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The real equiangular tight frames obtained from rank 3 graphs

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Abstract

We present all nontrivial real equiangular tight frames $\{\varphi_m\}_{m=1}^{M}$ in $\mathbb{R}^N$ obtained as spherical embeddings of primitive rank 3 graphs on M vertices, and those such that one of their associated M strongly regular graphs on $M - 1$ vertices is a primitive rank 3 graph.

1. Introduction

It is well known that a set $\{\varphi_m\}_{m=1}^{M}$ of M $(\geqslant N)$ unit vectors in $\mathbb{R}^N$ satisfies

$$\max_{m \neq n} |\langle \varphi_m, \varphi_n \rangle| \geqslant \sqrt{\frac{M - N}{N(M - 1)}},$$

and equality is attained if and only if $\{\varphi_m\}_{m=1}^M$ is *equiangular*, i.e., $|\langle\varphi_m, \varphi_n\rangle|$ is constant for $m \neq n$, and it is a *tight frame*, i.e., there is a constant $c > 0$ such that

$$\sum_{m=1}^M \langle\varphi, \varphi_m\rangle^2 = c\|\varphi\|^2$$

for all $\varphi \in \mathbb{R}^N$; see, e.g., [28], [7, §10.6.2]. It seems that the concept of real equiangular tight frames (real ETFs) first appeared in Van Lint and Seidel [21]; see also Lemmens and Seidel [20]. We refer the reader to [9, 12, 13, 14, 16, 17, 22, 27] for some of the recent activities on ETFs. This paper presents all nontrivial real ETFs obtained as spherical embeddings of primitive rank 3 graphs on M vertices, and those such that one of their associated M strongly regular graphs on $M-1$ vertices is a primitive rank 3 graph.

This research was started when Ei.B. received an email from Bob Griess dated July 29, 2012. The content of his email was essentially as follows.

> "There are interesting examples of spherical codes which come from VOA[1] theory. For instance, there is a code in 156-dimensional Euclidean space with 496 unit vectors and group $O_{10}^+(2)$. If x is one unit vector, it has inner product 0 with 255 others and inner product $1/8$ with 240 others. (The points of the spherical code are rescales of the so-called conformal vectors of central charge $1/2$). Do you know anything about this code? For example, is it close to being extremal in any sense?"

(He also mentioned that the above procedure is applied to the Monster simple group acting in 196884 dimensions.) The reply to his inquiry was essentially as follows.

> "The 496 points are on the hyperplane in $\mathbb{R}^{156}$ which is at distance $1/4$ from the origin. If we regard this 496 point set as a spherical code on the unit sphere in the 155-dimensional Euclidean space, then the inner products become $1/15$ and $-1/15$. So, we have an equiangular line system. Moreover, this gives an extremal property of being a real equiangular tight frame. Also, this construction of real equiangular tight frames is generalized for $O_{2m}^+(2)$ and $O_{2m}^-(2)$ (for all m) acting as transitive rank 3 groups (i.e., strongly regular graphs) on the set of non-isotropic points."

Then, Ei.B., Et.B., and H.T. quickly found four other infinite families by looking at the eigenmatrices of various primitive rank 3 graphs but did not publish the results. In Winter 2021, C.-Y.L. and W.-H.Y. joined the project of finding *all* nontrivial real ETFs obtained as spherical embeddings of primitive rank 3 graphs. The classification of primitive rank 3 graphs was the culmination of efforts by many researchers such as Foulser, Kallaher, Kantor, Liebler, Liebeck, and Saxl. Their results were inevitably scattered in a mass of papers and were not easy to follow. Our work was then greatly facilitated by the recent book written by Brouwer and Van Maldeghem [8], which includes descriptions of all primitive rank 3 graphs. Besides the above $2 + 4 = 6$ infinite families,

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[1]VOA stands for 'vertex operator algebra.'

it turns out that there are only two sporadic examples. Associated with every nontrivial real ETF $\{\varphi_m\}_{m=1}^M$ are M strongly regular graphs on $M - 1$ vertices having the same parameters. As a related problem, we also determine all nontrivial real ETFs such that one of these strongly regular graphs is a primitive rank 3 graph. Our main results are Theorems 3.2 and 5.1.

This paper is organized as follows. Section 2 recalls the necessary facts about strongly regular graphs and their spherical embeddings. In Section 3, we summarize the classification of primitive rank 3 graphs based on Brouwer and Van Maldeghem [8] and identify all those whose spherical embeddings give rise to real ETFs. Section 4 describes these six infinite families and two sporadic graphs. Finally, Section 5 discusses nontrivial real ETFs, one of whose associated strongly regular graphs is a primitive rank 3 graph.

