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Hamiltonicity in directed Toeplitz graphs $T_n\langle 1, 3, 6; t\rangle$

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Abstract

A directed Toeplitz graph, denoted as $T_n\langle s_1, \ldots, s_k; t_1, \ldots, t_l\rangle$ of order n, is a digraph in which an edge (i, j) exists if and only if $j - i = s_p$ or $i - j = t_q$ for some $1 \leq p \leq k$ and $1 \leq q \leq l$. The adjacency matrix of such a graph forms a Toeplitz matrix, characterized by constant values along all diagonals parallel to the main diagonal. In this paper, we explore the Hamiltonicity of directed Toeplitz graphs of the form $T_n\langle 1, 3, 6; t\rangle$. We establish that $T_n\langle 1, 3, 6; t\rangle$ is Hamiltonian for $t = 5, 10$ and for all $t \geq 12$, for every n. Additionally, we show that the graph remains Hamiltonian for all n, with only a finite number of exceptions when $t = 3, 4, 6, 7, 8, 9$ and 11. Specifically, for $t = 1$, the graph is Hamiltonian only when $n = 7$, while for $t = 2$, it is Hamiltonian under certain conditions on n, namely when $n \equiv 0, 1, 3 \pmod 4$.

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1. Introduction

For a graph G, let $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. An edge (u, v) is called an *increasing edge* if $u < v$ and a *decreasing edge* if $u > v$. The *length* of an edge (u, v) is defined as $|u - v|$. A *path* is a sequence of edges that connects a sequence of distinct vertices. A closed path is called a *cycle*. A *Hamiltonian path* is a path that visits each

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vertex of the graph exactly once. A graph that contains a Hamiltonian path is called *traceable*. Similarly, a *Hamiltonian cycle* is a cycle that visits each vertex exactly once, and a graph containing a Hamiltonian cycle is called *Hamiltonian*. A graph S is a *subgraph* of G if $V(S) \subset V(G)$ and $E(S) \subset E(G)$. If $V(S) = V(G)$, then S is said to *span* G. Consequently, if S is a spanning cycle of G, then S is *Hamiltonian*.

A *Toeplitz matrix*, named after Otto Toeplitz (1881-1940), is a square matrix in which each diagonal parallel to the main diagonal has constant values. That is, a Toeplitz matrix has the form:

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{-2} & a_{-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{-(n-1)} & a_{-(n-2)} & a_{-(n-3)} & \cdots & a_0 \end{bmatrix}$$

A Toeplitz matrix is called a *circulant matrix* if $a_i = a_{i-n}$ for $i = 1, \ldots, n-1$. Toeplitz matrices appear in various fields of applied mathematics and engineering, including queuing theory, signal processing, time series analysis, and integral equations. Circulant matrices, a notable subclass of Toeplitz matrices, also have significant applications in areas such as signal processing, error-correcting codes, image processing, cryptography, and numerical analysis.

A directed or undirected graph whose adjacency matrix is Toeplitz is called a *Toeplitz graph*, while one whose adjacency matrix is circulant is called a *circulant graph*.

For a Toeplitz matrix of order n, the main diagonal is labeled as 0, and the $n-1$ distinct diagonals above and below it are labeled $1,\ 2, \ldots, n-1$. Let $s_1,\ s_2, \ldots, s_k$ and $t_1,\ t_2, \ldots, t_l$ represent the upper and lower diagonals, respectively, that contain ones, where $0 < s_1 < s_2 < \cdots < s_k < n$, $0 < t_1 < t_2 < \cdots < t_l < n$. The corresponding Toeplitz graph is denoted by $T_n\langle s_1, s_2, \ldots, s_k; t_1, t_2, \ldots, t_l\rangle$, where $n > max\{s_k, t_l\}$. This graph consists of the vertex set $\{1,\ 2, \ldots, n\}$, , with an edge $(i,\ j)$ if and only if $j-i = s_p$ or $i-j = t_q$ for some p and q satisfying $1 \leq p \leq k$ and $1 \leq q \leq l$. Clearly, every increasing edge has length s_p for some p, and every decreasing edge has length t_q for some q. Additionally, note that the graphs $T_n\langle s_1, \ldots, s_i; t_1, \ldots, t_j\rangle$ and $T_n\langle t_1, \ldots, t_j; s_1, \ldots, s_i\rangle$ are obtained from each other by reversing the orientation of all edges.

Circulant graphs and their various properties have been extensively studied by several authors (see [8]-[11], [20]-[21], [23], [39]-[41], and [43], among others). In particular, Boesch and Tindell [8] conjectured that all undirected connected circulant graphs are Hamiltonian, a result later proved by Burkard and Sandholzer [10].

Various properties of Toeplitz graphs, including colorability, planarity, bipartiteness, connectivity, cycle discrepancy, edge irregularity strength, decomposition, labeling, and metric dimension, have been explored in [2]-[7], [13]-[17], [19], [22], [37], and [38]. The Hamiltonian properties of Toeplitz graphs were first investigated by R. van Dal et al. [12] and subsequently studied in [18, 36, 42]. The study of Hamiltonicity in directed Toeplitz graphs was initiated by S. Malik and T. Zamfirescu [35] and further examined by S. Malik [24], S. Malik and A.M. Qureshi [32, 33], as well as S. Malik in [25]-[31], and S. Malik and F. Ramezani in [34].

The Hamiltonicity of the Toeplitz graphs $T_n\langle 1, 3, 4; t\rangle$ was studied in [29, 30], while [33] examined the Hamiltonicity of $T_n\langle 1, 3, 5; t\rangle$. In this paper, we maintain $s_1 = 1$ and $s_2 = 3$ but consider

$s_3 = 6$, focusing on the Hamiltonicity of $T_n\langle 1,3,6;t\rangle$. Throughout, we consider finite directed graphs without multiple edges or loops.

Hamiltonicity results for undirected Toeplitz graphs directly influence the directed case. In particular, if $T_n\langle t_1, t_2, \ldots, t_i\rangle$ is Hamiltonian, then the corresponding directed Toeplitz graph $T_n\langle t_1, \ldots, t_i; t_1, \ldots, t_i\rangle$ is also Hamiltonian.

For a vertex a in $T_n\langle 1,3,6;t\rangle$, we define paths $P_{a\to a+4}$ and $Q_{a\to a+4}$ as follow:

$$P_{a\to a+4} = (a, a+3, a+4)$$

$$Q_{a\to a+4} = (a, a-2, a+4)$$

These paths are illustrated in Figure 1.

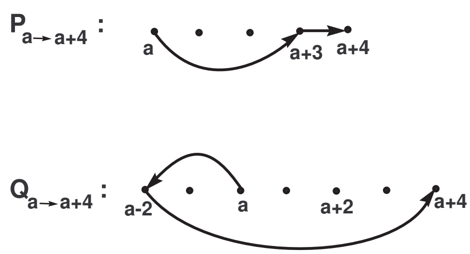

Figure 1. Paths $P_{a\to a+4}$ and $Q_{a\to a+4}$ in $T_n\langle 1,3,6;t\rangle$.

Remark 1.1. If the Toeplitz graph $T_n\langle 1,3,6;t\rangle$ has a Hamiltonian cycle containing the edge (n-2, n-1), then the extended graph $T_{n+(t-1)}\langle 1,3,6;t\rangle$ also possesses this property. This follows because a Hamiltonian cycle in $T_n\langle 1,3,6;t\rangle$ can be modified into a Hamiltonian cycle in $T_{n+(t-1)}\langle 1,3,6;t\rangle$ by replacing the edge (n-2, n-1) with the path (n-2, n+1, n+2,..., n+(t-1), n-1), thereby preserving the Hamiltonian property. For instance, as illustrated in Figure 2, a Hamiltonian cycle in $T_{10}\langle 1,3,6;5\rangle$ is transformed into a Hamiltonian cycle in $T_{14}\langle 1,3,6;5\rangle$ by replacing the edge (8, 9) with the path (8, 11, 12, 13, 14, 9). This process can be repeated to extend the Hamiltonian cycle to $T_{18}\langle 1,3,6;5\rangle$, and so forth.

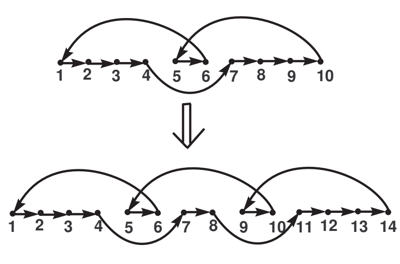

Figure 2. Hamiltonian cycles in $T_{10}\langle 1,3,6;5\rangle$ and $T_{14}\langle 1,3,6;5\rangle$.

2. Toeplitz Graphs $T_n\langle 1, 3, 6; t \leq 8\rangle$

For $t = 1$, it is clear that $T_n\langle 1, 3, 6; 1\rangle$ is Hamiltonian if and only if $n = 7$. This is because the only decreasing edges-edges of the form (a, b) where $a > b$-have length one, which is only possible when $n = 7$. Moreover, it is easily verified that $T_7\langle 1, 3, 6; 1\rangle$ has a unique Hamiltonian cycle given by $(1, 7, 6, 5, 4, 3, 2, 1)$.

Theorem 2.1. *The Toeplitz graph* $T_n\langle 1, 3, 6; 2\rangle$ *is Hamiltonian if and only if* $n \equiv 0, 1, 3 \pmod 4$.

