1. *C_1 contains exactly one vertex u.* The maximum number of vertices in $N[U] \cup F$ adjacent to u is $2|U| + |F| = n - 1 = deg_{MB^1_{n-1}}(u)$, then all the vertices of F must be in MB^1_{n-1}. By Lemma 4.2, a vertex of U cannot be adjacent to the outside neighbor of u, and since $deg_{LG_n}(u) = n$, then the outside neighbor of u must be in C.

Subcase 1.2. *C_1 contains at least two vertices.* Let u and v be two adjacent vertices in C_1. Since LG_n contains no odd cycles, then $|N(u) \cap N(v)| = 0$. The subgraph induced by $(N_{MB^1_{n-1}}[u] \cup N_{MB^1_{n-1}}[v]) - (N[U] \cup F)$ contains at least $2n-2-2|U|-|F|$ vertices. Since $-2|U|-|F| \geq 1-n$, then this subgraph contains at least $n-1$ vertices. A vertex in U can not have an outside neighbor that belongs to $N[U]$. On the other hand, $|F| \leq n-1-2|U|$, so $|F| \leq n-5$. Therefore, there must be a cross edge between C_1 and C, and hence $(LG_n \ominus U) - F$ is connected.

Case 2. $|U_i| \leq |U| - 1$*, for every* $i \in [n]$.

Subcase 2.1. *Assume that $(MB^i_{n-1} - N[U]) - F$ is connected for every $i \in [n]$.* Since $|U| \leq \lceil \frac{n}{2} \rceil - 1$, then at least half of the subgraphs MB^k_{n-1}, for $k \in [n]$, contain no elements of U. Suppose that $U_1 = U_2 = \emptyset$, it is easy to see that the subgraph C induced by $V(MB^1_{n-1}) \cup V(MB^2_{n-1})$ is connected. We want to show that there is always an edge between MB^i_{n-1} and C, for $i \in [n] - \{1,2\}$. The number of cross edges between MB^i_{n-1} and MB^1_{n-1} is $(n-2)! - (n - 1 - 2|U| + (n-2)|U_i| + |U| - |U_i|)$. We are removing all cross edges incident to vertices of F,

$N[U_i]$, and $U - U_i$. This number is equal to $(n-2)! - (n-1-|U| + (n-3)|U_i|)$. We have

$$\begin{aligned}(n-2)! - (n-1-|U| + (n-3)|U_i|) &\geq (n-2)! - (n-1-|U| + (n-3)(|U|-1)) \\ &\geq (n-2)! - (2 + (n-4)|U|) \\ &\geq 2\end{aligned}$$

then there is a an edge between C and MB^i_{n-1} for every $i \in [n] - \{1,2\}$, therefore $(LG_n \ominus U) - F$ is connected.

Subcase 2.2. *Assume that there exists $i \in [n]$ for which $(MB^i_{n-1} - N[U]) - F$ is disconnected.* Without loss of generality, assume $i = 1$. We have $|U_1| \leq \lceil \frac{n}{2} \rceil - 2$, and since $\lceil \frac{n}{2} \rceil - 2 \leq \lceil \frac{n-1}{2} \rceil - 1$, then by Lemma 4.5, $MB^1_{n-1} - N[U_1]$ is $(n-1-2|U_1|)$-connected, therefore

$$\begin{aligned}|F_1| + k_1 &\geq n-1-2|U_1| \\ |F_1| + |U| - |U_1| &\geq n-1-2|U_1| \\ |F_1| + |U| &\geq n-1-|U_1| \\ |F_1| + |U| &> n-1-|U| \\ |F_1| &> n-1-2|U| \\ |F_1| &> |F|\end{aligned}$$

this is a contradiction, therefore $(MB^i_{n-1} - N[U]) - F$ is connected for every $i \in [n]$, hence by Subcase 2.1, $(LG_n \ominus U) - F$ is connected. □

In the following lemma, we determine the value of $\kappa_{NB}(LG_n)$.

Lemma 4.8. *Let $n \geq 4$, then $\kappa_{NB}(LG_n) = \lceil \frac{n}{2} \rceil$.*