2. Spherical embeddings of strongly regular graphs

Let Γ be a finite simple graph with vertex set X, that is neither complete nor edgeless. Recall that Γ is called a *strongly regular graph* (*SRG*) *with parameters* (v, k, λ, μ) if it has v vertices and is k-regular, and every pair of adjacent (resp. distinct and nonadjacent) vertices has exactly λ (resp. μ) common neighbors. For the rest of this section, suppose that Γ is an SRG with parameters (v, k, λ, μ). The complement $\overline{\Gamma}$ of Γ is an SRG with parameters $(v, \bar{k}, \bar{\lambda}, \bar{\mu})$, where $\bar{k} = v - k - 1$, $\bar{\lambda} = v - 2k + \mu - 2$, and $\bar{\mu} = v - 2k + \lambda$; cf. [8, §1.1.2]. We also assume that Γ is *primitive*, i.e., Γ is neither a complete multipartite graph ($\mu = k$) nor a union of complete graphs ($\mu = 0$). Let A and $\overline{A} = J - I - A$ be the adjacency matrices of Γ and $\overline{\Gamma}$, respectively, where J is the all one's matrix. Then $\boldsymbol{A} := \text{span}\{I, A, \overline{A}\}$ is a three-dimensional algebra since $A^2 = kI + \lambda A + \mu \overline{A}$. The algebra $\boldsymbol{A}$ is known as the *Bose–Mesner algebra* of Γ; cf. [8, §1.3.2]. The matrix A has eigenvalue k with multiplicity one since Γ is connected and k-regular. It has two other distinct eigenvalues $r > s$ and these are the roots of the quadratic equation $\xi^2 + (\mu - \lambda)\xi + (\mu - k) = 0$; cf. [8, §1.1.1]. Let E_k, E_r, E_s be the orthogonal projections onto the eigenspaces V_k, V_r, V_s of A with eigenvalues k, r, s, respectively. We note that $E_k = v^{-1}J$ and that E_k, E_r, and E_s form a basis of $\boldsymbol{A}$. Define the 3×3 matrices P and Q by $(I, A, \overline{A}) = (E_k, E_r, E_s)P$ and $v(E_k, E_r, E_s) = (I, A, \overline{A})Q$. We call P and Q the *first* and *second eigenmatrices* of Γ, respectively. We have

$$P = \begin{pmatrix} 1 & k & \bar{k} \\ 1 & r & \bar{s} \\ 1 & s & \bar{r} \end{pmatrix},$$

where $\bar{r} = -s - 1$ and $\bar{s} = -r - 1$ are the distinct eigenvalues of $\overline{A}$ other than $\bar{k}$. Moreover, it follows that

$$Q = \begin{pmatrix} 1 & f & g \\ 1 & fr/k & gs/k \\ 1 & f\bar{s}/\bar{k} & g\bar{r}/\bar{k} \end{pmatrix},$$

where $f = \dim V_r$ and $g = \dim V_s = v - f - 1$. Observe in particular that

$$E_s = \frac{g}{v}\left(I + \frac{s}{k}A + \frac{\bar{r}}{\bar{k}}\overline{A}\right). \tag{1}$$

For every $x \in X$, let $(E_s)_x$ denote the xth column of E_s, and consider the set $\{\varphi_x : x \in X\}$ of v (column) vectors in the Euclidean space V_s ($\cong \mathbb{R}^g$) by

$$\varphi_x = \sqrt{\frac{v}{g}}(E_s)_x \qquad (x \in X).$$

Then it follows from (1) that the φ_x are unit vectors in V_s and that

$$\langle \varphi_x, \varphi_y \rangle = \begin{cases} s/k, & \text{if } x, y \text{ are adjacent,} \\ \bar{r}/\bar{k}, & \text{if } x, y \text{ are nonadjacent,} \end{cases} \qquad (x, y \in X, \ x \neq y). \qquad (2)$$

In this paper, we call $\{\varphi_x : x \in X\}$ the *spherical embedding* of Γ. Since

$$\sum_{x \in X} \langle \varphi, \varphi_x \rangle^2 = \frac{v}{g}\varphi^\top E_s \varphi = \frac{v}{g}\|\varphi\|^2 \qquad (\varphi \in V_s),$$

the spherical embedding of Γ gives a real ETF with $(M, N) = (v, g)$ if and only if it is equiangular, i.e.,

$$\frac{s}{k} = -\frac{\bar{r}}{\bar{k}}. \qquad (3)$$

If this is the case, then the columns of the matrix

$$\sqrt{\frac{v}{v-g}}(I - E_s) = \sqrt{\frac{v}{v-g}}(E_k + E_r)$$

also define a real ETF with $(M, N) = (v, v-g)$ in $V_k + V_r$ ($\cong \mathbb{R}^{v-g}$), known as the *Naimark complement* of $\{\varphi_x : x \in X\}$. We note that the spherical embedding of $\overline{\Gamma}$ consists of the normalized columns of E_r.