Proof. If $n \equiv 0 \pmod 4$, then a Hamiltonian cycle in $T_n\langle 1, 3, 6; 2\rangle$ is given by $(1, 4) \cup Q_{4\to 8} \cup Q_{8\to 12} \cup \cdots \cup Q_{n-4\to n} \cup (n, n-2, n-1, n-3, n-5, \ldots, 1)$, as illustrated in Figure 3. If $n \equiv 1\, mod\, 4$,

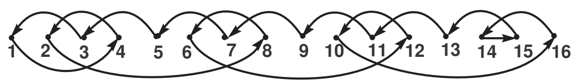

Figure 3. Hamiltonian cycles in $T_{16}\langle 1, 3, 6; 2\rangle$.

then a Hamiltonian cycle in $T_n\langle 1, 3, 6; 2\rangle$ is $(1, 4) \cup Q_{4\to 8} \cup Q_{8\to 12} \cup \cdots \cup Q_{n-5\to n-1} \cup (n-1, n-3, n, n-2, n-4, \ldots, 1)$, as shown in Figure 2. If $n \equiv 3\, mod\, 4$, then a Hamiltonian cycle in

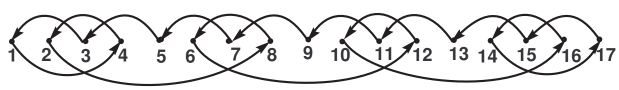

Figure 4. Hamiltonian cycles in $T_{17}\langle 1, 3, 6; 2\rangle$.

$T_n\langle 1, 3, 6; 2\rangle$ is $(1, 7) \cup Q_{7\to 11} \cup Q_{11\to 15} \cup \cdots \cup Q_{n-4\to n} \cup (n, n-2, n-1, n-3, n-5, \ldots, 2, 3, 1)$, as shown in Figure 5. Thus, $T_n\langle 1, 3, 6; 2\rangle$ is Hamiltonian for $n \equiv 0, 1, 3\, mod\, 4$. □

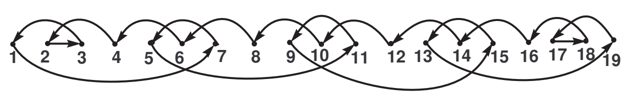

Figure 5. Hamiltonian cycles in $T_{19}\langle 1, 3, 6; 2\rangle$.

Note that the Hamiltonicity of $T_n\langle 1, 3, 6; 2\rangle$ for $n \cong 2\, mod\, 4$ remains undetermined.

Theorem 2.2. $T_n\langle 1, 3, 6; 3\rangle$ *is Hamiltonian for all* n *except for* $n = 7,\ 8,\ 9,\ 12,\ 14,\ 16$.

Proof. To prove this theorem, we will demonstrate that $T_n\langle 1, 3, 6; 3\rangle$ is Hamiltonian for all odd $n \geq 11$, and for all even $n \geq 18$, and for $n = 10$. A Hamiltonian cycle in $T_{11}\langle 1, 3, 6; 3\rangle$ is (1, 2, 3, 9, 10, 11, 8, 5, 6, 7, 4, 1). A Hamiltonian cycle in $T_{18}\langle 1, 3, 6; 3\rangle$ is (1, 2, 3, 9, 10, 16, 17, 18, 15, 12, 13, 14, 11, 8, 5, 6, 7, 4, 1). These Hamiltonian cycles contain the edge (n-2, n-1), which allows them to be extended to Hamiltonian cycles in $T_{n+2}\langle 1, 3, 6; 3\rangle$ by replacing the edge (n-2, n-1) with the path (n-2, n+1, n+2, n-1). This transformation preserves the Hamiltonicity property, proving that $T_n\langle 1,3,6;3\rangle$ is Hamiltonian for all even $n \geq 18$ and all odd $n \geq 11$. For $n = 10$, the unique Hamiltonian cycle in $T_{10}\langle 1,3,6;3\rangle$ is (1, 2, 8, 5, 6, 3, 9, 10, 7, 4, 1). This completes the proof. □

Note that the Hamiltonicity of $T_n\langle 1,3,6;3\rangle$ for $n = 7, 8, 9, 12, 14$ and 16 remains undetermined.

Theorem 2.3. *$T_n\langle 1,3,6;4\rangle$ is Hamiltonian for all n except for $n = 12$.*

Proof. We prove it by considering cases based on n mod 3.

Case 1. Let $n \equiv 0 \, mod \, 3$ and $n \neq 12$.

The smallest such n is 9. A Hamiltonian cycle in $T_9\langle 1,3,6;4\rangle$ is (1, 2, 8, 4, 7, 3, 6, 9, 5, 1), which does not include the edge (7, 8). The next representative in this class, different from 12, is 15. A Hamiltonian cycle in $T_{15}\langle 1,3,6;4\rangle$ is (1, 2, 8, 9, 10, 6, 12, 13, 14, 15, 11, 7, 3, 4, 5, 1).

Case 2. Let $n \equiv 1 \, mod \, 3$.

The smallest such n is 7. A Hamiltonian cycle in $T_7\langle 1,3,6;4\rangle$ is (1, 4, 7, 3, 6, 2, 5, 1). This cycle does not contain the edge $(5, 6)$. The next representative in this class is $n = 10$. A Hamiltonian cycle in $T_{10}\langle 1,3,6;4\rangle$ is (1, 2, 8, 9, 10, 6, 7, 3, 4, 5, 1).

Case 3. Let $n \equiv 2 \, mod \, 3$.

The smallest such n is 8. A Hamiltonian cycle in $T_8\langle 1,3,6;4\rangle$ is (1, 2, 3, 6, 7, 8, 4, 5, 1).

For $n = 8, 10, 15$, these Hamiltonian cycles contain the edge (n-2, n-1). By Remark 1.1, we can extend these Hamiltonian cycles to $T_{n+3}\langle 1,3,6;4\rangle$ by replacing the edge (n-2, n-1) with the path (n-2, n+1, n+2, n+3, n-1), which preserves the same property. Suppose, for some non-negative integer r, if $T_{n=n_o+3r\neq 12}\langle 1,3,6;4\rangle$ has a Hamiltonian cycle containing the edge (n-2, n-1), then $T_{n+3}\langle 1,3,6;4\rangle$ also has the same property. Thus we conclude that $T_n\langle 1,3,6;4\rangle$ is Hamiltonian for all $n \neq 12$. □

Note that the Hamiltonicity of $T_{12}\langle 1,3,6;4\rangle$ remains undetermined.

Theorem 2.4. *$T_n\langle 1,3,6;5\rangle$ is Hamiltonian for all n.*

Proof. We prove it by considering cases based on n mod 4.

Case 1. Let $n \equiv 0 \, mod \, 4$.

The smallest n in this case is 8. A Hamiltonian cycle in $T_8\langle 1,3,6;5\rangle$ is $($1, 7, 2, 8, 3, 4, 5, 6, 1$)$, which does not contain the edge (6, 7). The next representative in this class is 12, where a Hamiltonian cycle in $T_{12}\langle 1,3,6;5\rangle$ is (1, 4, 5, 8, 9, 10, 11, 12, 7, 2, 3, 6, 1).

Case 2. Let $n \equiv 1 \, mod \, 4$.

The first two representatives in this class are $n = 9$ and 13, but for both, the Hamiltonian cycles do not contain the edge $(n-2, n-1)$. A Hamiltonian cycle in $T_9\langle 1,3,6;5\rangle$ is (1, 7, 2, 8, 3, 9, 4, 5, 6, 1). A Hamiltonian cycle in $T_{13}\langle 1,3,6;5\rangle$ is $(1, 4, 10, 13, 8, 3, 9, 12, 7, 2, 5, 11, 6, 1)$. Now, the next representative in this class is $n = 17$. A Hamiltonian cycle in $T_{17}\langle 1,3,6;5\rangle$ is (1, 4, 5, 8, 9, 10, 13, 14, 15, 16, 11, 17, 12, 7, 2, 3, 6, 1).

Case 3. Let $n \equiv 2 \, mod \, 4$.

The smallest n in this case is 10. A Hamiltonian cycle in $T_{10}\langle 1,3,6;5\rangle$ is (1, 2, 3, 4, 7, 8, 9, 10, 5, 6, 1).

Case 4. Let $n \equiv 3\,mod\,4$.

The smallest n in this case is 7. A Hamiltonian cycle in $T_7\langle 1,3,6;5\rangle$ is (1, 7, 2, 3, 4, 5, 6, 1).

For $n = 7, 10, 12, 17$, these Hamiltonian cycles contain the edge (n-2, n-1). Thus, these cycles in $T_{n\in\{7,10,12,17\}}\langle 1,3,6;5\rangle$ can be transformed into Hamiltonian cycles in $T_{n+4}\langle 1,3,6;5\rangle$ by replacing the edge (n-2, n-1) with the path (n-2, n+1, n+2, n+3, n+4, n-1), which preserves the same property. Using the technique described in Remark 1.1, it follows that $T_n\langle 1,3,6;5\rangle$ is Hamiltonian for all n. □

Theorem 2.5. *$T_n\langle 1,3,6;6\rangle$ is Hamiltonian for all n except $n = 10$ and $n = 14$.*

Proof. We prove it by considering cases based on n mod 5.

Case 1. Let $n \equiv 0\,mod\,5$ and $n \neq 10$.

The smallest such n after 10 is $n = 15$. A Hamiltonian cycle in $T_{15}\langle 1,3,6;6\rangle$ is (1, 4, 5, 11, 12, 15, 9, 10, 13, 14, 8, 2, 3, 6, 7, 1), which contains the edge $(13, 14)$.

Case 2. Let $n \equiv 1\,mod\,5$.