Proof. From Lemma 4.7, we conclude that $\kappa_{NB}(LG_n) \geq \lceil \frac{n}{2} \rceil$. To show that $\kappa_{NB}(LG_n) \leq \lceil \frac{n}{2} \rceil$, we will construct a vertex-cut strategy of size $\lceil \frac{n}{2} \rceil$. Assume n is even, then the set of vertices $U = \{(i+2, i+3)(n-2-i, n-1-i);\ i = 1, \ldots, \frac{n}{2}-3\} \cup \{(\frac{n}{2}, \frac{n}{2}+1)(1,n), (1,2)(n-2,n-1), (2,3)(1,n-1)\}$ is a vertex-cut strategy of size $\frac{n}{2}$ because the vertex corresponding to the identity permutation $()$ is isolated in $LG_n \ominus U$. Similarly, if n is odd, then $U = \{(i, i+1)(n-1-i, n-i);\ i = 2, \ldots, \frac{n-1}{2} - 1\} \cup \{(1,2)(\frac{n+1}{2}, \frac{n+1}{2}+1), (1, n-1)(\frac{n-1}{2}, \frac{n-1}{2}+1), (1,n)(1,2)\}$ is a vertex-cut strategy of size $\frac{n+1}{2} = \lceil \frac{n}{2} \rceil$, because $()$ becomes an isolated vertex in $LG_n \ominus U$. Therefore, $\kappa_{NB}(LG_n) \leq \lceil \frac{n}{2} \rceil$. □

4.3. *Neighbor Connectivity of $\mathbb{U}_n$*

Let $\mathbb{U}_n = Cay(S_n, \mathcal{T})$ where $G(\mathcal{T})$ is not C_n nor $H_{n,n-1}$, then the generating graph $G(\mathcal{T})$ has always a vertex of degree 1, without loss of generality, let n be such vertex, and let j be the neighbor of n in $G(\mathcal{T})$. Let $\mathcal{T}' = \mathcal{T} - \{(j,n)\}$, then $\mathcal{T}'$ is a set of transpositions of S_{n-1}, and $G(\mathcal{T}')$ is a unicyclic graph on the vertex set $[n-1]$. Let $\mathbb{U}^i_{n-1}$ be the subgraph of $\mathbb{U}_n$ induced by the set of vertices $\{[p_1 p_2 \ldots p_{n-1} i];\ p_k \in [n] - \{i\}$ for $k = 1, \ldots, n-1\}$, then $\mathbb{U}^i_{n-1} \cong \mathbb{U}_{n-1}$. Therefore, $\mathbb{U}_n$ can be decomposed into n vertex disjoint subgraphs $\mathbb{U}^1_{n-1}, \ldots, \mathbb{U}^n_{n-1}$. The following proposition includes useful topological properties of $\mathbb{U}_n$.

Proposition 4.3. *[19] Let $\mathbb{U}_n = Cay(S_n, \mathcal{T})$ where $G(\mathcal{T})$ is a unicyclic graph of vertex set $[n]$ different from C_n and $H_{n,n-1}$, and let $\mathbb{U}^1_{n-1}, \ldots, \mathbb{U}^n_{n-1}$ be the subgraphs defined previously.*

(I) $\mathbb{U}_n$ *is n-regular bipartite graph.*

(II) If $u \in V(\mathbb{U}^i_{n-1})$, then u has exactly one neighbor, u', outside $\mathbb{U}^i_{n-1}$. u' is called the outside neighbor *of u, and $u' = u(j, n)$.*

(III) The outside neighbors of the vertices in $\mathbb{U}^i_{n-1}$ are all different.

(IV) There are exactly $(n-2)!$ independent edges between $\mathbb{U}^i_{n-1}$ and $\mathbb{U}^k_{n-1}$, for $i, k \in [n]$ and $i \neq k$. Such edges are called cross edges.

Lemma 4.9. *[19] Let m be the length of the unique cycle in $G(\mathcal{T})$. Let u and v be two distinct vertices of $\mathbb{U}_n = Cay(S_n, \mathcal{T})$. Then $|N(u) \cap N(v)| \leq 3$ if $m = 3$, and $|N(u) \cap N(v)| \leq 2$ if $m \geq 4$.*

Lemma 4.10. *Let $n \geq 4$ and let $\mathbb{U}_n = Cay(S_n, \mathcal{T})$, where $G(\mathcal{T})$ is a unicyclic graph on the vertex set $[n]$ and the length of its cycle is 3. Suppose $u \in V(\mathbb{U}^i_{n-1})$ for some $i \in [n]$. If u has its outside neighbor u' in $\mathbb{U}^j_{n-1}$ for some $j \in [n] - \{i\}$, then at most $n-3$ vertices in $N_{\mathbb{U}^i_{n-1}}(u)$ have their outside neighbors in $\mathbb{U}^j_{n-1}$.*

Proof. Since $\mathbb{U}_n$ is vertex transitive, then without loss of generality assume that $u = ()$. The generating graph $G(\mathcal{T})$ consists of a 3-cycle with edges corresponding to the transpositions $(1,2), (2,3)$, and $(3,1)$. At least one of the vertices $1, 2$ or 3 belongs to a tree that does not include the other two vertices. It is possible to have the following scenario; $(1, n)$ is an edge of $G(\mathcal{T})$ and vertices 2 and 3 are vertices that belong to disjoint trees. Then every vertex of $N_{\mathbb{U}_{n-1}}[u]$, except the vertices corresponding to $(1, 2)$ and $(1, 3)$ have their outside neighbors in $\mathbb{U}^1_{n-1}$. □