If $g = 1$, then it follows from (2) that Γ is a complete bipartite graph, which we exclude here. Hence we have $g \geqslant 2$. Likewise, we have $f \geqslant 2$. It follows that the real ETFs obtained in this way are always nontrivial, i.e., the pairs (M, N) satisfy $2 \leqslant N \leqslant M - 2$.[2]

It is known (cf. [8, §8.14]) that every dependent (i.e., $M > N$) real ETF gives rise to a *regular two-graph*, that is to say, a 2-design with block size three such that every 4-set of vertices (or points) contains an even number of blocks. Specifically, the vertices of the latter are indexed by the vectors of the former, where a 3-set of vertices is a block if and only if an odd number of angles between the corresponding three vectors are obtuse angles. Conversely, every regular two-graph is obtained in this way; see, e.g., [7, §§10.3, 10.6], [15, §11.4]. By [7, Proposition 10.3.2], one of the spherical embeddings of Γ and $\overline{\Gamma}$ is a real ETF if and only if

$$v = 2(2k - \lambda - \mu) = 2(2\bar{k} - \bar{\lambda} - \bar{\mu}). \qquad (4)$$

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[2]The *trivial* real ETFs are those with (a) $N = M$ (orthonormal bases); (b) $N = M - 1$ (vertices of regular simplices); or (c) $N = 1$. Cases (b) and (c) are the Naimark complements of each other.

3. Primitive rank 3 graphs

Brouwer and Van Maldeghem [8] described all *primitive rank* 3 *graphs*, i.e., those SRGs admitting primitive rank 3 automorphism groups. Recall that the *socle* of a group G is the subgroup generated by the minimal normal subgroups of G.

Theorem 3.1 ([8, Theorem 11.1.1])**.** *Let Γ be a primitive strongly regular graph on v vertices, and let G be a primitive rank 3 permutation group acting as an automorphism group of Γ. Then one of the following holds.*

(i) $T \times T \vartriangleleft G \leq T_0 \wr 2$, where T_0 is a 2-transitive group of degree v_0, the socle T of T_0 is simple, and $v = v_0^2$.

(ii) The socle L of G is (nonabelian) simple.

(iii) The group G is an affine group, i.e., G has a regular elementary abelian normal subgroup and v is a power of a prime.

See also [10]. Case (i) is discussed in [8, §11.2]. There are 25 kinds of groups, but all the groups generate the same class of the lattice graphs $L_2(q)$ (cf. [8, §1.1.8]). Case (ii) has further four subcases: alternating socle, classical simple socle, exceptional simple socle, and sporadic simple socle; cf. [8, §11.3]. Case (iii) has further three subcases: infinite classes, extraspecial classes, and exceptional classes; cf. [8, §11.4]. The infinite families from Cases (ii) and (iii) are summarized in Tables 1 and 2, respectively.[3] Most[4] of the sporadic examples are listed in [8, §11.5].

After many pages of cumbersome, case-by-case, and largely computer-assisted verifications of (3) or (4),[5] we obtained the following result:

Theorem 3.2. *The primitive rank 3 graphs whose spherical embeddings give rise to nontrivial real equiangular tight frames are given in Table 3.*

We also note the following isomorphisms: $T(5) \cong \overline{NO_4^-(2)}$ (cf. [8, §10.3]), $\overline{T(8)} \cong \overline{NO_3^+(7)} \cong NO_6^+(2)$ (cf. [8, §§3.1.4, 3.6.1]), $\overline{L_2(4)} \cong \overline{O_4^+(3)} \cong H_2(2,2) \cong \overline{VO_2^+(4)} \cong VO_4^+(2)$ (cf. [8, §§2.6.4, 3.3.1, 3.4.1]), $NO_5^-(3) \cong \overline{NO_6^-(2)}$ (cf. [8, §10.15]), and $VSz(2) \cong VO_4^-(2)$ (cf. [8, §2.5.5]).

4. Rank 3 graphs yielding equiangular tight frames

In this section, we describe the primitive rank 3 graphs found in Theorem 3.2 for the convenience of the reader.