The smallest such n is $n = 11$. A Hamiltonian cycle in $T_{11}\langle 1,3,6;6\rangle$ is (1, 2, 8, 9, 3, 4, 10, 11, 5, 6, 7, 1), which does not contain the edge $(9, 10)$. The next representative in this class is $n = 16$. A Hamiltonian cycle in $T_{16}\langle 1,3,6;6\rangle$ is (1, 2, 5, 8, 11, 14, 15, 9, 3, 6, 12, 13, 16, 10, 4, 7, 1), which contains the edge $(14, 15)$.

Case 3. Let $n \equiv 2\,mod\,5$.

The smallest such n is $n = 7$. A Hamiltonian cycle in $T_7\langle 1,3,6;6\rangle$ is (1, 2, 3, 4, 5, 6, 7, 1), which contains the edge $(5, 6)$

Case 4. Let $n \equiv 3\,mod\,5$.

The smallest such n is $n = 8$. A Hamiltonian cycle in $T_8\langle 1,3,6;6\rangle$ is (1, 4, 5, 8, 2, 3, 6, 7, 1), which contains the edge $(6, 7)$.

Case 5. Let $n \equiv 4\,mod\,5$ and $n \neq 14$.

The smallest such n is $n = 9$. A Hamiltonian cycle in $T_9\langle 1,3,6;6\rangle$ is (1, 2, 8, 9, 3, 4, 5, 6, 7, 1), which does not contain the edge $(7, 8)$. The next representative in this class after 14 is $n = 19$. A Hamiltonian cycle in $T_{19}\langle 1,3,6;6\rangle$ is (1, 2, 3, 4, 10, 11, 5, 8, 14, 17, 18, 12, 6, 9, 15, 16, 19, 13, 7, 1), which contains the edge $(17, 18)$.

For $n = 7, 8, 15, 16, 19$, the Hamiltonian cycles above contain the edge (n-2, n-1). This allows us to extend the cycles to $T_{n+5}\langle 1,3,6;6\rangle$ by replacing the edge (n-2, n-1) with the path $(n-2, n+1, n+2, n+3, n+4, n-1)$, which preserves the same property. By applying the technique in Remark 1.1, it follows that $T_n\langle 1,3,6;6\rangle$ is Hamiltonian for all n except $n = 10$ and $n = 14$. □

Note that for $n \in \{10, 14\}$, the Hamiltonicity of $T_n\langle 1,3,6;6\rangle$ remains undecided.

Theorem 2.6. *$T_n\langle 1,3,6;7\rangle$ is Hamiltonian for all n except $n = 9$ and $n = 12$.*

Proof. We prove it by considering cases based on n mod 6.

Case 1. Let $n \equiv 0,\ 1,\ 2,\ 3,\ 5\,mod\,6$ and $n \notin \{9, 12\}$.

The smallest such values of $n \notin \{9, 12\}$ are $n = 18, 13, 8, 15, 11$, respectively. Hamiltonian cycles for these values are:

$n = 18:$ (1, 4, 5, 6, 12, 15, 18, 11, 14, 7, 13, 16, 17, 10, 3, 9, 2, 8,1),
$n = 13:$ (1, 2, 3, 4, 7, 13, 6, 9, 10, 11, 12, 5, 8, 1),
$n = 8:$ (1, 2, 3, 4, 5, 6, 7, 8, 1),
$n = 15:$ (1, 2, 3, 6, 9, 12, 5, 11, 4, 7, 10, 13, 14, 15, 8, 1),
$n = 11:$ (1, 2, 3, 9, 10, 11, 4, 5, 6, 7, 8, 1).
Each of these cycles contains the edge (n-2,n-1). Thus, these cycles can be extended to $T_{n+6}\langle 1, 3, 6; 7\rangle$ by replacing the edge (n-2, n-1) with the path (n-2, n+1, n+2, n+3, n+4, n-1), which preserves the same property.

Case 2. Let $n \equiv 4 \, mod \, 6$.

The smallest such n is $n = 10$. A Hamiltonian cycle in $T_{10}\langle 1, 3, 6; 7\rangle$ is (1, 2, 5, 6, 9, 10, 3, 4, 7, 8, 1), which does not contain the edge $(8, 9)$. The next representative in this class is $n = 16$. A Hamiltonian cycle in $T_{16}\langle 1, 3, 6; 7\rangle$ is (1, 4, 7, 10, 13, 16, 9, 2, 3, 6, 12, 5, 11, 14, 15, 8, 1), which contains the edge (14, 15). Thus, this cycle can also be extended to $T_{n+6}\langle 1, 3, 6; 7\rangle$ using the same extension technique.

By applying the technique in Remark 1.1, it follows that $T_n\langle 1, 3, 6; 7\rangle$ is Hamiltonian for all n except $n = 9$ and $n = 12$. □

Note that for $n \in \{9, 12\}$, the Hamiltonicity of $T_n\langle 1, 3, 6; 7\rangle$ has not yet been determined.

Theorem 2.7. $T_n\langle 1, 3, 6; 8\rangle$ *is Hamiltonian for all* $n \neq 10$.

Proof. We prove it by considering cases based on n mod 7.

Case 1. $n \equiv 0,\ 1,\ 2,\ 3,\ 5,\ 6 \, mod \, 7$ and $n \neq 10$.
The smallest such values of $n \neq 10$ are $n = 14, 15, 9, 17, 12, 13$, respectively. Hamiltonian cycles for these values are:
$n = 14:$ (1, 4, 5, 8, 11, 12, 13, 14, 6, 7, 10, 2, 3, 9, 1),
$n = 15:$ (1, 4, 5, 8, 11, 12, 13, 14, 15, 7, 10, 2, 3, 6, 9, 1),
$n = 9:$ (1, 2, 3, 4, 5, 6, 7, 8, 9, 1),
$n = 17:$ (1, 2, 5, 6, 12, 15, 16, 8, 11, 3, 4, 7, 10, 13, 14, 17, 9, 1),
$n = 12:$ (1, 2, 3, 6, 7, 10, 11, 12, 4, 5, 8, 9, 1),
$n = 13:$ (1, 2, 3, 4, 10, 11, 12, 13, 5, 6, 7, 8, 9, 1).

Each of these cycles contains the edge (n-2, n-1). Thus, these cycles can be extended to $T_{n+7}\langle 1, 3, 6; 8\rangle$ by replacing the edge $(n-2, n-1)$ with the path $(n-2, n+1, n+2, n+3, n+4, n-1)$, which preserves the same property.

Case 2. $n \equiv 4 \, mod \, 7$.
The smallest such n is $n = 11$. A Hamiltonian cycle in $T_{11}\langle 1, 3, 6; 8\rangle$ is (1, 7, 8, 11, 3, 4, 10, 2, 5, 6, 9, 1), which does not contain the edge $(9, 10)$. The next representative in this class is $n = 18$. A Hamiltonian cycle in $T_{18}\langle 1, 3, 6; 8\rangle$ is (1, 4, 5, 11, 12, 13, 14, 15, 16, 17, 18, 10, 2, 3, 6, 7, 8, 9, 1), which contains the edge $(16, 17)$. Thus, this cycle can also be extended to $T_{n+7}\langle 1, 3, 6; 8\rangle$ using the same extension technique.

By applying the technique in Remark 1.1, it follows that $T_n\langle 1, 3, 6; 8\rangle$ is Hamiltonian for all $n \neq 10$. □

Note that the Hamiltonicity of $T_{10}\langle 1, 3, 6; 8\rangle$ has not yet been determined.

3. Toeplitz Graphs $T_n\langle 1,3,6;t\geq 9\rangle$

Theorem 3.1. *Let $t\in\{9,11\}$. For all $n\neq 13$, the graph $T_n\langle 1,3,6;t\rangle$ is Hamiltonian.*

Proof. **Case 1.** $t=9$
We analyze values of n mod 8.

(i) If $n\equiv 0,\ 1,\ 2,\ 3,\ 5,\ 6,\ 7\, mod\, 8$ and $n\neq 13$.
The smallest such values of $n\neq 13$ are $n=16,17,10,11,21,14,15$, respectively. Hamiltonian cycles for these values are:
$n=16:$ (1, 5, 6, 12, 13, 14, 15, 16, 7, 8, 11, 2, 3, 9, 10, 1),
$n=17:$ (1, 4, 7, 13, 14, 15, 16, 17, 8, 11, 2, 5, 6, 12, 3, 9, 10, 1),
$n=10:$ (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 1),
$n=11:$ (1, 4, 5, 11, 2, 3, 6, 7, 8, 9, 10, 1),
$n=21:$ (1, 7, 8, 14, 15, 16, 17, 18, 19, 20, 21, 12, 13, 4, 5, 11, 2, 3, 6, 9, 10, 1),
$n=14:$ (1, 2, 3, 4, 7, 8, 11, 12, 13, 14, 5, 6, 9, 10, 1),
$n=15:$ (1, 2, 3, 4, 5, 11, 12, 13, 14, 15, 6, 7, 8, 9, 10, 1). Each of these cycles contains the edge (n-2, n-1). Thus, these cycles can be extended to $T_{n+8}\langle 1,3,6;9\rangle$ by replacing the edge (n-2, n-1) with the path (n-2, n+1, n+2, n+3, n+4, n-1), which preserves the same property.

