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Further results on the total vertex irregularity strength of trees

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Abstract

We investigate the total vertex irregularity strength of trees with specific characteristics. Initially, we categorize trees into three distinct groups: types A, B, and C. Subsequently, we calculate $\mathrm{tvs}(T)$ for all type A trees T where the maximum degree is at least three. Additionally, we provide the value of $\mathrm{tvs}(T)$ whenever T is a tree of types B or C with maximum degree at least three and large number of exterior vertices. Finally, we propose a conjecture related to $\mathrm{tvs}(T)$ where T is a non-path tree of types B or C with few exterior vertices.

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1. Introduction

Throughout the paper, all graphs are assumed to be simple, finite, and undirected. We follow [6] for terminologies and definitions not mentioned here. For a graph $G = (V, E)$, V represents its set of vertices and E its set of edges. We denote the degree of a vertex v in G by $\deg(v)$. The minimum and maximum vertex degrees in G are denoted by δ and Δ, respectively. For a graph G

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and a positive integer k, a function $\varphi : V \cup E \to \{1, 2, \ldots, k\}$ is called a *vertex irregular total k-labeling* of G if the *vertex-weights*,

$$wt(v) = \varphi(v) + \sum_{uv \in E} \varphi(uv)$$

are all distinct. The *total vertex irregularity strength* of G, denoted by $\mathrm{tvs}(G)$, is the smallest integer k such that G admits a vertex irregular total k-labeling.

The concept of total vertex irregularity strength of graphs was proposed by Bača et al. [3] as a variant of the irregularity strength initially described by Chartrand et al. [5]. In connection with these irregularity parameters, studies have also examined the modular irregularity strength, which was introduced by Bača et al. [4], as an alternative means of evaluating the irregularity of graphs based on modular arithmetic constraints. For recent developments in this area, refer to [11, 12]. A comprehensive overview of irregularity in graphs can be found in [2], while a detailed survey on graph labelings is available in [7].

Determining the total vertex irregularity strength for any given tree is generally an intricate and complex task. Nurdin et al. [8] established a general lower bound for the total vertex irregularity strength of a tree with maximum degree Δ as follows:

$$\mathrm{tvs}(T) \geqslant \max\{t_i : i \in [1, \Delta]\},$$

where

$$t_i = \left\lceil \frac{\sum_{j=1}^{i} n_j + 1}{i+1} \right\rceil \text{ and } n_j = |\{v \in V : \deg(v) = j\}|.$$

Subsequent work by Susanto et al. [13] refined this lower bound for $\mathrm{tvs}(T)$ by expressing it solely in terms of the number of vertices with degrees one, two, and three.

Lemma 1.1 ([13]). *Let T be a tree. Then* $\mathrm{tvs}(T) \geqslant \max\{t_1, t_2, t_3\}$.

Moreover, Nurdin et al. [8] conjectured that the lower bounds described in Lemma 1.1 apply universally to all trees, that is, for any tree T,

$$\mathrm{tvs}(T) = \max\{t_1, t_2, t_3\}. \tag{1}$$

Although this condition holds for many trees such as complete n-ary trees [9], subdivisions of various trees like stars, caterpillars, firecrackers, and amalgamations of stars [9, 16], irregular subdivisions of trees [15], trees without vertices of degree two [8], certain trees with a significant number of vertices of degree two [10], and some trees with maximum degrees four and five [17, 18], the conjecture is generally false as Susanto et al. [14] discovered infinite counterexamples disproving it.

In this work, we investigate the total vertex irregularity strength of trees with specific properties. To begin, let us define three families of trees as follows.

(a) A tree of type A is a tree that satisfies either ($n_2 \leqslant \frac{n_1+1}{2}$ and $n_3 \leqslant \frac{n_1+1}{2}$) or ($n_2 \leqslant n_1 - n_3 + 1$ and $n_3 > \frac{n_1+1}{2}$).

(b) A tree of type B is a tree that satisfies either ($n_2 > \frac{n_1+1}{2}$ and $n_3 \leqslant \frac{n_1+1}{2}$) or ($n_2 > 3n_3 - n_1 - 1$ and $n_3 > \frac{n_1+1}{2}$).
(c) A tree of type C is a tree that satisfies $n_1 - n_3 + 1 < n_2 \leqslant 3n_3 - n_1 - 1$ and $n_3 > \frac{n_1+1}{2}$.

It is evident that trees of types A, B, and C form a partition of all trees. In this study, we demonstrate that the Equation (1) is valid for all trees of types B or C with a maximum degree of at least three and a significant number of exterior vertices. Additionally, we establish the same result for all trees of type A with a maximum degree of no less than three.