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[3]For some families, there are additional conditions to be of rank 3.

[4]The sporadic examples missing in [8, §11.5] are those in the classical simple socle subcase of Case (ii), i.e., Theorem 11.3.2 (v), (vi), (vii), (x), and Theorem 11.3.3 (iii) in [8, §11.3.2]. See also https://homepages.cwi.nl/~aeb/math/srg/rk3/.

[5] Details are available at https://zenodo.org/records/10849838. We will use (* *) to separate the cases. For instance, (*Lattice graph*) denotes new cases of calculation of lattice graphs.

Graph	Reference
$T(n)$	[8, §1.1.7]
$\mathsf{Sp}_{2n}(q)$	[8, §2.5]
$\mathsf{O}_{2n+1}(q)$	[8, §2.6]
$\mathsf{O}_{2n}^{+}(q)$	[8, §2.6]
$\mathsf{O}_{2n+2}^{-}(q)$	[8, §2.6]
$\mathsf{U}_{2n}(\sqrt{q})$	[8, §2.7]
$\mathsf{U}_{2n+1}(\sqrt{q})$	[8, §2.7]
dual polar	[8, §2.2.11]
half dual polar	[8, §2.2.12]
$NU_n(q)$	[8, §3.1.6]
$NO_{2n}^{\varepsilon}(2)$	[8, §3.1.2]
$NO_{2n}^{\varepsilon}(3)$	[8, §3.1.3]
$NO_{2n+1}^{\varepsilon}(q)$	[8, §3.1.4]
$J_q(n,2)$	[8, §3.5.1]
$\mathsf{E}_{6,1}(q)$	[8, §4.9]

Table 1: Case (ii) families

Graph	Reference
$P(q)$	[8, §1.1.9]
$P^*(q)$	[8, §7.3.6]
Van Lint-Schrijver	[8, §7.3.1]
$L_2(q)$	[8, §1.1.8]
$H_q(2,e)$	[8, §3.4.1]
$VO_{2n}^{\varepsilon}(q)$	[8, §3.3.1]
alternating forms	[8, §3.4.2]
$VD_{5,5}(q)$	[8, §3.3.3]
$VSz(q)$	[8, §3.3.1]

Table 2: Case (iii) families

Example 4.1 ($NO_{2n}^{+}(2)$)**.** Equip $\mathbb{F}_2^{2n}$ with a nondegenerate quadratic form of Witt index n and let X be the set of nonsingular points. Let $\Gamma = NO_{2n}^{+}(2)$ have vertex set X, two vertices being adjacent when they are orthogonal. See [8, §3.1.2]. For $n \geqslant 3$, the graph $NO_{2n}^{+}(2)$ is a primitive rank 3 graph with parameters $(2^{n-1}(2^n - 1), 2^{2n-2} - 1, 2^{2n-3} - 2, 2^{n-2}(2^{n-1} + 1))$ and eigenmatrices

$$
P = \begin{pmatrix} 1 & 2^{2n-2} - 1 & 2^{n-1}(2^{n-1} - 1) \\ 1 & 2^{n-2} - 1 & -2^{n-2} \\ 1 & -2^{n-1} - 1 & 2^{n-1} \end{pmatrix}, \quad
Q = \begin{pmatrix} 1 & \frac{2^{2n}-4}{3} & \frac{(2^{n-1}-1)(2^n-1)}{3} \\ 1 & \frac{2^n-4}{3} & -\frac{2^n-1}{3} \\ 1 & -\frac{2^n+2}{3} & \frac{2^n-1}{3} \end{pmatrix}.
$$

Example 4.2 ($\overline{NO_{2n}^{-}(2)}$)**.** Equip $\mathbb{F}_2^{2n}$ with a nondegenerate quadratic form of Witt index $n - 1$ and let X be the set of nonsingular points. Let $\Gamma = \overline{NO_{2n}^{-}(2)}$ have vertex set X, two vertices being adjacent when they are nonorthogonal. See [8, §3.1.2]. For $n \geqslant 2$, the graph $\overline{NO_{2n}^{-}(2)}$ is a primitive rank 3 graph with parameters $(2^{n-1}(2^n + 1), 2^{n-1}(2^{n-1} + 1), 2^{n-2}(2^{n-1} + 1), 2^{n-1}(2^{n-2} + 1))$ and eigenmatrices

$$
P = \begin{pmatrix} 1 & 2^{n-1}(2^{n-1} + 1) & 2^{2n-2} - 1 \\ 1 & 2^{n-2} & -2^{n-2} - 1 \\ 1 & -2^{n-1} & 2^{n-1} - 1 \end{pmatrix}, \quad
Q = \begin{pmatrix} 1 & \frac{2^{2n}-4}{3} & \frac{(2^{n-1}+1)(2^n+1)}{3} \\ 1 & \frac{2^n-2}{3} & -\frac{2^n+1}{3} \\ 1 & -\frac{2^n+4}{3} & \frac{2^n+1}{3} \end{pmatrix}.
$$