(ii) If $n\equiv 4\, mod\, 8$.
The smallest such n is $n=12$. A Hamiltonian cycle in $T_{12}\langle 1,3,6;9\rangle$ is (1, 7, 8, 11, 2, 5, 6, 9, 12, 3, 4, 10, 1), which does not contain the edge $(10,11)$. The next representative in this class is $n=20$. A Hamiltonian cycle in $T_{20}\langle 1,3,6;9\rangle$ is (1, 4, 5, 8, 9, 12, 13, 14, 15, 16, 17, 18, 19, 20, 11, 2, 3, 6, 7, 10, 1), which contains the edge $(18,19)$. Thus, this cycle can also be extended to $T_{n+8}\langle 1,3,6;9\rangle$ using the same extension technique.

Case 2. $t=11$
We analyze values of n mod 10.

(i) If $n\equiv 0,\ 1,\ 2,\ 3,\ 5,\ 7,\ 8,\ 9\, mod\, 10$ and $n\neq 13$.
The smallest such values of $n\neq 13$ are $n=20,21,12,23,15,17,18,19$, respectively. Hamiltonian cycles for these values are:
$n=20:$ (1, 2, 3, 4, 5, 6, 7, 10, 13, 14, 20, 9, 15, 16, 17, 18, 19, 8, 11, 12, 1),
$n=21:$ (1, 4, 5, 6, 7, 8, 14, 15, 16, 17, 18, 19, 20, 21, 10, 13, 2, 3, 9, 12, 1),
$n=10:$ (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 1),
$n=23:$ (1, 2, 8, 9, 15, 21, 22, 11, 14, 3, 4, 5, 6, 7, 10, 13, 16, 17, 18, 19, 20, 23, 12, 1),
$n=15:$ (1, 2, 3, 13, 14, 15, 4, 5, 8, 9, 10, 11, 12, 1),
$n=17:$ (1, 2, 3, 4, 10, 13, 14, 15, 16, 5, 11, 17, 6, 7, 8, 9, 12, 1),
$n=18:$ (1, 2, 3, 4, 5, 6, 9, 10, 13, 14, 15, 16, 17, 18, 7, 8, 11, 12, 1),
$n=19:$ (1, 2, 3, 4, 5, 6, 7, 13, 14, 15, 16, 17, 18, 19, 8, 9, 10, 11, 12, 1).
Each of these cycles contains the edge (n-2,n-1), allowing extension to $T_{n+10}\langle 1,3,6;11\rangle$ using the same method.

(ii) If $n\equiv 4,\ 6\, mod\, 10$.
The smallest such values are $n=14$ and $n=16$, with Hamiltonian cycle (1, 2, 5, 6, 9, 10, 13, 14, 3, 4, 7, 8, 11, 12, 1) and (1, 2, 8, 9, 15, 16, 5, 6, 7, 13, 14, 3, 4, 10, 11, 12, 1), respectively. These cycles do not contain the edge (n-2, n-1). The next representative in these classes are $n=24$ and $n=16$, with Hamiltonian cycles (1, 7, 8, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 13, 2, 3, 4, 5,

6, 9, 10, 11, 12, 1) and (1, 2, 8, 9, 10, 11, 17, 18, 24, 25, 14, 3, 4, 5, 6, 7, 13, 16, 19, 20, 26, 15, 21, 22, 23, 12, 1), respectively, containing the edge (n-2, n-1), and thus allow extension using the same technique.

By applying the extension technique in Remark 1.1, we conclude that for $t \in \{9, 11\}$, the graph $T_n\langle 1,3,6;t\rangle$ is Hamiltonian for all $n \neq 13$. □

Note that for $n = 13$, the Hamiltonicity of $T_n\langle 1,3,6;9\rangle$ and $T_n\langle 1,3,6;11\rangle$ has not yet been determined.

Now we prove that for $t = 10$ or $t \geq 12$, the graph $T_n\langle 1,3,6;t\rangle$ is Hamiltonian for all n.

Theorem 3.2. *Let* $t = 10$ *or* $t \geq 12$*. Then* $T_n\langle 1,3,6;t\rangle$ *is Hamiltonian for all* n*.*

Proof. Let $t = 10$ or $t \geq 12$. We prove it by considering cases based on n mod $t-1$.

Case 1. $n \equiv 0 \, mod \, (t-1)$.
The smallest value of n distinct from $t-1$, is $n = 2t-2$. We analyze values of t mod 4.

(i) If $t \equiv 0 \, mod \, 4$, we construct a Hamiltonian cycle in the graph $T_{n=2t-2}\langle 1,3,6;t\rangle$ as $P_{1\to5} \cup P_{5\to9} \cup \cdots \cup P_{t-7\to t-3} \cup (t-3, t, t+3, t+4, \ldots, n-2, n-1, n, n-t = t-2, t-1, t+2, 2, 3) \cup P_{3\to7} \cup P_{7\to11} \cup \cdots \cup P_{t-9\to t-5} \cup (t-5, t+1, 1)$. See Figure 6 for an illustration of the cycle.

(ii) If $t \equiv 1 \, mod \, 4$, a Hamiltonian cycle in $T_{n=2t-2}\langle 1,3,6;t\rangle$ is given by $P_{1\to5} \cup P_{5\to9} \cup \cdots \cup P_{t-8\to t-4} \cup (t-4, t+2, 2, 3) \cup P_{3\to7} \cup P_{7\to11} \cup \cdots \cup P_{t-10\to t-6} \cup (t-6, t-3, t+3, t+4, \ldots, n-2, n-1, n, n-t = t-2, t-1, t, t+1, 1)$. See Figure 7 for an illustration.

(iii) If $t \equiv 2 \, mod \, 4$, a Hamiltonian cycle in $T_{n=2t-2}\langle 1,3,6;t\rangle$ is given by $(1,2,3) \cup P_{3\to7} \cup P_{7\to11} \cup \cdots \cup P_{t-11\to t-7} \cup (t-7, t-1, t+5) \cup P_{t+5\to t+10} \cup P_{t+10\to t+14} \cup \cdots \cup P_{n-7\to n-3} \cup (n-3, n, n-t = t-2, t+4, 4, 5) \cup P_{5\to9} \cup P_{9\to13} \cup \cdots \cup P_{t-9\to t-5} \cup (t-5, t-4, t+2, t+3) \cup P_{t+3\to t+7} \cup P_{t+7\to t+11} \cup \cdots \cup P_{n-5\to n-1} \cup (n-1, n-1-t = t-3, t, t+1, 1)$. See Figure 8 for an illustration.

(iv) If $t \equiv 3 \, mod \, 4$, a Hamiltonian cycle in the graph $T_{n=2t-2}\langle 1,3,6;t\rangle$ is given by $P_{1\to5} \cup P_{5\to9} \cup \cdots \cup P_{t-14\to t-10} \cup (t-10, t-4, t-3, t, t+3, t+4, \ldots, n-2, n-1, n, n-t = t-2, t-1, t+2, 2, 3) \cup P_{3\to7} \cup P_{7\to11} \cup \cdots \cup P_{t-12\to t-8} \cup (t-8, t-7, t-6, t-5, t+1, 1)$. See Figure 9 for an illustration.

These Hamiltonian cycles in each class contain the edge (n-2, n-1). Suppose that for some non-negative integer r, the graph $T_{n=(2t-2)+r(t-1)}\langle 1,3,6;t\rangle$ has a Hamiltonian cycle containing the edge (n-2, n-1). Then, by Remark 1.1, the graph $T_{n+t-1}\langle 1,3,6;t\rangle$ also has a Hamiltonian cycle with the same property. Thus, $T_n\langle 1,3,6;t\rangle$ is Hamiltonian for all $n \equiv 0 \, mod \, (t-1)$.

Case 2. $n \equiv 1 \, mod \, (t-1)$.
The smallest n, distinct from t, is $n = 2t-1$. We analyze values of t mod 4.

(i) If $t \equiv 0 \, mod \, 4$, a Hamiltonian cycle in the graph $T_{n=2t-1}\langle 1,3,6;t\rangle$ is given by $P_{1\to5} \cup P_{5\to9} \cup \cdots \cup P_{t-7\to t-3} \cup (t-3, t, t+3, t+4, \ldots, n-2, n-1, n, n-t = t-1, t+2, 2, 3) \cup P_{3\to7} \cup P_{7\to11} \cup \cdots \cup P_{t-9\to t-5} \cup (t-5, t-2, t+1, 1)$. See Figure 10 for an illustration.

(ii) If $t \equiv 1 \, mod \, 4$, a Hamiltonian cycle in the graph $T_{n=2t-1}\langle 1,3,6;t\rangle$ is given by $(1,2) \cup P_{2\to6} \cup P_{6\to10} \cup \cdots \cup P_{t-15\to t-11} \cup (t-11, t-5, t-4, t+2, t+3, 3, 4) \cup P_{4\to8} \cup P_{8\to12} \cup \cdots \cup P_{t-13\to t-9} \cup (t-9, t-8, t-7, t-6, t-3, t-2, t+4, t+5, \ldots, n-2, n-1, n, n-t = t-1, t, t+1, 1)$. See Figure 11 for an illustration.

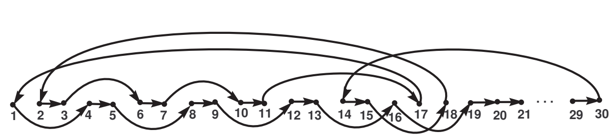

Figure 6. A Hamiltonian cycle in $T_{30}\langle 1,3,6;16\rangle$.

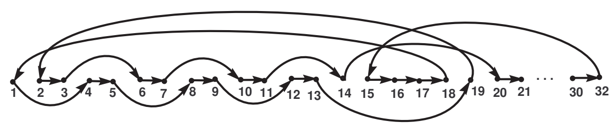

Figure 7. A Hamiltonian cycle in $T_{32}\langle 1,3,6;17\rangle$.