2. Preliminaries

In a tree T, a vertex is called a *pendant* if it has degree one. An edge incident to a pendant vertex is a *pendant edge*. A vertex in T with degree at least two is an *exterior vertex* if adjacent to a pendant, otherwise an *interior vertex*. Any non-pendant edge is called an *interior edge*.

An *exterior path* is a path from a pendant vertex to the nearest vertex with a degree of at least three. An *interior path* is a path between two vertices of degree at least three, passing only through vertices of degree two. Note that every pendant edge incident to an exterior vertex of degree at least three is an exterior path of length one, and every interior edge that connects two vertices of degree at least three is an interior path of length one.

Given an edge (total) labeling ϕ of a tree T, the *temporary weight* of a vertex v of T under the labeling ϕ is given by

$$\omega_\phi(v) = \sum_{uv \in E} \phi(uv).$$

For $i \in [1, \Delta]$ and $j \in [1, n_i]$, denote by v_{ij}^{ϕ} a vertex of degree i with the jth smallest temporary weight under ϕ.

The next three lemmas establish key properties of trees of types A, B, and C, which are essential for evaluating their total vertex irregularity strengths.

Lemma 2.1. *Let T be a tree of type A. Then $t_1 = \max\{t_1, t_2, t_3\}$.*

Proof. We divide two cases.

Case 1: $n_2 \leqslant \frac{n_1+1}{2}$ and $n_3 \leqslant \frac{n_1+1}{2}$. Because $n_3 \leqslant \frac{n_1+1}{2}$ we obtain $3n_3 - n_1 - 1$. Then

$$t_2 = \left\lceil \frac{n_1 + n_2 + 1}{3} \right\rceil \leqslant \left\lceil \frac{n_1 + (\frac{n_1+1}{2}) + 1}{3} \right\rceil = \left\lceil \frac{n_1 + 1}{2} \right\rceil = t_1$$

and

$$t_3 = \left\lceil \frac{n_1 + n_2 + n_3 + 1}{4} \right\rceil \leqslant \left\lceil \frac{n_1 + (\frac{n_1+1}{2}) + (\frac{n_1+1}{2}) + 1}{4} \right\rceil = \left\lceil \frac{n_1 + 1}{2} \right\rceil = t_1.$$

Therefore, $\max\{t_1, t_2, t_3\} = t_1$.

Case 2: $n_2 \leqslant n_1 - n_3 + 1$ and $n_3 > \frac{n_1+1}{2}$. Since $n_2 \leqslant n_1 - n_3 + 1$, we get $n_3 \leqslant n_1 - n_2 + 1$. Then

$$t_3 = \left\lceil \frac{n_1 + n_2 + n_3 + 1}{4} \right\rceil \leqslant \left\lceil \frac{n_1 + n_2 + (n_1 - n_2 + 1) + 1}{4} \right\rceil = \left\lceil \frac{n_1 + 1}{2} \right\rceil = t_1.$$

Furthermore, as $n_3 > \frac{n_1+1}{2}$, it follows that $3n_3 - n_1 - 1 > n_1 - n_3 + 1 \geqslant n_2$. So

$$\begin{aligned} t_2 = \left\lceil \frac{n_1+n_2+1}{3} \right\rceil &= \left\lceil \frac{4n_1+3n_2+n_2+4}{12} \right\rceil \\ &\leqslant \left\lceil \frac{4n_1+3n_2+(3n_3-n_1-1)+4}{12} \right\rceil = \left\lceil \frac{n_1+n_2+n_3+1}{4} \right\rceil = t_3 \leqslant t_1. \end{aligned}$$

Hence, $\max\{t_1, t_2, t_3\} = t_1$. □

Lemma 2.2. *Let T be a tree of type B. Then $t_2 = \max\{t_1, t_2, t_3\}$.*

Proof. We consider two cases.

Case 1. $n_2 > \frac{n_1+1}{2}$ and $n_3 \leqslant \frac{n_1+1}{2}$. Because $n_2 > \frac{n_1+1}{2}$ we have $n_1 < 2n_2 - 1$, and so

$$t_1 = \left\lceil \frac{n_1+1}{2} \right\rceil = \left\lceil \frac{2n_1+n_1+3}{6} \right\rceil \leqslant \left\lceil \frac{2n_1+(2n_2-1)+3}{6} \right\rceil = \left\lceil \frac{n_1+n_2+1}{3} \right\rceil = t_2.$$

Moreover, since $n_3 \leqslant \frac{n_1+1}{2}$, we obtain $\frac{n_1+1}{2} \leqslant n_1 - n_3 + 1$ and $3n_3 - n_1 - 1 \leqslant n_1 - n_3 + 1$.