Example 4.3 ($NO_{2n+1}^{+}(4)$)**.** Equip $\mathbb{F}_4^{2n+1}$ with a nondegenerate quadratic form and let X be the set of nonsingular hyperbolic hyperplanes. Let $\Gamma = NO_{2n+1}^{+}(4)$ have vertex set X, two vertices

Graph	M	N	$M - N$
$NO_{2n}^+(2)$ $(n \geqslant 3)$	$2^{n-1}(2^n - 1)$	$\frac{(2^{n-1}-1)(2^n-1)}{3}$	$\frac{2^{2n}-1}{3}$
$\overline{NO_{2n}^-(2)}$ $(n \geqslant 2)$	$2^{n-1}(2^n + 1)$	$\frac{(2^{n-1}+1)(2^n+1)}{3}$	$\frac{2^{2n}-1}{3}$
$NO_{2n+1}^+(4)$ $(n \geqslant 1)$	$\frac{4^n(4^n+1)}{2}$	$\frac{4^{2n}-1}{3}$	$\frac{(4^n+1)(4^n+2)}{6}$
$\overline{NO_{2n+1}^-(4)}$ $(n \geqslant 2)$	$\frac{4^n(4^n-1)}{2}$	$\frac{4^{2n}-1}{3}$	$\frac{(4^n-1)(4^n-2)}{6}$
$VO_{2n}^+(2)$ $(n \geqslant 2)$	2^{2n}	$2^{n-1}(2^n - 1)$	$2^{n-1}(2^n + 1)$
$\overline{VO_{2n}^-(2)}$ $(n \geqslant 2)$	2^{2n}	$2^{n-1}(2^n + 1)$	$2^{n-1}(2^n - 1)$
$\overline{G_2(2)}$	36	21	15
$\overline{M_{22}}$	176	154	22

Table 3: Primitive rank 3 graphs yielding nontrivial real ETFs

being adjacent when the restriction of the quadratic form to their intersection is degenerate. See [8, §3.1.4]. For $n \geqslant 1$, the graph $NO_{2n+1}^+(4)$ is a primitive rank 3 graph with parameters $(4^n(4^n + 1)/2, (4^{n-1} + 1)(4^n - 1), (4^{n-1} + 2)(4^n - 2)/2, 4^n(4^{n-1} + 1)/2)$ and eigenmatrices

$$P = \begin{pmatrix} 1 & (4^{n-1} + 1)(4^n - 1) & 4^{n-1}(4^n - 1) \\ 1 & 2 \cdot 4^{n-1} - 1 & -2 \cdot 4^{n-1} \\ 1 & -4^{n-1} - 1 & 4^{n-1} \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & \frac{2(4^{n-1}+1)(4^n-1)}{3} & \frac{4^{2n}-1}{3} \\ 1 & \frac{4^n-2}{3} & -\frac{4^n+1}{3} \\ 1 & -\frac{4(4^{n-1}+1)}{3} & \frac{4^n+1}{3} \end{pmatrix}.$$

Example 4.4 ($\overline{NO_{2n+1}^-(4)}$)**.** Equip $\mathbb{F}_4^{2n+1}$ with a nondegenerate quadratic form and let X be the set of nonsingular elliptic hyperplanes. Let $\Gamma = \overline{NO_{2n+1}^-(4)}$ have vertex set X, two vertices being adjacent when the restriction of the quadratic form to their intersection is nondegenerate. See [8, §3.1.4]. For $n \geqslant 2$, the graph $\overline{NO_{2n+1}^-(4)}$ is a primitive rank 3 graph with parameters $(4^n(4^n - 1)/2, 4^{n-1}(4^n + 1), 4^n(4^{n-1} + 1)/2, 4^{n-1}(4^n + 2)/2)$ and eigenmatrices

$$P = \begin{pmatrix} 1 & 4^{n-1}(4^n + 1) & (4^{n-1} - 1)(4^n + 1) \\ 1 & 2 \cdot 4^{n-1} & -2 \cdot 4^{n-1} - 1 \\ 1 & -4^{n-1} & 4^{n-1} - 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & \frac{2(4^{n-1}-1)(4^n+1)}{3} & \frac{4^{2n}-1}{3} \\ 1 & \frac{4(4^{n-1}-1)}{3} & -\frac{4^n-1}{3} \\ 1 & -\frac{4^n+2}{3} & \frac{4^n-1}{3} \end{pmatrix}.$$