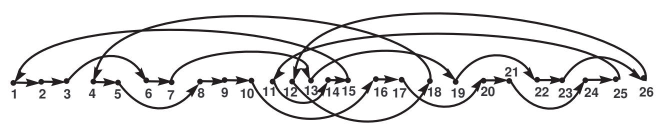

Figure 8. A Hamiltonian cycle in $T_{26}\langle 1,3,6;14\rangle$.

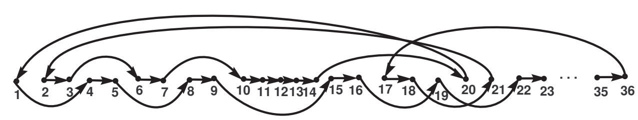

Figure 9. A Hamiltonian cycle in $T_{36}\langle 1,3,6;19\rangle$.

(iii) If $t \equiv 2 \, mod \, 4$, a Hamiltonian cycle in $T_{n=2t-1}\langle 1,3,6;t\rangle$ is given by $(1,2)\cup P_{2\to 6}\cup P_{6\to 10}\cup \cdots \cup P_{t-8\to t-4}\cup (t-4,t+2,t+3,3,4)\cup P_{4\to 8}\cup P_{8\to 12}\cup \cdots \cup P_{t-6\to t-2}\cup\ (t-2,t+4,t+5,\ldots,n-2,n-1,n,n-t=t-1,t,t+1,1)$. See Figure 12 for an illustration.

(iv) If $t \equiv 3 \, mod \, 4$, a Hamiltonian cycle in $T_{n=2t-1}\langle 1,3,6;t\rangle$ is given by $(1,2)\cup P_{2\to 6}\cup P_{6\to 10}\cup \cdots \cup P_{t-9\to t-5}\cup (t-5,t-2,t+4,t+5,\ldots,n-2,n-1,n,n-t=t-1,t+2,t+3,3,4)\cup P_{4\to 8}\cup P_{8\to 12}\cup \cdots \cup P_{t-3\to t+1}\cup\ (t+1,1)$. See Figure 13 for an illustration.

The Hamiltonian cycles in $T_{n=(2t-1)}\langle 1,3,6;t\rangle$ that contain the edge (n-2, n-1) can be extended to cycles in $T_{(2t-1)+(t-1)}\langle 1,3,6;t\rangle$ while preserving this property. Suppose that for some non-negative integer r, the graph $T_{n=(2t-1)+r(t-1)}\langle 1,3,6;t\rangle$ has a Hamiltonian cycle containing the edge (n-2, n-1). Then, by Remark 1.1, the graph $T_{n+t-1}\langle 1,3,6;t\rangle$ also has a Hamiltonian cycle with the same property. Thus, $T_n\langle 1,3,6;t\rangle$ is Hamiltonian for all $n \equiv 1 \, mod \, (t-1)$.

Case 3. $n \equiv 2 \, mod \, (t-1)$.
The smallest n in this case is $t+1$. A Hamiltonian cycle in $T_{n=t+1}\langle 1,3,6;t\rangle$ is given by $(1,2,3,\ldots,n-2,n-1,n,1)$, which contains the edge $(n-2,n-1)$. Suppose that for some non-negative integer r, the graph $T_{n=(t+1)+r(t-1)}\langle 1,3,6;t\rangle$ has a Hamiltonian cycle containing the edge $(n-2,n-1)$.

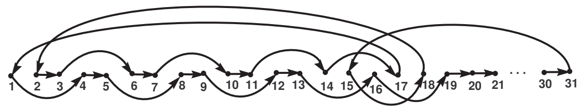

Figure 10. A Hamiltonian cycle in $T_{31}\langle 1,3,6;16\rangle$.

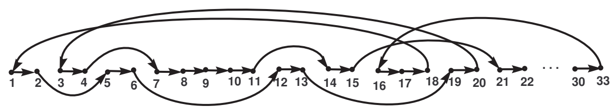

Figure 11. A Hamiltonian cycle in $T_{33}\langle 1,3,6;17\rangle$.

Then, by Remark 1.1, the graph $T_{n+t-1}\langle 1,3,6;t\rangle$ also preserves this property. Thus, $T_n\langle 1,3,6;t\rangle$ is Hamiltonian for all $n \equiv 2\,mod\,(t-1)$.

Case 4. $n \equiv 4\,mod\,(t-1)$.

The smallest n in this case is $t+3$. We analyze values of t mod 4.

(i) If $t \equiv 0\,mod\,4$ and $t \neq 12$, then a Hamiltonian cycle in $T_{n=t+3}\langle 1,3,6;t\rangle$ is $(1,2) \cup P_{2\to 6} \cup P_{6\to 10} \cup \cdots \cup P_{t-18\to t-14} \cup (t-14, t-8, t-7, t-6, t-5, t-4, t+2, n=t+3, 3, 4) \cup P_{4\to 8} \cup P_{8\to 12} \cup \cdots \cup P_{t-16\to t-12} \cup\ (t-12, t-11, t-10, t-9, t-3, t-2, t-1, t, t+1, 1)$, as illustrated in Figure 14.

A Hamiltonian cycle in $T_{15}\langle 1,3,6;12\rangle$ is given by $(1,7,10,11,14,2,8,9,15,3,4,5,6,12,13,1)$.

(ii) If $t \equiv 1\,mod\,4$, a Hamiltonian cycle in $T_{n=t+3}\langle 1,3,6;t\rangle$ is given by $(1,2) \cup P_{2\to 6} \cup P_{6\to 10} \cup \cdots \cup P_{t-11\to t-7} \cup (t-7, t-6, t-5, t-4, t+2, n=t+3, 3, 4) \cup P_{4\to 8} \cup P_{8\to 12} \cup \cdots \cup P_{t-13\to t-9} \cup (t-9, t-3, t-2, t-1, t, t+1, 1)$, see Figure 15.

(iii) If $t \equiv 2\,mod\,4$, a Hamiltonian cycle in $T_{n=t+3}\langle 1,3,6;t\rangle$ is given by $(1,2) \cup P_{2\to 6} \cup P_{6\to 10} \cup \cdots \cup P_{t-8\to t-4} \cup (t-4, t+2, n=t+3, 3, 4) \cup P_{4\to 8} \cup P_{8\to 12} \cup \cdots \cup P_{t-6\to t-2} \cup\ (t-2, t-1, t, t+1, 1)$, as illustrated in Figure 16.

(iv) If $t \equiv 3\,mod\,4$, a Hamiltonian cycle in $T_{n=t+3}\langle 1,3,6;t\rangle$ is given by $(1,2) \cup P_{2\to 6} \cup P_{6\to 10} \cup \cdots \cup P_{t-1\to t+3} \cup (t+3, 3, 4) \cup P_{4\to 8} \cup P_{8\to 12} \cup \cdots \cup P_{t-3\to t+1} \cup\ (t+1, 1)$, as illustrated in Figure 17.

All these Hamiltonian cycles in $T_{n=t+3}\langle 1,3,6;t\rangle$ do not contain the edge (n-2, n-1). The next representative in this class is $n = 2t+2$. We again analyze values of t mod 4.

(i) If $t \equiv 0\,mod\,4$, a Hamiltonian cycle in $T_{n=2t+2}\langle 1,3,6;t\rangle$ is given by $P_{1\to 5} \cup P_{5\to 9} \cup \cdots \cup P_{t-11\to t-7} \cup (t-7, t-1, t, t+3, t+4, \ldots, n-2, n-1, n, n-t=t+2, 2, 3) \cup P_{3\to 7} \cup P_{7\to 11} \cup \cdots \cup P_{t-9\to t-5} \cup\ (t-5, t-4, t-3, t-2, t+1, 1)$, as illustrated in Figure 18.

(ii) If $t \equiv 1\,mod\,4$, a Hamiltonian cycle in $T_{n=2t+2}\langle 1,3,6;t\rangle$ is given by $P_{1\to 5} \cup P_{5\to 9} \cup \cdots \cup P_{t\to t+4} \cup (t+4, t+5, \ldots, n-2, n-1, n, n-t=t+2, 2, 3) \cup P_{3\to 7} \cup P_{7\to 11} \cup \cdots \cup P_{t-6\to t-2} \cup (t-2, t+1, 1)$, as illustrated in Figure 19.

(iii) If $t \equiv 2\,mod\,4$, a Hamiltonian cycle in $T_{n=2t+2}\langle 1,3,6;t\rangle$ is given by $P_{1\to 5} \cup P_{5\to 9} \cup \cdots \cup P_{t-5\to t-1} \cup (t-1, t+5, t+6)) \cup P_{t+6\to t+10} \cup P_{t+10\to t+14} \cup \cdots \cup P_{n-10\to n-6} \cup\ (n-6, n, n-t=$

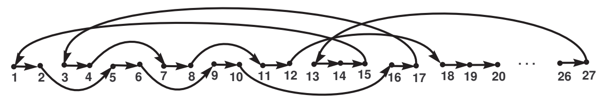

Figure 12. A Hamiltonian cycle in $T_{27}\langle 1,3,6;14\rangle$.

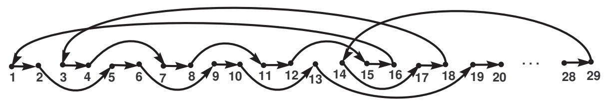

Figure 13. A Hamiltonian cycle in $T_{29}\langle 1,3,6;15\rangle$.