If $n_2 \leqslant n_1 - n_3 + 1$ then $\frac{n_1+1}{2} < n_2 \leqslant n_1 - n_3 + 1$, and so

$$t_3 = \left\lceil \frac{n_1+n_2+n_3+1}{4} \right\rceil \leqslant \left\lceil \frac{n_1+n_2+(n_1-n_2+1)+1}{4} \right\rceil = \left\lceil \frac{n_1+1}{2} \right\rceil = t_1 \leqslant t_2.$$

If $n_2 > n_1 - n_3 + 1$ then $n_2 > 3n_3 - n_1 - 1$, thus

$$\begin{aligned} t_3 = \left\lceil \frac{n_1+n_2+n_3+1}{4} \right\rceil &= \left\lceil \frac{3n_1+3n_2+3n_3+3}{12} \right\rceil \\ &\leqslant \left\lceil \frac{3n_1+3n_2+(n_1+n_2+1)+3}{12} \right\rceil = \left\lceil \frac{n_1+n_2+1}{3} \right\rceil = t_2. \end{aligned}$$

Thus, $\max\{t_1, t_2, t_3\} = t_2$.

Case 2. $n_2 > 3n_3 - n_1 - 1$ and $n_3 > \frac{n_1+1}{2}$. As $n_2 > 3n_3 - n_1 - 1$, we get $3n_3 < n_1 + n_2 + 1$. Then

$$\begin{aligned} t_3 = \left\lceil \frac{n_1+n_2+n_3+1}{4} \right\rceil &= \left\lceil \frac{3n_1+3n_2+3n_3+3}{12} \right\rceil \\ &\leqslant \left\lceil \frac{3n_1+3n_2+(n_1+n_2+1)+3}{12} \right\rceil = \left\lceil \frac{n_1+n_2+1}{3} \right\rceil = t_2. \end{aligned}$$

Also, since $n_3 > \frac{n_1+1}{2}$, it holds that $n_1 - n_3 + 1 < 3n_3 - n_1 - 1 < n_2$. Hence

$$t_1 = \left\lceil \frac{n_1+1}{2} \right\rceil = \left\lceil \frac{n_1+n_1+2}{4} \right\rceil \leqslant \left\lceil \frac{n_1+(n_2+n_3-1)+2}{4} \right\rceil = t_3 \leqslant t_2.$$

Consequently, $\max\{t_1, t_2, t_3\} = t_2$. □

Lemma 2.3. *Let T be a tree of type C. Then $t_3 = \max\{t_1, t_2, t_3\}$.*

Proof. We know that $n_1 - n_3 + 1 < n_2 \leqslant 3n_3 - n_1 - 1$. Then

$$t_1 = \left\lceil \frac{n_1+1}{2} \right\rceil = \left\lceil \frac{n_1+n_1+2}{4} \right\rceil \leqslant \left\lceil \frac{n_1+(n_2+n_3-1)+2}{4} \right\rceil = t_3.$$

Further, we also know that $n_3 > \frac{n_1+1}{2}$. Then $n_1 - n_3 + 1 < \frac{n_1+1}{2}$ and $\frac{n_1+1}{2} < 3n_3 - n_1 - 1$.

If $n_2 \leqslant \frac{n_1+1}{2}$ then $n_1 - n_3 + 1 < n_2 \leqslant \frac{n_1+1}{2}$, and so

$$t_2 = \left\lceil \frac{n_1+n_2+1}{3} \right\rceil \leqslant \left\lceil \frac{n_1+(\frac{n_1+1}{2})+1}{3} \right\rceil = \left\lceil \frac{n_1+1}{2} \right\rceil = t_1 \leqslant t_3.$$

If $n_2 > \frac{n_1+1}{2}$ then $\frac{n_1+1}{2} < n_2 \leqslant 3n_3 - n_1 - 1$, thus

$$\begin{aligned} t_2 = \left\lceil \frac{n_1+n_2+1}{3} \right\rceil &= \left\lceil \frac{4n_1+3n_2+n_2+4}{12} \right\rceil \\ &\leqslant \left\lceil \frac{4n_1+3n_2+(3n_3-n_1-1)+4}{12} \right\rceil = \left\lceil \frac{n_1+n_2+n_3+1}{4} \right\rceil = t_3. \end{aligned}$$