Lemma 4.11. *Let $n \geq 4$ and let $\mathbb{U}_n = Cay(S_n, \mathcal{T})$, where $G(\mathcal{T})$ is a unicyclic graph on the vertex set $[n]$ and the length of its cycle is 3. Let $U \subseteq V(\mathbb{U}_n)$, such that $1 \leq |U| \leq \lfloor \frac{n}{2} \rfloor - 1$. Then $\mathbb{U}_n \ominus U$ is $(n-1-2|U|)$-connected.*

Proof. When $n = 4$, then $\mathbb{U}_4$ is the same as LG_4, then by Lemma 4.7 the result holds. We proceed by mathematical induction on n. Suppose that $\mathbb{U}_{n-1} \ominus W$ is $(n-2-2|W|)$-connected, for every set $W \subseteq V(\mathbb{U}_{n-1})$ such that $1 \leq |W| \leq \lfloor \frac{n-1}{2} \rfloor - 1$. Let $U \subseteq V(\mathbb{U}_n)$, such that $1 \leq |U| \leq \lfloor \frac{n}{2} \rfloor - 1$ and let $F \subseteq V(\mathbb{U}_n)$, such that $|F| \leq n-2-2|U|$. Our aim is to show that $(\mathbb{U}_n \ominus U) - F$ is connected. We consider cases depending on the distribution of the elements of U. Let $U_i = U \cap V(\mathbb{U}^i_{n-1})$, $k_i = |N[U - U_i] \cap V(\mathbb{U}^i_{n-1})|$, and $F_i = F \cap V(\mathbb{U}^i_{n-1})$, for $i \in [n]$.

Case 1. $|U| = |U_1|$. For $i \geq 2$, $|F_i| + k_i \leq (n-2-2|U|) + |U| \leq n-3 < \delta(\mathbb{U}^i_{n-1})$, then by Theorem 3.1, the subgraph induced by the vertices of $(\mathbb{U}^i_{n-1} - N[U]) - F$ is connected. The number of cross edges between the subgraphs induced by $(\mathbb{U}^i_{n-1} - N[U]) - F$ and $(\mathbb{U}^j_{n-1} - N[U]) - F$, for $2 \leq i < j \leq n$, is greater than $(n-2)! - [(n-2-2|U|) + |U|] \geq (n-2)! - (n-3) \geq 1$, for $n \geq 5$, hence there is always a cross edge between these two subgraphs. Then the subgraph C induced by the vertices of $\bigcup_{i=2}^{n} V(\mathbb{U}^i_{n-1}) - (N[U] \cup F)$ is connected. If $(\mathbb{U}^1_{n-1} - N[U]) - F$ is connected, then the graph $(\mathbb{U}_n \ominus U) - F$ becomes connected since there is at least one cross edge connecting a vertex from $(\mathbb{U}^1_{n-1} - N[U]) - F$ and a vertex from C. In fact, the number of cross edges between $(\mathbb{U}^1_{n-1} - N[U]) - F$ and C is

$$\begin{aligned}(n-1)(n-2)! - (|F| + (n-2)|U|) &\geq (n-1)! - (n-2-2|U| + (n-2)|U|)\\ &\geq (n-1)! - n + 2 - (n-4)|U|\\ &\geq (n-1)! - n + 2 - (n-4)\frac{n}{2}\\ &\geq 4.\end{aligned}$$

Suppose that $(\mathbb{U}^1_{n-1} - N[U]) - F$ is not connected. Let C_1 be a connected component in $(\mathbb{U}^1_{n-1} - N[U]) - F$.

Subcase 1.1. *C_1 contains exactly one vertex u*, then u is isolated in $\mathbb{U}^1_{n-1}$. The maximum number of vertices of $N_{\mathbb{U}^1_{n-1}}(u)$ that are adjacent to $N[U_1]$ is $3 + 2(|U| - 1)$, then

$$\begin{aligned}3 + 2(|U| - 1) + |F_1| &\geq deg_{\mathbb{U}^1_{n-1}}(u)\\ 2|U| + 1 + |F_1| &\geq n - 1\\ |F_1| &\geq n - 2 - 2|U|\\ |F_1| &\geq |F|\end{aligned}$$

In this situation $|F_i| = 0$ for every $i \in [n] - \{1\}$, then the outside neighbor of u is a vertex in C, therefore there is an edge between C_1 and C.