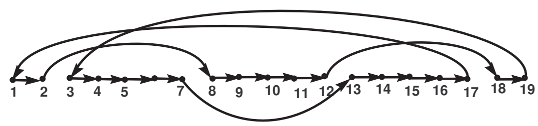

Figure 14. A Hamiltonian cycle in $T_{19}\langle 1,3,6;16\rangle$.

$t+2,2,3)\cup P_{3\to 7}\cup P_{7\to 11}\cup\cdots\cup P_{t-7\to t-3}\cup(t-3,t,t+3,t+4)\cup P_{t+4\to t+8}\cup P_{t+8\to t+12}\cup\cdots\cup P_{n-8\to n-4}\cup(n-4,n-3,n-2,n-1,n-1-t=t+1,1)$. See Figure 20 for reference.
(iv) If $t\equiv 3\,mod\,4$, a Hamiltonian cycle in $T_{n=2t+2}\langle 1,3,6;t\rangle$ is given by $P_{1\to 5}\cup P_{5\to 9}\cup\cdots\cup P_{t-14\to t-10}\cup(t-10,t-4,t-3,t-2,t-1,t,t+3,t+4,\ldots,n-2,n-1,n,n-t=t+2,2,3)\cup P_{3\to 7}\cup P_{7\to 11}\cup\cdots\cup P_{t-16\to t-8}\cup(t-8,t-7,t-6,t-5,t+1,1)$, see Figure 21 for reference.

Since, in all these cases, the edge (n-2, n-1) appears in the Hamiltonian cycles of $T_{2t+2}\langle 1,3,6;t\rangle$, the cycle can be extended to a Hamiltonian cycle in $T_{2t+2+(t-1)}\langle 1,3,6;t\rangle$. Thus, by applying the technique described in Remark 1.1, we conclude that $T_n\langle 1,3,6;t\rangle$ is Hamiltonian for all $n\equiv 4\,mod\,(t-1)$.

Case 5. $n\equiv t-8\,mod\,(t-1)$.
Clearly, this applies for $t\geq 13$, since the cases for $t\in\{10,12\}$, have already been analyzed in Cases 2 and Case 4, respectively. The smallest possible n satisfying this condition is $n=2t-9$. We analyze values of t mod 4.

(i) If $t\equiv 0\,mod\,4$, a Hamiltonian cycle in $T_{n=2t-9}\langle 1,3,6;t\rangle$ is given by $P_{1\to 5}\cup P_{5\to 9}\cup\cdots\cup P_{t-15\to t-11}\cup(t-11,t-10,t-4,t+2,2,3)\cup P_{3\to 7}\cup P_{7\to 11}\cup\cdots\cup P_{t-17\to t-13}\cup(t-13,t-7,t-6,t-3,t-2,t-1,t,t+3,t+4,\ldots,n-2,n-1,n,n-t=t-9,t-8,t-5,t+1,1)$, see Figure 22 for reference.

(ii) If $t\equiv 1\,mod\,4$ and $t\neq 13$, a Hamiltonian cycle in $T_{n=2t-9}\langle 1,3,6;t\rangle$ is given by $P_{1\to 5}\cup P_{5\to 9}\cup\cdots\cup P_{t-16\to t-12}\cup(t-12,t-11,t-5,t-4,t-3,t+3,t+4,t+5)\cup P_{t+5\to t+9}\cup P_{t+9\to t+13}\cup\cdots\cup P_{n-7\to n-3}\cup(n-3,n,n-t=t-9,t-6,t,t+6,t+7)\cup P_{t+7\to t+11}\cup P_{t+11\to t+15}\cup\cdots\cup P_{n-5\to n-1}\cup(n-1,n-1-t=t-10,t-7,t-1,t+2,2,3)\cup P_{3\to 7}\cup P_{7\to 11}\cup\cdots\cup P_{t-18\to t-14}\cup(t-14,t-8,t-2,t+1,1)$, see Figure 23 for reference.

For $t=13$, a Hamiltonian cycle in $T_{17}\langle 1,3,6;13\rangle$ is $(1,7,10,16,3,9,15,2,8,11,17,4,5,6,12,13,14,1)$,

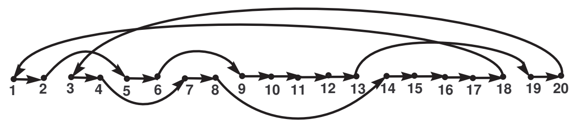

Figure 15. A Hamiltonian cycle in $T_{20}\langle 1,3,6;17\rangle$.

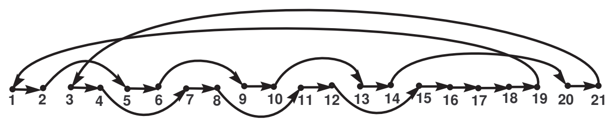

Figure 16. A Hamiltonian cycle in $T_{21}\langle 1,3,6;18\rangle$.

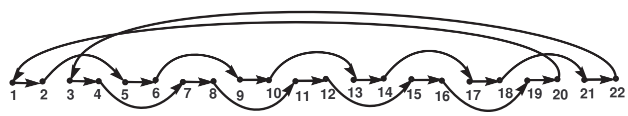

Figure 17. A Hamiltonian cycle in $T_{22}\langle 1,3,6;19\rangle$.

which does not contain the edge $(15,16)$. The next representative in this class is $n=29$, where a Hamiltonian cycle in $T_{29}\langle 1,3,6;13\rangle$ is $(1,4,5,8,9,10,11,12,18,19,20,21,22,25,26,29,16,17,23,24,27,28,15$ which contains the edge $(27,28)$.

(iii) If $t\equiv 2\,mod\,4$ and $t\neq 14$, a Hamiltonian cycle in $T_{n=2t-9}\langle 1,3,6;t\rangle$ is given by $P_{1\to 5}\cup P_{5\to 9}\cup\cdots\cup P_{t-21\to t-17}\cup(t-17,t-11,t-10,t-7,t-1,t,t+3,t+4,\ldots,n-2,n-1,n,n-t=t-9,t-8,t-5,t-4,t+2,2,3)\cup P_{3\to 7}\cup P_{7\to 11}\cup\cdots\cup P_{t-19\to t-15}\cup(t-15,t-14,t-13,t-12,t-6,t-3,t-2,t+1,1)$, see Figure 24.

For $t=14$, a Hamiltonian cycle in $T_{19}\langle 1,3,6;14\rangle$ is $(1,7,13,19,5,8,11,14,17,18,4,10,16,2,3,6,9,12,15,1$

(iv) If $t\equiv 3\,mod\,4$, a Hamiltonian cycle in $T_{n=2t-9}\langle 1,3,6;t\rangle$ is given by $P_{1\to 5}\cup P_{5\to 9}\cup\cdots\cup P_{t-14\to t-10}\cup(t-10,t-7,t-1,t,t+3,t+4,\ldots,n-2,n-1,n,n-t=t-9,t-8,t-5,t-4,t+2,2,3)\cup P_{3\to 7}\cup P_{7\to 11}\cup\cdots\cup P_{t-16\to t-12}\cup(t-12,t-6,t-3,t-2,t+1,1)$, see Figure 25.

All these Hamiltonian cycles in $T_n\langle 1,3,6;t\rangle$, which contain the edge (n-2, n-1), can be extended to Hamiltonian cycles in $T_{n+(t-1)}\langle 1,3,6;t\rangle$. Thus, by applying the technique from Remark 1.1, it follows that $T_n\langle 1,3,6;t\rangle$ is Hamiltonian for all $n\equiv t-8\,mod\,(t-1)$.

Case 6. $n\equiv t-5\,mod\,(t-1)$.

The smallest n in this case is $n=2t-6$. We analyze values of t mod 4.

(i) If $t\equiv 0\,mod\,4$, a Hamiltonian cycle in $T_{n=2t-6}\langle 1,3,6;t\rangle$ is given by $P_{1\to 5}\cup P_{5\to 9}\cup\cdots\cup P_{t-11\to t-7}\cup(t-7,t-1,t,t+3,t+4,\ldots,n-2,n-1,n,n-t=t-6,t-5,t-4,t+2,2,3)\cup P_{3\to 7}\cup P_{7\to 11}\cup\cdots\cup P_{t-13\to t-9}\cup(t-9,t-3,t-2,t+1,1)$, see Figure 26.

(ii) If $t\equiv 1\,mod\,4$, a Hamiltonian cycle in $T_{n=2t-6}\langle 1,3,6;t\rangle$ is $P_{1\to 5}\cup P_{5\to 9}\cup\cdots\cup P_{t-12\to t-8}\cup$

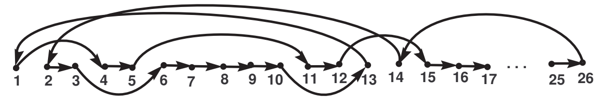

Figure 18. A Hamiltonian cycle in $T_{26}\langle 1,3,6;12\rangle$.

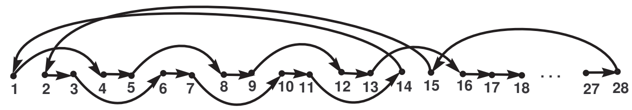

Figure 19. A Hamiltonian cycle in $T_{28}\langle 1,3,6;13\rangle$.

$(t-8,t-5,t-2,t-1,t+2,2,3)\cup P_{3\to 7}\cup P_{7\to 11}\cup\cdots\cup P_{t-14\to t-10}\cup(t-10,t-7,t-4,t-3,t+3,t+4,\ldots,n-2,n-1,n,n-t=t-6,t,t+1,1)$, see Figure 27.