Example 4.5 ($VO_{2n}^+(2)$)**.** Equip $\mathbb{F}_2^{2n}$ with a nondegenerate quadratic form of Witt index n and let $X = \mathbb{F}_2^{2n}$. Let $\Gamma = VO_{2n}^+(2)$ have vertex set X, two vertices being adjacent when their difference is isotropic. See [8, §3.3.1]. For $n \geqslant 2$, the graph $VO_{2n}^+(2)$ is a primitive rank 3 graph with parameters $(2^{2n}, (2^{n-1} + 1)(2^n - 1), (2^{n-1} + 2)(2^{n-1} - 1), 2^{n-1}(2^{n-1} + 1))$ and eigenmatrices

$$P = Q = \begin{pmatrix} 1 & (2^{n-1} + 1)(2^n - 1) & 2^{n-1}(2^n - 1) \\ 1 & 2^{n-1} - 1 & -2^{n-1} \\ 1 & -2^{n-1} - 1 & 2^{n-1} \end{pmatrix}.$$

Example 4.6 ($\overline{VO_{2n}^-(2)}$)**.** Equip $\mathbb{F}_2^{2n}$ with a nondegenerate quadratic form of Witt index $n-1$ and let $X = \mathbb{F}_2^{2n}$. Let $\Gamma = \overline{VO_{2n}^-(2)}$ have vertex set X, two vertices being adjacent when their difference is nonisotropic. See [8, §3.3.1]. For $n \geqslant 2$, the graph $\overline{VO_{2n}^-(2)}$ is a primitive rank 3 graph with parameters $(2^{2n}, 2^{n-1}(2^n+1), 2^{n-1}(2^{n-1}+1), 2^{n-1}(2^{n-1}+1))$ and eigenmatrices

$$P = \begin{pmatrix} 1 & 2^{n-1}(2^n+1) & (2^{n-1}-1)(2^n+1) \\ 1 & 2^{n-1} & -2^{n-1}-1 \\ 1 & -2^{n-1} & 2^{n-1}-1 \end{pmatrix},$$

$$Q = \begin{pmatrix} 1 & (2^{n-1}-1)(2^n+1) & 2^{n-1}(2^n+1) \\ 1 & 2^{n-1}-1 & -2^{n-1} \\ 1 & -2^{n-1}-1 & 2^{n-1} \end{pmatrix}.$$

Example 4.7 ($\overline{G_2(2)}$)**.** Let X be the set consisting of the 7 points, 7 lines, and 21 flags of the Fano plane, together with an additional element denoted by ∞. The graph $\overline{G_2(2)}$ has vertex set X and the adjacency is defined as follows: The vertex ∞ is adjacent to the flags. The points form a clique, and so do the lines. A point and a line are adjacent when they are incident. A point (resp. line) is adjacent to the flags whose lines (resp. points) are not on it. Finally, two flags are adjacent when either they are not disjoint, or they are disjoint and the point of each of them is not on the line of the other. See [8, §10.14]. The graph $\overline{G_2(2)}$ is a primitive rank 3 graph with parameters $(36, 21, 12, 12)$ and eigenmatrices

$$P = \begin{pmatrix} 1 & 21 & 14 \\ 1 & 3 & -4 \\ 1 & -3 & 2 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 14 & 21 \\ 1 & 2 & -3 \\ 1 & -4 & 3 \end{pmatrix}.$$

Example 4.8 ($\overline{M_{22}}$)**.** Let X be the set of 176 blocks of the unique quasi-symmetric 2-$(22, 7, 16)$ design with block intersection numbers 1 and 3. Let $\Gamma = \overline{M_{22}}$ have vertex set X, two vertices being adjacent when they intersect in 3 points. See [8, §10.51]. The graph $\overline{M_{22}}$ is a primitive rank 3 graph with parameters $(176, 105, 68, 54)$ and eigenmatrices

$$P = \begin{pmatrix} 1 & 105 & 70 \\ 1 & 17 & -18 \\ 1 & -3 & 2 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 21 & 154 \\ 1 & \frac{17}{5} & -\frac{22}{5} \\ 1 & -\frac{27}{5} & \frac{22}{5} \end{pmatrix}.$$

Remark. The six infinite families of SRGs in Examples 4.1–4.6 are part of more general families of association schemes obtained from actions of classical groups, and their eigenmatrices were extensively studied. See, e.g., [1, 2, 3, 6, 18, 19, 23, 24, 25, 26]. It may be interesting to point out that these SRGs and association schemes were used to construct Ramanujan graphs; cf. [4, 5].