Subcase 1.2. *C_1 contains more than one vertex*. Let $u, v \in V(C_1)$ such that u and v are adjacent in C_1. Since $\mathbb{U}_n$ is bipartite, then it does not contain an odd cycle, therefore u and v have no common neighbors. The subgraph induced by $(N[u] \cup N[v]) \cap V(\mathbb{U}^1_{n-1})$ contains $2n-2$ vertices. A vertex of U can not be adjacent to a neighbor of u and a neighbor of v at the same time because this would create an odd cycle. Then the maximum number of vertices of $N_{\mathbb{U}^1_{n-1}}[u] \cup N_{\mathbb{U}^1_{n-1}}[v]$ that are in $N[U_1]$ is $3+3+2(|U|-2) = 2|U|+2$. Then there are at least $(2n-2)-(2|U|+2) = 2n-2|U|$ vertices in the subgraph induced by $[(N[u] \cup N[v]) \cap V(\mathbb{U}^1_{n-1})] - N[U]$. Since $|F| < 2n - 2|U|$, then there exists at least one cross edge incident to a vertex of C_1 and a vertex of C.

Case 2. *$|U_i| \leq |U| - 1$, for $i \in [n]$.* We have $|U_i| \leq \lfloor \frac{n}{2} \rfloor - 2$, then at least two of the $\mathbb{U}^i_{n-1}$ subgraphs contain no elements of U, for $i \in [n]$. Let $\mathbb{U}^1_{n-1}$ and $\mathbb{U}^2_{n-1}$ be these subgraphs. By Theorem 3.1, $\mathbb{U}^1_{n-1}$ and $\mathbb{U}^2_{n-1}$ are $(n-1)$-connected. The maximum number of vertices of $N[U] \cup F$ in $\mathbb{U}^1_{n-1}$ is $|U| + |F| \leq n - 2 - |U| \leq n - 2$, then $(\mathbb{U}^1_{n-1} \ominus U) - F$ is connected. Since $|U^i_{n-1}| \leq \lfloor \frac{n}{2} \rfloor - 2 \leq \lfloor \frac{n-1}{2} \rfloor - 1$, for $n \geq 5$, then by the induction hypothesis, $\mathbb{U}^i_{n-1}$ is $(n-2-2|U_i|)$-connected. On the other hand,

$$\begin{aligned}|F| + N[U - U_i] &\leq n - 2 - 2|U| + |U| - |U_i|\\ &\leq n - 2 - |U| - |U_i|\\ &\leq n - 2 - |U_i| - 1 - |U_i|\\ &\leq n - 3 - 2|U_i|\\ &< n - 2 - 2|U_i|\end{aligned}$$

then $(\mathbb{U}_{n-1}^{i} \ominus U) - F$ is connected, for $i \in [n] - \{1, 2\}$. By Proposition 4.3, there are $(n-2)!$ cross edges between $\mathbb{U}_{n-1}^{i}$ and $\mathbb{U}_{n-1}^{1}$, for $i \in [n] - \{1, 2\}$. If $x \in U_i$, then by Lemma 4.10 $N_{\mathbb{U}_{n-1}^{i}}[x]$ contributes to at most $n-2$ cross edges between $\mathbb{U}_{n-1}^{i}$ and $\mathbb{U}_{n-1}^{1}$. When $n \geq 5$, we have

$$\begin{aligned}(n-2)! - (n-2)|U_i| - |F| &\geq (n-2)! - (n-2)(|U|-1) - (n-2-2|U|)\\ &\geq (n-2)! - n|U|\\ &\geq (n-2)! - n(\lfloor n/2 \rfloor - 1)\\ &\geq 1\end{aligned}$$

then there exists at least one cross edge between $(\mathbb{U}_{n-1}^{i} \ominus U) - F$ and $(\mathbb{U}_{n-1}^{1} \ominus U) - F$ for $i \in [n] - \{1, 2\}$, and since there are enough edges between $(\mathbb{U}_{n-1}^{1} \ominus U) - F$ and $(\mathbb{U}_{n-1}^{2} \ominus U) - F$, then $(\mathbb{U}_n \ominus U) - F$ is connected. □

The previous lemma implies that when the length of the cycle in $G(\mathcal{T})$ is 3, then the value of $\kappa_{NB}(\mathbb{U}_n)$ is greater than $\lfloor \frac{n}{2} \rfloor - 1$. The next lemma provides an upper bound for $\kappa_{NB}(\mathbb{U}_n)$ when the length of the cycle in $G(\mathcal{T})$ is 3.