(iii) If $t\equiv 2\, mod\, 4$ and $t\neq 10$, a Hamiltonian cycle in $T_{n=2t-6}\langle 1,3,6;t\rangle$ is $P_{1\to 5}\cup P_{5\to 9}\cup\cdots\cup P_{t-13\to t-9}\cup(t-9,t-3,t+3,t+4,t+5)\cup P_{t+5\to t+9}\cup P_{t+9\to t+13}\cup\cdots\cup P_{n-7\to n-3}\cup(n-3,n,n-t=t-6,t,t+6,t+7)\cup P_{t+7\to t+11}\cup P_{t+11\to t+15}\cup\cdots\cup P_{n-5\to n-1}\cup(n-1,n-1-t=t-7,t-4,t-1,t+2,2,3)\cup P_{3\to 7}\cup P_{7\to 11}\cup\cdots\cup P_{t-15\to t-11}\cup(t-11,t-8,t-5,t-2,t+1,1)$, see Figure 28.

For $t=10$, a Hamiltonian cycle in $T_{14}\langle 1,3,6;10\rangle$ is $(1,7,10,13,3,6,9,12,2,8,14,4,5,11,1)$, which does not contain the edge $(12,13)$. The next representative in this class is $n=23$, where a Hamiltonian cycle in $T_{23}\langle 1,3,6;10\rangle$ is $(1,2,8,14,20,21,22,12,18,19,9,15,16,17,23,13,3,4,5,6,7,10,11,1)$, which contains the edge $(21,22)$.

(iv) If $t\equiv 3\, mod\, 4$, a Hamiltonian cycle in $T_{n=2t-6}\langle 1,3,6;t\rangle$ is $P_{1\to 5}\cup P_{5\to 9}\cup\cdots\cup P_{t-18\to t-14}\cup(t-14,t-8,t-7,t-1,t,t+3,t+4,\ldots,n-2,n-1,n,n-t=t-6,t-5,t-4,t+2,2,3)\cup P_{3\to 7}\cup P_{7\to 11}\cup\cdots\cup P_{t-16\to t-12}\cup(t-12,t-11,t-10,t-9,t-3,t-2,t+1,1)$, see Figure 29.

All these Hamiltonian cycles in $T_n\langle 1,3,6;t\rangle$, which include the edge (n-2, n-1), can be extended to form Hamiltonian cycles in $T_{n+(t-1)}\langle 1,3,6;t\rangle$. Therefore, by applying the technique from Remark 1.1, $T_n\langle 1,3,6;t\rangle$ is Hamiltonian for all $n\equiv t-5\, mod\,(t-1)$.

Case 7. $n\equiv t-4\, mod\,(t-1)$.
The smallest possible value of n is $n=2t-5$. We analyze values of t mod 4.

(i) If $t\equiv 0\, mod\, 4$, a Hamiltonian cycle in $T_{n=2t-5}\langle 1,3,6;t\rangle$ is $P_{1\to 5}\cup P_{5\to 9}\cup\cdots\cup P_{t-11\to t-7}\cup(t-7,t-1,t,t+3,t+4,\ldots,n-2,n-1,n,n-t=t-5,t-4,t+2,2,3)\cup P_{3\to 7}\cup P_{7\to 11}\cup\cdots\cup P_{t-13\to t-9}\cup(t-9,t-6,t-3,t-2,t+1,1)$, see Figure 30.

(ii) If $t\equiv 1\, mod\, 4$ and $t\notin\{13,17\}$, a Hamiltonian cycle in $T_{n=2t-5}\langle 1,3,6;t\rangle$ is $P_{1\to 5}\cup P_{5\to 9}\cup\cdots\cup P_{t-20\to t-16}\cup(t-16,t-15,t-14,t-13,t-7,t-6,t-3,t+3,t+4,\ldots,n-2,n-1,n,n-t=t-5,t-4,t+2,2,3)\cup P_{3\to 7}\cup P_{7\to 11}\cup\cdots\cup P_{t-22\to t-18}\cup(t-18,t-12,t-11,t-10,t-9,t-8,t-2,t-1,t,t+1,1)$, see Figure 31.

For $t\in\{13,17\}$, Hamiltonian cycles in $T_{21}\langle 1,3,6;13\rangle$ and $T_{29}\langle 1,3,6;17\rangle$ are $(1,4,7,10,$

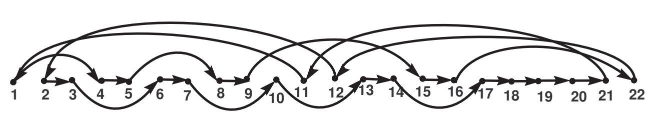

Figure 20. A Hamiltonian cycle in $T_{22}\langle 1,3,6;10\rangle$.

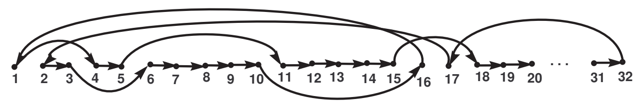

Figure 21. A Hamiltonian cycle in $T_{32}\langle 1,3,6;15\rangle$.

$13,16,3,6,9,12,15,2,5,11,17,18,19,20,21,8,14,1)$ and $(1,4,7,10,13,16,19,2,5,8,11,14,17,20,3,6,9,15,21,22,23,24,25,26,27,28,29,12,18,1)$, respectively.

(iii) If $t \equiv 2\, mod\, 4$ and $t \neq 10$, a Hamiltonian cycle in $T_{n=2t-5}\langle 1,3,6;t\rangle$ is $P_{1\to 5} \cup P_{5\to 9} \cup \cdots \cup P_{t-17\to t-13} \cup (t-13, t-7, t-6, t-3, t+3, t+4, \ldots, n-2, n-1, n, n-t = t-5, t-4, t+2, 2, 3) \cup P_{3\to 7} \cup P_{7\to 11} \cup \cdots \cup P_{t-15\to t-11} \cup (t-11, t-10, t-9, t-8, t-2, t-1, t, t+1, 1)$, see Figure 32.

For $t = 10$, a Hamiltonian cycle in $T_{15}\langle 1,3,6;10\rangle$ is $(1,2,8,14,15,5,6,9,12,13,3,4,7,10,11,1)$, which does not include the edge $(13,14)$. The next representative in this class is $n = 24$, where a Hamiltonian cycle in $T_{24}\langle 1,3,6;10\rangle$ is $(1,2,3,4,5,6,12,15,18,24,14,20,21,22,23,13,19,9,10,16,17,7,8,11,1)$ which includes the edge $(22,23)$.

(iv) If $t \equiv 3\, mod\, 4$, a Hamiltonian cycle in $T_{n=2t-5}\langle 1,3,6;t\rangle$ is $P_{1\to 5} \cup P_{5\to 9} \cup \cdots \cup P_{t-10\to t-6} \cup (t-6, t-3, t+3, t+4, \ldots, n-2, n-1, n, n-t = t-5, t-4, t+2, 2, 3) \cup P_{3\to 7} \cup P_{7\to 11} \cup \cdots \cup P_{t-12\to t-8} \cup (t-8, t-2, t-1, t, t+1, 1)$, see Figure 33.

All Hamiltonian cycles in $T_n\langle 1,3,6;t\rangle$ that include the edge (n-2, n-1) can be extended to Hamiltonian cycles in $T_{n+(t-1)}\langle 1,3,6;t\rangle$. Thus, by applying the technique from Remark 1.1, $T_n\langle 1,3,6;t\rangle$ is Hamiltonian for all $n \equiv t-4\, mod\, (t-1)$.

Case 8. $n \equiv s\, mod\, (t-1)$, where $s \in \{0,1,2,\ldots,t-2\} \setminus \{0,1,2,4,t-8,t-5,t-4\}$. The smallest n for each s is $n = s+t-1$. We further analyze values of n mod 4, with even and odd t.

(i) If ($n \equiv 0\, mod\, 4$ and t is even) or ($n \equiv 2\, mod\, 4$ and t is odd). Then a Hamiltonian cycle in $T_n\langle 1,3,6;t\rangle$ is $(1,2,\ldots,s-2) \cup P_{s-2\to s+2} \cup P_{s+2\to s+6} \cup \cdots \cup P_{t-5\to t-1} \cup (t-1, t+2, t+3, \ldots, n-2, n-1, n, n-t = s-1, s) \cup P_{s\to s+4} \cup P_{s+4\to s+8} \cup \cdots \cup P_{t-3\to t+1} \cup (t+1, 1)$, see Figure 34.

(ii) If ($n \equiv 1\, mod\, 4$ and t is even) or ($n \equiv 3\, mod\, 4$ and t is odd), then a Hamiltonian cycle in $T_n\langle 1,3,6;t\rangle$ is $(1,2,\ldots,s-2) \cup P_{s-2\to s+2} \cup P_{s+2\to s+6} \cup \cdots \cup P_{t-8\to t-4} \cup (t-4, t+2, t+3, \ldots, n-2, n-1, n, n-t = s-1, s) \cup P_{s\to s+4} \cup P_{s+4\to s+8} \cup \cdots \cup P_{t-6\to t-2} \cup (t-2, t-1, t, t+1, 1)$, see Figure 35.