Observe that some of the real ETFs in Theorem 3.2 have the same parameters up to Naimark complements (see also Table 5 in Section 5). We wonder if, for example, the real ETFs obtained from $NO_{2n+1}^+(4)$ and $\overline{NO_{4n}^-(2)}$ are equivalent, i.e., the graphs $NO_{2n+1}^+(4)$ and $\overline{NO_{4n}^-(2)}$ are switching equivalent.

We also note that the real ETFs in Theorem 3.2 are still realized only as Gram matrices since the isomorphism $V_s \cong \mathbb{R}^N$ (where $N = g$) is not canonical. However, the families $VO_{2n}^+(2)$ and $\overline{VO_{2n}^-(2)}$ are Cayley graphs on elementary abelian 2-groups, and their real ETFs can be given as explicit N-dimensional real unit column vectors. For $VO_{2n}^+(2)$, we may assume that the quadratic form is given by

$$q(x) = x_1 x_2 + \cdots + x_{2n-1} x_{2n}$$

for $x = (x_1, \ldots, x_{2n}) \in \mathbb{F}_2^{2n}$. The associated bilinear form is then given by

$$B(x, y) = (x_1 y_2 + x_2 y_1) + \cdots + (x_{2n-1} y_{2n} + x_{2n} y_{2n-1})$$

for $x = (x_1, \ldots, x_{2n}), y = (y_1, \ldots, y_{2n}) \in \mathbb{F}_2^{2n}$. Let X_1 be the set of nonisotropic points in $X = \mathbb{F}_2^{2n}$. Then $|X_1| = g = N = 2^{n-1}(2^n - 1)$, and the following column vectors form an orthonormal basis of V_s:

$$\frac{1}{2^n} \big((-1)^{B(x,z)} \big)_{x \in X} \quad (z \in X_1).$$

See, e.g., [26]. Hence, with respect to this basis, the vectors φ_x $(x \in X)$ in the real ETF are expressed as $2^{n-1}(2^n - 1)$-dimensional column vectors

$$\frac{1}{\sqrt{2^{n-1}(2^n - 1)}} \big((-1)^{B(x,z)} \big)_{z \in X_1} \quad (x \in X).$$

A similar discussion applies to $\overline{VO_{2n}^-(2)}$.

5. Real equiangular tight frames having rank 3 descendants

We mentioned earlier that every dependent real ETF corresponds to a regular two-graph. If, moreover, the former is nontrivial, then the derived designs of the latter are SRGs with $v = M - 1$ and $k = 2\mu$. The derived designs are called the *descendants* [7, §10.2] and also the *neighborhoods* [15, §11.5] of the latter. Here, let us temporarily call them the *descendants of the real ETF*. Conversely, if Γ is an SRG with $k = 2\mu$, then the matrix

$$I + \frac{1}{1 + 2r} \left(\begin{array}{c|c} 0 & \mathbf{1}^\top \\ \hline \mathbf{1} & J - I - 2A \end{array} \right)$$

is the Gram matrix of a nontrivial real ETF with $M = v + 1$ and $N = g + 1$, where $\mathbf{1}$ denotes the all one's vector. The graphs Γ and $\overline{\Gamma}$ are then descendants of this real ETF and its Naimark complement, respectively. See [7, §10.3], [15, §11.6], and also [27, §5]. Our findings are as follows:[6]

Theorem 5.1. *The primitive rank* 3 *graphs, up to taking complements, that are descendants of nontrivial real equiangular tight frames, are given in Table 4.*

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[6]See Footnote 5.

Graph	M	N	$M - N$
$\mathsf{Sp}_{2n}(2)$ $(n \geqslant 2)$	2^{2n}	$2^{n-1}(2^n - 1)$	$2^{n-1}(2^n + 1)$
$\mathsf{O}_{2n}^+(2)$ $(n \geqslant 2)$	$2^{n-1}(2^n + 1)$	$\frac{2^{2n}-1}{3}$	$\frac{(2^{n-1}+1)(2^n+1)}{3}$
$\mathsf{O}_{2n+2}^-(2)$ $(n \geqslant 2)$	$2^n(2^{n+1} - 1)$	$\frac{(2^n-1)(2^{n+1}-1)}{3}$	$\frac{2^{2n+2}-1}{3}$
$P(q)$	$q + 1$	$\frac{q+1}{2}$	$\frac{q+1}{2}$
$P^*(q)$	$q + 1$	$\frac{q+1}{2}$	$\frac{q+1}{2}$
McLaughlin	276	23	253
$P^{**}(529)$	530	265	265
$(2209, 1104, 551, 552)$	2210	1105	1105