Lemma 4.12. *Let $n \geq 4$ and let $\mathbb{U}_n = Cay(S_n, \mathcal{T})$, where $G(\mathcal{T})$ is a unicyclic graph on the vertex set $[n]$, where the length of the cycle in $G(\mathcal{T})$ is 3. Then $\lfloor \frac{n}{2} \rfloor \leq \kappa_{NB}(\mathbb{U}_n) \leq n-2$.*

Proof. By Lemma 4.11, we have $\kappa_{NB}(\mathbb{U}_n) \geq \lfloor \frac{n}{2} \rfloor$. Let $\{(1,2), (2,3), (3,1)\}$ be the edges of the 3-cycle of $G(\mathcal{T})$. The vertex $()$ is adjacent to $(1,2), (2,3), (3,1)$ and $n-3$ other vertices corresponding to the remaining edges of $G(\mathcal{T})$. Let $u_1, \ldots, u_{n-3}$ be these vertices, and let u_i' be a vertex adjacent to u_i such that $u_i' \neq ()$, for $i = 1, \ldots, n-3$. If U consists of $\{(1,2,3), u_i';\ \text{for } i = 1, \ldots n-3\}$ then the vertex $()$ is isolated in $\mathbb{U}_n \ominus U$ because $(1,2,3)$ is adjacent to $(1,2), (1,3), (2,3)$, and u_i' is adjacent to u_i for $i = 1, \ldots n-3$. Therefore $\kappa_{NB}(\mathbb{U}_n) \leq n-2$. □

Now we will show that bounds of $\kappa_{NB}(\mathbb{U}_n)$ are tight. In the next lemma, we find a generating graph for which the lower bound of $\kappa_{NB}(\mathbb{U}_n)$ is attained.

Lemma 4.13. *Let $n \geq 4$ and let $\mathbb{U}_n = Cay(S_n, \mathcal{T})$, where $G(\mathcal{T})$ is the graph consisting of vertex set $[n]$ and edge set $\{(1,2), (2,3), (1,3), (i, i+1);\ \text{for } i = 3, \ldots, n-1\}$. Then $\kappa_{NB}(\mathbb{U}_n) = \lfloor \frac{n}{2} \rfloor$.*

Proof. We want to construct a vertex-cut strategy U such that $|U| \leq \lfloor \frac{n}{2} \rfloor$.

Case 1. *n is even.* Let $U = \{(1,2,3), \left(\frac{n}{2}+1, \frac{n}{2}+2\right)(1,2), (i+3, i+4)(n-1-i, n-i);\ \text{for } i = 0, \ldots, \frac{n}{2}-3\}$. The vertex $()$ is isolated in $\mathbb{U}_n \ominus U$, and $|U| = 2 + \frac{n}{2} - 2 = \frac{n}{2}$.

Case 2. *n is odd.* Let $U = \{(1,2,3), (i+3, i+4)\left(\frac{n+3}{2}+i, \frac{n+5}{2}+i\right);\ \text{for } i = 0, \ldots, \frac{n-5}{2}\}$. The vertex $()$ is isolated in $\mathbb{U}_n \ominus U$, and $|U| = 1 + \frac{n-5}{2} + 1 = \frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor$. Then $\kappa_{NB}(\mathbb{U}_n) = \lfloor \frac{n}{2} \rfloor$ when $G(\mathcal{T})$ is the graph of vertex set $[n]$ and edge set $\{(1,2), (2,3), (1,3), (i, i+1);\ \text{for } i = 3, \ldots, n-1\}$. □

Lemma 4.14. *Let $n \geq 4$ and let $\mathbb{U}_n = Cay(S_n, \mathcal{T})$, where $G(\mathcal{T})$ is the graph consisting of vertex set $[n]$ and edge set $\{(2,3), (1,i);\ \text{for } i = 2, \ldots, n\}$. Let $x \in V(\mathbb{U}_{n-1}^{i})$, for some $i \in [n]$. Then $N_{\mathbb{U}_{n-1}^{i}}(x) = \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$, where x_j has its outside neighbor x_j' in $\mathbb{U}_{n-1}^{j}$, for $j \in [n] - \{i\}$. Moreover, if the outside neighbor of x, x', is in $\mathbb{U}_{n-1}^{k}$, for $k \in [n] - \{i\}$, then (x, x', x_k', x_k, x) is a 4-cycle.*

Proof. Since $\mathbb{U}_n$ is a Cayley graph, then it is vertex transitive. Without loss of generality, assume that $x = ()$ is a vertex in $\mathbb{U}_{n-1}^n$, then $N_{\mathbb{U}_{n-1}^1}(x) = \{(2,3),(1,2),(1,3),\ldots,(1,n-1)\}$. Let $x_1 = (2,3)$ and $x_i = (1,i)$, for $i \in [n-1]-\{1\}$. The outside neighbor of x_1 is x_1' and it corresponds to the permutation $(2,3)(1,n)$ which is a vertex in $\mathbb{U}_{n-1}^1$. The outside neighbor of x_i is x_i' and it corresponds to the permutation $(1,i)(1,n) = (1,n,i)$ which is a vertex in $\mathbb{U}_{n-1}^i$, for $i \in [n]-\{1\}$. In addition, the outside neighbor of x is the vertex x' in $\mathbb{U}_{n-1}^1$, and $x' = (1,n)$. It is easy to see that (x, x', x_1', x_1, x) is a 4-cycle. □

