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The competition numbers of Johnson graphs with diameter four

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Abstract

In 2010, Kim, Park and Sano studied the competition numbers of Johnson graphs. They gave the competition numbers of $J(n,2)$ and $J(n,3)$. In this note, we consider the competition number of $J(n,4)$.

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1. Introduction

The notion of a competition graph was introduced by Cohen [1] as a means of determining the smallest dimension of ecological phase space. The *competition graph* $C(D)$ of a digraph D is a simple undirected graph which has the same vertex set as D and an edge between vertices x and y if and only if there exists a vertex $u \in D$ such that (x,u) and (y,u) are arcs of D. For any graph G, G together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. Roberts [7] defined the *competition number* $k(G)$ of a graph G to be the smallest number k such that G together with k isolated vertices is the competition graph of an acyclic digraph. Opsut [4] showed that the computation of the competition number of a graph is an NP-hard problem. In the study of competition graphs, it has been one of important problems to determine the competition numbers for various graph classes. In [3], Kim, Park and Sano studied the competition numbers of Johnson graphs. In particular, they gave the following results.

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Theorem 1.1 (See [3]). *For $n \geq 4$, we have $k(J(n,2)) = 2$.*

Theorem 1.2 (See [3]). *For $n \geq 6$, we have $k(J(n,3)) = 4$.*

They also asked about the exact value of the competition number of $J(n,4)$. In this note, we give a partial answer to the question. Our result is the following.

Theorem 1.3. *For $n \geq 8$, we have $k(J(n,4)) \in \{7,8,9\}$.*

2. Preliminaries

Throughout this note, we use the notations given in [3]. We denote an n-set $\{1,\ldots,n\}$ by $[n]$ and the set of all d-subsets of n-set by $\binom{[n]}{d}$. The *Johnson graph* $J(n,d)$ is an undirected graph whose vertex set is $\{v_X \mid X \in \binom{[n]}{d}\}$, and two vertices v_{X_1} and v_{X_2} are adjacent if and only if $|X_1 \cap X_2| = d-1$. Since $J(n,d)$ is isomorphic to $J(n,n-d)$, we always assume $n \geq 2d$.

For a digraph D, a sequence $v_1,\ldots,v_n$ of the vertex set $V(D)$ is called an *acyclic ordering* of D if $(v_i,v_j) \in A(D)$ implies $i<j$. It is well known that a digraph D is acyclic if and only if there exists an acyclic ordering of D.

For a digraph D and a vertex v of D, we define the *out-neighborhood* $P_D(v)$ of v in D to be the set $\{w \in V(D) \mid (v,w) \in A(D)\}$. A vertex in the out-neighborhood of a vertex v in a digraph D is called a *prey* of v in D.

For a graph G and a vertex v of G, we define the *neighborhood* $N_G(v)$ of v in G to be the set $\{u \in V(G) \mid uv \in E(G)\}$. We also use $N_G(v)$ to stand for the subgraph induced by its vertices.

For a clique S of a graph G and an edge e of G, we say *e is covered by S* if both of the endpoints of e are contained in S. An *edge clique cover* of a graph G is a family of cliques such that each edge of G is covered by some clique in the family. The *edge clique cover number* $\theta_E(G)$ of a graph G is the minimum size of an edge clique cover of G. An edge clique cover of G is called a *minimum edge clique cover* of G if its size is equal to $\theta_E(G)$. A *vertex clique cover* of a graph G is a family of cliques such that each vertex of G is contained in some clique in the family. The *vertex clique cover number* $\theta_V(G)$ of a graph G is the minimum size of a vertex clique cover of G.

A minimum edge clique cover of $J(n,d)$ is given in [3] as follows. For each $Y \in \binom{[n]}{d-1}$, we define

$$S_Y = \{v_X \mid X = Y \cup \{j\} \text{ for } j \in [n] \setminus Y\}.$$

Then $\{S_Y \mid Y \in \binom{[n]}{d-1}\}$ is the collection of cliques of maximum size. We denote it by $\mathcal{F}_d^n$. Note that $\mathcal{F}_d^n$ is an edge clique cover of $J(n,d)$.