(iii) If ($n \equiv 2\, mod\, 4$ and t is even) or ($n \equiv 0\, mod\, 4$ and t is odd), then a Hamiltonian cycle in $T_n\langle 1,3,6;t\rangle$ is $(1,2,\ldots,s-2) \cup P_{s-2\to s+2} \cup P_{s+2\to s+6} \cup \cdots \cup P_{t-15\to t-11} \cup (t-11, t-5, t-4, t+2, t+3, \ldots, n-2, n-1, n, n-t = s-1, s) \cup P_{s\to s+4} \cup P_{s+4\to s+8} \cup \cdots \cup P_{t-13\to t-9} \cup$

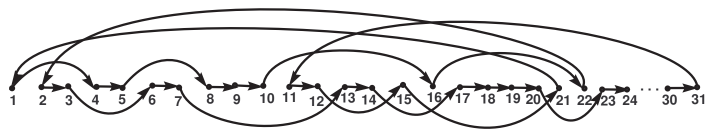

Figure 22. A Hamiltonian cycle in $T_{31}\langle 1,3,6;20\rangle$.

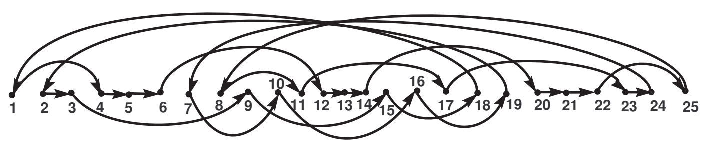

Figure 23. A Hamiltonian cycle in $T_{25}\langle 1,3,6;17\rangle$.

$(t-9, t-8, t-7, t-6, t-3, t-2, t-1, t, t+1, 1)$, see Figure 36.

(iv) If ($n \equiv 3\, mod\, 4$ and t is even) or ($n \equiv 1\, mod\, 4$ and t is odd), then a Hamiltonian cycle in $T_n\langle 1,3,6;t\rangle$ is $(1,2,\ldots,s-2) \cup P_{s-2\to s+2} \cup P_{s+2\to s+6} \cup \cdots \cup P_{t-18\to t-14} \cup (t-14, t-8, t-7, t-6, t-5, t-4, t+2, t+3, \ldots, n-2, n-1, n, n-t = s-1, s) \cup P_{s\to s+4} \cup P_{s+4\to s+8} \cup \cdots \cup P_{t-16\to t-12} \cup (t-12, t-11, t-10, t-9, t-3, t-2, t-1, t, t+1, 1)$, see Figure 37.

Since in all these cases the edge (n-2, n-1) is present in the Hamiltonian cycles of $T_n\langle 1,3,6;t\rangle$, applying the technique from Remark 1.1 ensures that $T_n\langle 1,3,6;t\rangle$ remains Hamiltonian for all $n \equiv s\, mod\,(t-1)$, where $s \in \{0,1,2,\ldots,t-2\} \setminus \{0,1,2,4,t-8,t-5,t-4\}$.

This completes the proof. □

Conjectures and concluding remark

We propose the following conjectures.

1. $T_n\langle 1,3,6;2\rangle$ is non-Hamiltonian for $n \equiv 2\, mod\, 4$.
2. $T_n\langle 1,3,6;3\rangle$ is non-Hamiltonian for $n \in \{7,8,9,12,14,16\}$.
3. $T_{12}\langle 1,3,6;4\rangle$ is non-Hamiltonian.
4. $T_n\langle 1,3,6;6\rangle$ is non-Hamiltonian for $n \in \{10,14\}$.
5. $T_n\langle 1,3,6;7\rangle$ is non-Hamiltonian for $n \in \{9,12\}$.
6. $T_{10}\langle 1,3,6;8\rangle$, $T_{13}\langle 1,3,6;9\rangle$ and $T_{13}\langle 1,3,6;11\rangle$ are non-Hamiltonian.

The next step, in our view, is to complete the investigation of Hamiltonicity in Toeplitz graphs $T_n\langle 1,3,6,s_4,s_5,\ldots,s_k;t_1,t_2,\ldots,t_l\rangle$ by solving the stated conjectures.

Acknowledgement

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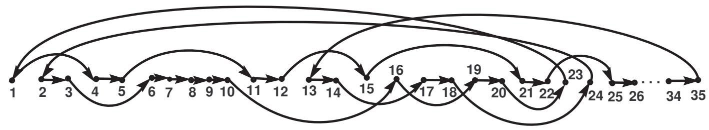

Figure 24. A Hamiltonian cycle in $T_{35}\langle 1,3,6;22\rangle$.

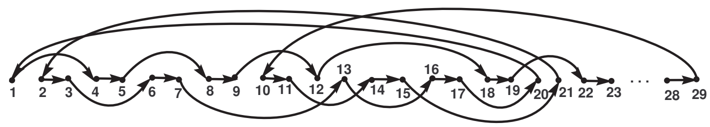

Figure 25. A Hamiltonian cycle in $T_{29}\langle 1,3,6;19\rangle$.

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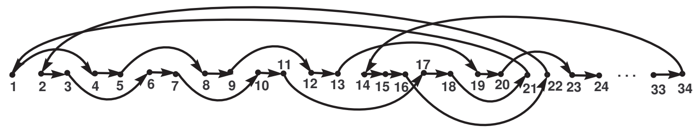

Figure 26. A Hamiltonian cycle in $T_{34}\langle 1,3,6;20\rangle$.

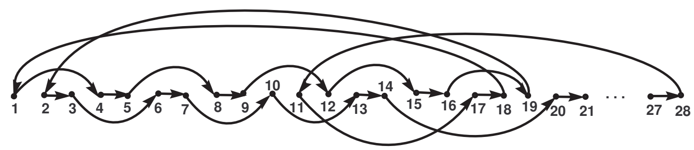

Figure 27. A Hamiltonian cycle in $T_{28}\langle 1,3,6;17\rangle$.

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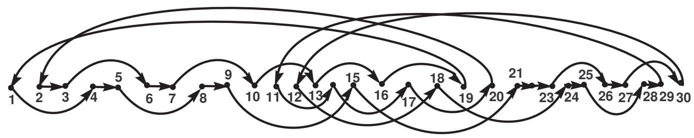

Figure 28. A Hamiltonian cycle in $T_{30}\langle 1, 3, 6; 18\rangle$.

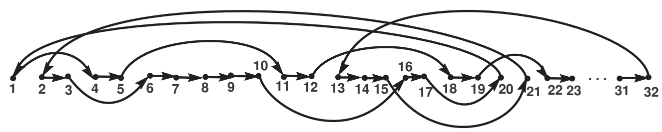

Figure 29. A Hamiltonian cycle in $T_{32}\langle 1, 3, 6; 19\rangle$.

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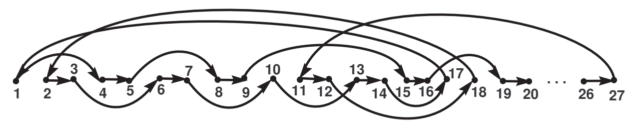

Figure 30. A Hamiltonian cycle in $T_{27}\langle 1,3,6;16\rangle$.

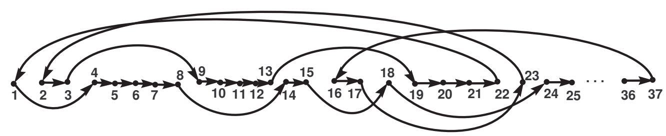

Figure 31. A Hamiltonian cycle in $T_{37}\langle 1,3,6;21\rangle$.

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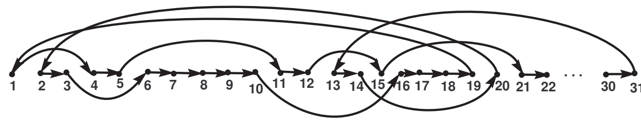

Figure 32. A Hamiltonian cycle in $T_{31}\langle 1,3,6;18\rangle$.

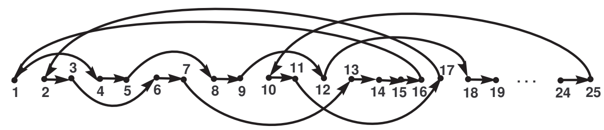

Figure 33. A Hamiltonian cycle in $T_{25}\langle 1,3,6;15\rangle$.

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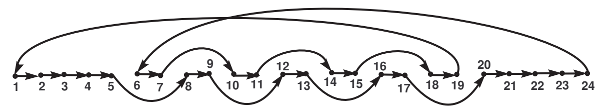

Figure 34. A Hamiltonian cycle in $T_{24}\langle 1, 3, 6; 18\rangle$, where $s = 7$.

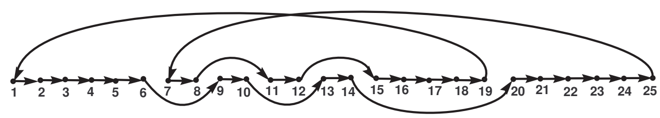

Figure 35. A Hamiltonian cycle in $T_{25}\langle 1, 3, 6; 18\rangle$, where $s = 8$.

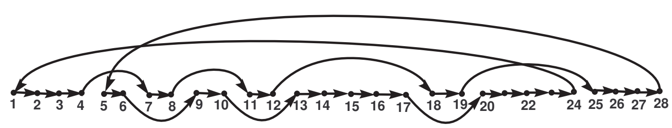

Figure 36. A Hamiltonian cycle in $T_{28}\langle 1, 3, 6; 23\rangle$, where $s = 6$

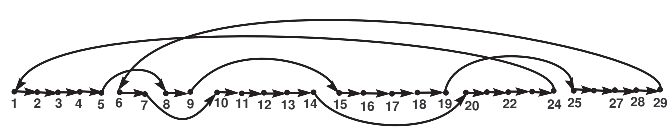

Figure 37. A Hamiltonian cycle in $T_{29}\langle 1, 3, 6; 23\rangle$, where $s = 7$.