Lemma 4.15. *Let $n \geq 4$ and let $\mathbb{U}_n = Cay(S_n, \mathcal{T})$, where $G(\mathcal{T})$ is the graph consisting of vertex set $[n]$ and edge set $\{(2,3),(1,i);$ for $i = 2,\ldots,n\}$. Let $U \subseteq V(\mathbb{U}_n)$, such that $1 \leq |U| \leq n-3$. Then $\mathbb{U}_n \ominus U$ is $(n-2-|U|)$-connected.*

Proof. When $n = 4$, then $|U| = 1$ and the case is the same as the base case of the proof of Lemma 4.11. We proceed using mathematical induction. Suppose that $\mathbb{U}_{n-1} \ominus U'$ is $(n-3-|U'|)$-connected, where $\mathbb{U}_{n-1} = Cay(S_{n-1}, \mathcal{T}')$, and $G(\mathcal{T}')$ is the graph consisting of vertex set $[n-1]$ and edge set $\{(2,3),(1,i);$ for $i = 2,\ldots,n-1\}$, and $U' \subseteq V(\mathbb{U}_{n-1})$ such that $1 \leq |U'| \leq n-4$. Let $F \subseteq V(\mathbb{U}_n)$, such that $|F| \leq n-3-|U|$. We want to show that $(\mathbb{U}_n \ominus U) - F$ is connected. We consider cases depending on the distribution of the elements of U. Let $U_i = U \cap V(\mathbb{U}_{n-1}^i)$, $k_i = |N[U-U_i] \cap V(\mathbb{U}_{n-1}^i)|$, and $F_i = F \cap V(\mathbb{U}_{n-1}^i)$, for $i \in [n]$.

Case 1. $|U_1| = |U|$. For $i \in [n]-\{1\}$, $|F_i| + k_i \leq n-3-|U|+|U| \leq n-3$. Then $(\mathbb{U}_{n-1}^i - N[U]) - F$ is connected for $i \in [n]-\{1\}$. The number of cross edges between $\mathbb{U}_{n-1}^i$ and $\mathbb{U}_{n-1}^j$ is $(n-2)!$, for $i,j \in [n]-\{1\}$ and $i \neq j$, at most $|F|+|U|$ of these edges are incident to vertices of F or to vertices of $N[U]$. Since $|F|+|U| \leq n-3$ and $(n-2)! > n-3$ for $n \geq 5$, then there is always a cross edge between $(\mathbb{U}_{n-1}^i - N[U]) - F$ and $(\mathbb{U}_{n-1}^j - N[U]) - F$. Let C be the graph induced by $\bigcup_{i=2}^n (V(\mathbb{U}_{n-1}^i) - N[U]) - F$, then C is connected. If $(\mathbb{U}_{n-1}^1 - N[U]) - F$ *is connected*, then the number of cross edges between $(\mathbb{U}_{n-1}^1 - N[U]) - F$ and C is $(n-1)(n-2)! - (n|U|+|F|)$; for $n \geq 5$, we have

$$\begin{aligned} n|U| + |F| &\leq n|U| + n - 3 - |U| \\ &\leq (n-1)|U| + (n-3) \\ &\leq (n-1)(n-3) + (n-3) \\ &\leq n(n-3) \\ &< (n-1)! \end{aligned}$$

then there is always a vertex in $(\mathbb{U}_{n-1}^1 - N[U]) - F$ having its outside neighbor in C, therefore $(\mathbb{U}_n - N[U]) - F$ is connected. Suppose that $(\mathbb{U}_{n-1}^1 - N[U]) - F$ *is not connected*, we will show that there is a cross edge between every connected component of $(\mathbb{U}_{n-1}^1 - N[U]) - F$ and C. Let C_1 be a connected component in $(\mathbb{U}_{n-1}^1 - N[U]) - F$.

Subcase 1.1. $|V(C_1)| = 1$. C_1 consists of one vertex u that is isolated in $(\mathbb{U}_{n-1}^1 - N[U]) - F$, then

$$\begin{aligned} |U| + 2 + |F_1| &\geq deg_{\mathbb{U}_{n-1}^1}(u) \\ |F_1| &\geq n - 3 - |U| \\ |F_1| &\geq |F| \end{aligned}$$

then $|F| = |F_1|$, therefore the outside neighbor of u is not in F and by Lemma 4.3 u is not in $N[U]$, then it must be in C, and hence there is a cross edge between C_1 and C.