Lemma 2.1 (See Section 3 of [3]). *We have $\theta_E(J(n,d)) = \binom{n}{d-1}$, and $\mathcal{F}_d^n$ is a minimum edge clique cover of $J(n,d)$.*

3. Main results

In this section, we give a lower bound for the competition number of $J(n,d)$ and an upper bound for the competition number of $J(n,4)$.

Lemma 3.1 (See Lemma 3 of [3]). *We have $\theta_V(N_{J(n,d)}(x)) = d$.*

Lemma 3.2 (See Theorem 4 of [3]). *For any two adjacent vertices v_{X_1} and v_{X_2} of $J(n,d)$, we have $|P_D(v_{X_1}) \setminus P_D(v_{X_2})| \geq d-1$.*

Theorem 3.1. *For $n \geq 2d \geq 8$, we have $k(J(n,d)) \geq 2d-1$.*

Proof. We denote $k(J(n,d))$ by k. Then there exists an acyclic digraph D such that $C(D) = J(n,d) \cup I_k$, where $I_k = \{z_1, z_2, \ldots, z_k\}$ is a set of isolated vertices.

Let $x_1, x_2, \ldots, x_{\binom{n}{d}}, z_1, z_2, \ldots, z_k$ be an acyclic ordering of D. Put $v_1 = x_{\binom{n}{d}}$, $v_2 = x_{\binom{n}{d}-1}$ and $v_3 = x_{\binom{n}{d}-2}$. It follows from Lemma 3.1 that $\theta_V(N_{J(n,d)}(x_i)) = d$ for $1 \leq i \leq \binom{n}{d}$. So, v_i has at least d distinct prey in D, that is,

$$|P_D(v_i)| \geq d. \tag{1}$$

Since $x_1, x_2, \ldots, x_{\binom{n}{d}}, z_1, z_2, \ldots, z_k$ is an acyclic ordering of D, we have

$$P_D(v_1) \cup P_D(v_2) \cup P_D(v_3) \subseteq I_k \cup \{v_1, v_2\}. \tag{2}$$

First of all, we assume that v_1 and v_2 are not adjacent in $J(n,d)$. Then v_1 and v_2 do not have a common prey in D, that is,

$$P_D(v_1) \cap P_D(v_2) = \emptyset. \tag{3}$$

It follows from (1), (2) and (3) that

$$k+1 \geq |P_D(v_1) \cup P_D(v_2)| = |P_D(v_1)| + |P_D(v_2)| \geq 2d.$$

So, we have $k \geq 2d-1$.

Next, we assume that v_1 and v_2 are adjacent in $J(n,d)$. Then v_1 and v_2 have at least one common prey in D, that is,

$$|P_D(v_1) \cap P_D(v_2)| \geq 1. \tag{4}$$

Now we divide our consideration into four cases:

1. v_1 and v_3 are not adjacent, and v_2 and v_3 are not adjacent;
2. v_1 and v_3 are adjacent, and v_2 and v_3 are not adjacent;
3. v_1 and v_3 are not adjacent, and v_2 and v_3 are adjacent;
4. v_1 and v_3 are adjacent, and v_2 and v_3 are adjacent.

In the first case, we have

$$\begin{aligned}
k+2 &\geq |P_D(v_1) \cup P_D(v_2) \cup P_D(v_3)| \quad \text{(by (2))} \\
&= |P_D(v_3)| + |P_D(v_1) \setminus P_D(v_2)| + |P_D(v_2) \setminus P_D(v_1)| + |P_D(v_1) \cap P_D(v_2)| \\
&\geq d + d - 1 + d - 1 + 1 \quad \text{(by (1), Lemma 3.2 and (4))} \\
&= 3d - 1.
\end{aligned}$$

So, we have $k \geq 3d-3$.

In the second case, we have

$$\begin{aligned} k+2 &\geq |P_D(v_1)\cup P_D(v_2)\cup P_D(v_3)| \quad \text{(by (2))} \\ &= |P_D(v_3)\setminus P_D(v_1)|+|P_D(v_2)\setminus P_D(v_1)|+|P_D(v_1)\setminus P_D(v_2)|+|P_D(v_1)\cap P_D(v_2)| \\ &\geq d-1+d-1+d-1+1 \quad \text{(by Lemma 3.2 and (4))} \\ &= 3d-2. \end{aligned}$$

So, we have $k \geq 3d-4$.