Table 4: Primitive rank 3 descendants of nontrivial real ETFs

For $P(q)$, q is a prime power and $q \equiv 1 \pmod 4$. For $P^*(q)$, q is an even power of a prime $p \equiv 3 \pmod 4$. For the McLaughlin graph, see [8, §10.61]. The graph $P^{**}(529)$ is the sporadic Peisert graph [8, §10.70]. There are precisely three rank 3 graphs with the last parameters, two of which are $P(2209)$ and $P^*(2209)$; cf. [8, §10.86].

We note the following isomorphisms: $\mathsf{O}_{2n+1}(2) \cong \mathsf{Sp}_{2n}(2)$ (cf. [8, §2.6.3]), $\overline{T(6)} \cong NO_4^-(3) \cong \overline{NO_3^+(5)} \cong \mathsf{Sp}_4(2)$ (cf. [8, §§3.1.3, 3.1.4, 10.5]), $J_2(4,2) \cong \mathsf{O}_6^+(2)$ (cf. [8, §10.13]), and $L_2(3) \cong VO_2^+(3) \cong P(9) \cong \mathsf{O}_4^+(2)$ (cf. [8, §10.2]). It should be mentioned that the graph $NU_3(3)$ satisfies $k = 2\mu$ but is of rank 4; cf. [8, §10.22]. We do not describe the graphs in Table 4 to keep the paper concise. See the references given.

The Paley graphs $P(q)$ and the Peisert graphs $P^*(q)$ give rise to real ETFs having the same parameters, so we ask if these real ETFs are equivalent, i.e., the disjoint union $K_1 + P(q)$ is switching equivalent to $K_1 + P^*(q)$. We may ask the same question for $P^{**}(529)$ and the last graph in Table 4.

Thm. 3.2	Thm. 5.1	M	$\{N, M - N\}$
$NO_{2n}^+(2)$	$\mathsf{O}_{2n}^-(2)$	$2^{n-1}(2^n - 1)$	$\left\{ \frac{(2^{n-1}-1)(2^n-1)}{3}, \frac{2^{2n}-1}{3} \right\}$
$\overline{NO_{2n}^-(2)}$	$\mathsf{O}_{2n}^+(2)$	$2^{n-1}(2^n + 1)$	$\left\{ \frac{(2^{n-1}+1)(2^n+1)}{3}, \frac{2^{2n}-1}{3} \right\}$
$NO_{2n+1}^+(4)$	$\mathsf{O}_{4n}^+(2)$	$2^{2n-1}(2^{2n} + 1)$	$\left\{ \frac{2^{4n}-1}{3}, \frac{(2^{2n-1}+1)(2^{2n}+1)}{3} \right\}$
$\overline{NO_{2n+1}^-(4)}$	$\mathsf{O}_{4n}^-(2)$	$2^{2n-1}(2^{2n} - 1)$	$\left\{ \frac{2^{4n}-1}{3}, \frac{(2^{2n-1}-1)(2^{2n}-1)}{3} \right\}$
$VO_{2n}^+(2), \overline{VO_{2n}^-(2)}$	$\mathsf{Sp}_{2n}(2)$	2^{2n}	$\left\{ 2^{n-1}(2^n - 1), 2^{n-1}(2^n + 1) \right\}$
$\overline{\mathsf{G}_2(2)}$	$\mathsf{O}_6^+(2)$	36	$\{21, 15\}$

Table 5: Comparison of the parameters of real ETFs

Some of the real ETFs found in Theorems 3.2 and 5.1 also have the same parameters; see Table 5. We again wonder if some of these real ETFs are equivalent. If the spherical embedding of an SRG Γ is a real ETF, then its descendants are obtained by first removing a vertex from Γ and then switching it with respect to the neighbors and the nonneighbors of the removed vertex. For example, if $\Gamma = NO_{2n}^+(2)$, then we ask if the resulting graph is isomorphic to $O_{2n}^-(2)$. It seems that this problem was already addressed in [11, Remark 4.6] for the first, second, and fifth rows in Table 5. We may remark that if Γ has parameters (v, k, λ, μ) (cf. (4)), then the parameters of the descendants are given by $(v - 1, 2(k - \mu), k + \lambda - 2\mu, k - \mu)$; see [7, Proposition 10.3.2].

Acknowledgements

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