Subcase 1.2. $|V(C_1)| > 1$. Let x and y be two adjacent vertices in C_1. Since $\mathbb{U}_n$ is bipartite, then it does not contain odd cycles, therefore x and y have no common neighbor in $\mathbb{U}_n$. The subgraph induced by $N_{\mathbb{U}^1_{n-1}}[x] \cup N_{\mathbb{U}^1_{n-1}}[y]$ contains $2n-2$ vertices. A vertex of $N[U]$ can not be adjacent to a vertex in $N_{\mathbb{U}^1_{n-1}}(x)$ and a vertex in $N_{\mathbb{U}^1_{n-1}}(y)$ since this will create an odd cycle. The maximum number of vertices in U^1_{n-1} adjacent to x that are adjacent to vertices of $N[U]$ is $|U|+2$. Similarly, the maximum number of vertices in $\mathbb{U}^1_{n-1}$ adjacent to y that are adjacent to vertices of $N[U]$ is $|U|+2$. There is at most two vertices in U such that, one of them is adjacent to three neighbors of x and the other is adjacent to three neighbors of y, and every other element of U can be adjacent to at most one neighbor of x or to at most one neighbor of y. Then the maximum number of vertices of $N_{\mathbb{U}^1_{n-1}}[x] \cup N_{\mathbb{U}^1_{n-1}}[y]$ that are in $N[U]$ is $6+(|U|-2) = |U|+4 \leq n+1$. Then the subgraph induced by $(N_{\mathbb{U}^1_{n-1}}[x] \cup N_{\mathbb{U}^1_{n-1}}[y]) - N[U]$ contains at least $2n-2-(n+1) = n-3$ vertices, and since $|F| < n-3$, then there is always a vertex in C_1 that has an outside neighbor in C, hence there is a cross edge between C and C_1 in $(\mathbb{U}_n - N[U]) - F$. As a result $(\mathbb{U}_n - N[U]) - F$ is connected.

Case 2. $|U_i| < |U|$ *for every* $i \in [n]$. By the induction hypothesis, $\mathbb{U}^i_{n-1} \ominus U_i$ is $(n-3-|U_i|)$-connected, for $i \in [n]$. If $(\mathbb{U}^i_{n-1} - N[U]) - F$ is connected for every $i \in [n]$, then $(\mathbb{U}_n - N[U]) - F$ is connected. Suppose that $(\mathbb{U}^i_{n-1} - N[U]) - F$ is disconnected for some $i \in [n]$. Without loss of generality, suppose that $i = 1$, then $|F_1| + (|U| - |U_1|) \geq n-3-|U_1|$, then $|F_1| + |U| \geq n-3$, then $|F_1| \geq n-3-|U| \geq |F|$, therefore $|F_1| = |F|$. Then all the elements of F are in $\mathbb{U}^1_{n-1}$ and $|F_i| = 0$ for $i \in [n] - \{1\}$. Let C_1 be a connected component of $(\mathbb{U}^1_{n-1} - N[U]) - F$.

Subcase 2.1. $|C_1| = 1$. Let $C_1 = \{x\}$, where x is an isolated vertex in $\mathbb{U}^1_{n-1} - (N[U] \cup F)$. Then,

$$\begin{aligned} |F_1| + (|U_1|+2) + k_1 &\geq n-1 \\ k_1 &\geq n-1-|F_1|-|U_1|-2 \\ k_1 &\geq n-3-|F_1|-|U_1| \\ k_1 &\geq |U|-|U_1| \end{aligned}$$

then $k_1 = |U| - |U_1|$, which means that every element of $U - U_1$ has its outside neighbor in $N_{\mathbb{U}^1_{n-1}(x)}$. However, By Lemma 4.14 this can only happen if no two vertices of $U - U_1$ belong to the same $\mathbb{U}^i_{n-1}$-subgraph. Then $|U_i| \leq 1$ for $i \in [n] - \{1\}$, and by the induction hypothesis $\mathbb{U}^i_{n-1} \ominus U_i$ is $(n-4)$-connected. Since $|U_1| < |U|$, then $|U_1| \leq n-4$. There could be at most one subgraph of $\mathbb{U}^i_{n-1} \ominus U_i$ for $i \in [n] - \{1\}$, say $\mathbb{U}^2_{n-1} \ominus U_2$, that is disconnected. Let C be the subgraph induced by the vertices of $\bigcup_{i=3}^n (\mathbb{U}^i_{n-1} - N[U]) - F$, then C is connected. Let x' be the outside neighbor of x. If x' is in C, then the case is done. Suppose that x' is in $\mathbb{U}^2_{n-1} - N[U]$, since $|U_2| \leq 1$, then U_2 contains at most one vertex a which has its outside neighbor in $N_{\mathbb{U}^1_{n-1}}(x)$, and a cannot be adjacent to a neighbor of x', because if this is the case then we will have a 5-cycle. By Lemma 4.14, all the neighbors of x' except one (which is x'_2, the outside neighbor of x_2) are in $(\mathbb{U}^2_{n-1} - N[U]) - F$, then x' has a neighbor that has its outside neighbor in C, therefore there exists a path from x to C in $(\mathbb{U}_n \ominus U) - F$.