In the third case, we have

$$\begin{aligned} k+2 &\geq |P_D(v_1)\cup P_D(v_2)\cup P_D(v_3)| \quad \text{(by (2))} \\ &= |P_D(v_3)\setminus P_D(v_2)|+|P_D(v_1)\setminus P_D(v_2)|+|P_D(v_2)\setminus P_D(v_1)|+|P_D(v_1)\cap P_D(v_2)| \\ &\geq d-1+d-1+d-1+1 \quad \text{(by Lemma 3.2 and (4))} \\ &= 3d-2. \end{aligned}$$

So, we have $k \geq 3d-4$.

In the fourth case, we have

$$\begin{aligned} k+2 &\geq |P_D(v_1)\cup P_D(v_2)\cup P_D(v_3)| \quad \text{(by (2))} \\ &\geq |P_D(v_3)\setminus P_D(v_2)|+|P_D(v_2)\setminus P_D(v_1)|+|P_D(v_1)\setminus P_D(v_3)| \\ &= d-1+d-1+d-1 \quad \text{(by Lemma 3.2)} \\ &= 3d-3. \end{aligned}$$

So, we have $k \geq 3d-5$.

Since $d \geq 4$, it holds $3d-5 \geq 2d-1$. Therefore, we have $k(J(n,d)) \geq 2d-1$. □

Now we give an order $\prec$ on the vertex set of $J(n,d)$ as follows. Take two distinct elements v_{X_1} and v_{X_2} in $\{v_X \mid X \in \binom{[n]}{d}\}$. Let $X_1 = \{i_1,\ldots,i_d\}$ and $X_2 = \{j_1,\ldots,j_d\}$, where $i_1 < \cdots < i_d$ and $j_1 < \cdots < j_d$. Then we define $v_{X_1} \prec v_{X_2}$ if there exists $t \in \{1,\ldots,d\}$ such that $i_s = j_s$ for $1 \leq s \leq t-1$ and $i_t < j_t$.

Theorem 3.2. *For $n \geq 8$, we have $k(J(n,4)) \leq 9$.*

Proof. We define a digraph D as follows:

$$V(D) = V(J(n,4)) \cup I_9$$

where $I_9 = \{z_1, \ldots, z_9\}$, and

$$
\begin{aligned}
A(D) &= \bigcup_{i=1}^{n-4}\bigcup_{j=i+1}^{n-3}\bigcup_{k=j+1}^{n-2}\{(x, v_{\{i,j,k+1,k+2\}}) \mid x \in S_{\{i,j,k\}} \in \mathcal{F}_4^n\} \\
&\cup \bigcup_{i=1}^{n-4}\bigcup_{j=i+1}^{n-3}\{(x, v_{\{i,j+1,j+2,j+3\}}) \mid x \in S_{\{i,j,n-1\}} \in \mathcal{F}_4^n\} \\
&\cup \bigcup_{i=1}^{n-5}\bigcup_{j=i+1}^{n-4}\{(x, v_{\{i,j+1,j+2,j+4\}}) \mid x \in S_{\{i,j,n\}} \in \mathcal{F}_4^n\} \\
&\cup \bigcup_{i=1}^{n-4}\{(x, v_{\{i+1,i+2,i+3,i+4\}}) \mid x \in S_{\{i,n-3,n\}} \in \mathcal{F}_4^n\} \\
&\cup \bigcup_{i=1}^{n-6}\{(x, v_{\{i+1,i+2,i+3,i+6\}}) \mid x \in S_{\{i,n-1,n\}} \in \mathcal{F}_4^n\} \\
&\cup \{(x, z_8) \mid x \in S_{\{n-5,n-1,n\}} \in \mathcal{F}_4^n\} \\
&\cup \bigcup_{i=1}^{n-6}\{(x, v_{\{i+1,i+2,i+4,i+6\}}) \mid x \in S_{\{i,n-2,n\}} \in \mathcal{F}_4^n\} \\
&\cup \{(x, z_9) \mid x \in S_{\{n-5,n-2,n\}} \in \mathcal{F}_4^n\} \\
&\cup \bigcup_{i=1}^{n-5}\{(x, v_{\{i+1,i+2,i+3,i+5\}}) \mid x \in S_{\{i,n-2,n-1\}} \in \mathcal{F}_4^n\} \\
&\cup \bigcup_{i=1}^{3}\{(x, z_i) \mid x \in S_{\{n-5+i,n-1,n\}} \in \mathcal{F}_4^n\} \\
&\cup \bigcup_{i=1}^{2}\{(x, z_{i+3}) \mid x \in S_{\{n-5+i,n-2,n\}} \in \mathcal{F}_4^n\} \\
&\cup \bigcup_{i=1}^{2}\{(x, z_{i+5}) \mid x \in S_{\{n-5+i,n-2,n-1\}} \in \mathcal{F}_4^n\}.
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
\mathcal{F}_4^n &= \{S_{\{i,j,k\}} \mid i = 1, \ldots, n-4; j = i+1, \ldots, n-3; k = j+1, \ldots, n-2\} \\
&\cup \{S_{\{i,j,n-1\}}, S_{\{i,j,n\}} \mid i = 1, \ldots, n-4; j = i+1, \ldots, n-3\} \\
&\cup \{S_{\{i,n-1,n\}}, S_{\{i,n-2,n\}}, S_{\{i,n-2,n-1\}} \mid i = 1, \ldots, n-5\} \\
&\cup \{S_{\{n-4,n-1,n\}}, S_{\{n-3,n-1,n\}}, S_{\{n-2,n-1,n\}}\} \\
&\cup \{S_{\{n-4,n-2,n\}}, S_{\{n-3,n-2,n\}}\} \cup \{S_{\{n-4,n-2,n-1\}}, S_{\{n-3,n-2,n-1\}}\}.
\end{aligned}
$$

By the definition of $\prec$, for x in the cliques in $\mathcal{F}_4^n$ one can check that $(x, y) \in A(D)$ if and only if either $x = v_X$ and $y = V_Y$ with $X \prec Y$, or $x = v_X$ and $y = z_i$ with $X \in S_{\{n-4,n-1,n\}} \cup$

$S_{\{n-3,n-1,n\}} \cup S_{\{n-2,n-1,n\}} \cup S_{\{n-4,n-2,n\}} \cup S_{\{n-3,n-2,n\}} \cup S_{\{n-4,n-2,n-1\}} \cup S_{\{n-3,n-2,n-1\}} \cup S_{\{n-5,n-1,n\}} \cup S_{\{n-5,n-2,n\}}$ and $1 \leq i \leq 9$. Thus, we have $C(D) = J(n,4) \cup I_9$. This completes the proof. $\square$

By Theorems 3.1 and 3.2, we have Theorem 1.3.

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References

[1] J.E. Cohen, *Interval graphs and food webs: a finding and a problem*, Document 17696-PR, RAND Corporation, Santa Monica, CA, 1968.

[2] R.D. Dutton and R.C. Brigham, A characterization of competition graphs, *Discrete Appl. Math.* **6** (1983), 315–317.

[3] S.-R. Kim, B. Park and Y. Sano, The competition numbers of Johnson graphs, *Discussiones Mathematicae Graph Theory* **30** (2010), 449–459.

[4] R.J. Opsut, On the computation of the competition number of a graph, *SIAM J. Algebraic Discrete Methods* **3** (4) (1982), 420–428.

[5] B. Park and Y. Sano, The competition numbers of Hamming graphs with diameter at most three, *J. Korean Math. Soc.* **48** (4) (2011), 691–702.

[6] B. Parkand Y. Sano, The competition numbers of ternary Hamming graphs, *Appl. Math. Lett.* **24** (2011), 1608–1613.

[7] F.S. Roberts, Food webs, competition graphs, and the boxicity of ecological phase space, Theory and applications of graphs (Proc. Internat. Conf., Western Mich. Univ., Kalamazoo, Mich., 1976), pp. 477–490. *Lecture Notes in Math.*, Vol. 642, Springer, Berlin, 1978.