Subcase 2.2. $|C_1| > 1$. let x and y be two vertices of C_1 such that x and y are adjacent. Let H be the subgraph induced by the vertices of $(N_{\mathbb{U}^1_{n-1}}[x] \cup N_{\mathbb{U}^1_{n-1}}[y]) - (N[U] \cup F)$. H has at least $(2n-2) - (|U_1|+4) - |F|$ vertices. Suppose that there exists $i \in [n] - \{1\}$ such that $(\mathbb{U}^i_{n-1} - N[U]) - F$ is not connected. Without loss of generality, suppose that $(\mathbb{U}^2_{n-1} - N[U]) - F$ is not connected, then $k_2 \geq n-3-|U_2|$, then $|U| - |U_2| \geq n-3-|U_2|$, then $|U| \geq n-3$, hence $|U| = n-3$ and $|F| = 0$. We have the following inequalities $k_1 \geq n-3-|U_1|$ and $k_2 \geq n-3-|U_2|$, then

$$\begin{aligned} k_1 + k_2 &\geq 2(n-3) - (|U_1| + |U_2|) \\ &\geq 2|U| - |U| \\ &\geq |U| \end{aligned}$$

this means that $k_1 + k_2 = |U|$, and $k_i = 0$ for $i \in [n] - \{1,2\}$. Let C be the graph induced by the $\bigcup_{i=3}^{n} V(\mathbb{U}^i_{n-1}) - (N[U] \cup F)$. Since $|F_i| + k_i = 0$, then by the induction hypothesis $(\mathbb{U}^i_{n-1} - N[U]) - F$ is connected for $i \in [n] - \{1,2\}$, and hence C is connected. Since $|F| = 0$, then H has at least $n-2$ vertices because

$$\begin{aligned} (2n-2) - (|U_1|+4) - |F| &> (2n-2) - (|U|+4) - |F| \\ &\geq 2n-2-|U|-4 \\ &\geq 2n-6-(n-3) \\ &\geq n-3. \end{aligned}$$

For $n \geq 5$, H contains at least three vertices and by Lemma 4.14 at most two of them can be in $\mathbb{U}^2_{n-1}$, then a vertex of H has outside neighbor in C. The same approach can be used to show that for every connected component of $(\mathbb{U}_{n-2} - N[U]) - F$ there exists an edge (or path) connecting a vertex of C_1 with a vertex in C. □

Lemmas 4.11, 4.13, and 4.15 provide the following result.

Theorem 4.2. *Let $n \geq 4$ and let $\mathbb{U}_n = Cay(S_n, \mathcal{T})$, where $G(\mathcal{T})$ is a unicyclic graph on n where the length of the cycle in $G(\mathcal{T})$ is 3. Then $\lfloor \frac{n}{2} \rfloor \leq \kappa_{NB}(\mathbb{U}_n) \leq n-2$. Moreover, the bounds are tight.*

5. Conclusion

In this paper, we determined the neighbor connectivity of $Cay(S_n, \mathcal{T})$, where $G(\mathcal{T})$ is a tree with n vertices, a unicyclic graph with n vertices where the unique cycle is of length 3, $n-1$, or n. The methods employed to derive the outcomes presented in this paper can be extended to determine the neighbor connectivity in cases where the length of the cycle in a unicyclic graph falls between 3 and $n-1$. We put forth the following conjecture.

Conjecture 1. *Let $n \geq 6$ and let $\mathbb{U}_n = Cay(S_n, \mathcal{T})$, where $G(\mathcal{T})$ is a unicyclic graph on the vertex set $[n]$. Let m be the length of the cycle in $G(\mathcal{T})$ such that $4 \leq m \leq n-1$.*

1. *If* $n \geq 2m-4$, *then* $\lceil n/2 \rceil \leq \kappa_{NB}(\mathbb{U}_n) \leq n-m+2$
2. *If* $n < 2m-4$, *then* $\lceil n/2 \rceil \leq \kappa_{NB}(\mathbb{U}_n) \leq n-m+2+\lceil \frac{2m-n-4}{2} \rceil$

Moreover, the bounds are tight.

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