So, $\max\{t_1, t_2, t_3\} = t_3$. □

Lemma 2.4. *Let T be a tree.*

1. *If T is of type A then $t_1 < n_1$.*
2. *If T is of type B then $t_2 \leqslant n_1$ if and only if $n_2 < 2n_1$.*
3. *If T is of type C then $t_3 < n_1$.*

Proof. If T is of type A then it is clear that $t_1 < n_1$, and if T is of type C, we have $n_2 \leqslant 3n_3 - n_1 - 1$ by the definition, so

$$t_3 = \left\lceil \frac{n_1+n_2+n_3+1}{4} \right\rceil \leqslant \left\lceil \frac{n_1+(3n_3-n_1-1)+n_3+1}{4} \right\rceil = n_3 < n_1.$$

Now let T be a tree of type B. Then

$$t_2 = \left\lceil \frac{n_1+n_2+1}{3} \right\rceil \leqslant n_1 \quad \Leftrightarrow \quad n_2 < 2n_1.$$

□

A vertex of degree two in a tree T with maximum degree at least three is called a *special vertex* if it meets one of the following conditions:

(a) It belongs to an exterior path of length two.

(b) It belongs to an exterior path of length at least three, and both of its neighbors have degree at most two.

(c) It belongs to an interior path, and both of its neighbors have degree two.

For positive integers k and ℓ, let P_k^I represents an interior path of length $k-1$ in T, and P_ℓ^E denotes an exterior path of length $\ell-1$ in T. The terms $n(P_k^I)$ and $n(P_\ell^E)$ indicate the number of P_k^I and P_ℓ^E in T, respectively.

Lemma 2.5. *Let T be a tree of type B with $n_2 \geqslant 2n_1$. Then the number of special vertices must be greater than $t_2 - n_1$.*

Proof. By definition, the number of special vertices in T is

$$S = (k-4)\sum_{k\geqslant 4} n(P_k^I) + n(P_3^E) + (\ell-3)\sum_{\ell\geqslant 4} n(P_\ell^E).$$

As $n_2 \geqslant 2n_1$, we may write $n_2 = 2n_1 + i$ for $i \geqslant 0$. Our aim is to show that $S > t_2 - n_1 = \lceil \frac{i+1}{3} \rceil$ for $i \geqslant 0$. We know that for $i \geqslant 0$,

$$2n_1 + i = (k-2)\sum_{k\geqslant 3} n(P_k^I) + (\ell-2)\sum_{\ell\geqslant 3} n(P_\ell^E) = S + S' + n(P_3^I) + 2(n(P_4^I) + n(P_5^I)),$$

where $S' = 2\sum_{k\geqslant 6} n(P_k^I) + \sum_{\ell\geqslant 4} n(P_\ell^E)$. Note that the number of interior paths in T must be at least half the sum $n(P_3^I) + 2(n(P_4^I) + n(P_5^I))$, that is,

$$\begin{aligned} n(P_3^I) + 2(n(P_4^I) + n(P_5^I)) &\leqslant 2\sum_{k\geqslant 2} n(P_k^I) \\ &= 2\left(\sum_{i=3}^{\Delta} n_i - 1\right) = 2n_1 + \sum_{i=3}^{\Delta}(6-2i)n_i - 6 \leqslant 2n_1 - 6, \end{aligned}$$

and so $S + S' \geqslant i + 6$. Note also that $S - S' \geqslant 0$. Combining both of these inequalities, $S \geqslant \frac{i+6}{2} > \lceil \frac{i+1}{3} \rceil$. □

Suppose that $V_{Ex} = \{v_1, v_2, \ldots, v_\varepsilon\}$ be the set of exterior vertices of a tree such that for $1 \leqslant i < j \leqslant \varepsilon$, the following properties hold:

(a) $\deg(v_i) \leqslant \deg(v_j)$.
(b) If $\deg(v_i) = \deg(v_j) = 2$ then $d(v_i, z_i) \geqslant d(v_j, z_j)$, where z_i is a vertex of degree at least three closest to v_i.
(c) If $\deg(v_i) = \deg(v_j) \geqslant 3$ then $|E_P(v_i)| \geqslant |E_P(v_j)|$, where $E_P(v_i)$ denotes the set of all pendant edges incident to v_i.

For $i \in [1, \varepsilon]$ and $j \in [1, |E_P(v_i)|]$, let e_{ij} denotes the jth pendant edge incident to the exterior vertex v_i. Set $s = \max\{|E_P(v_i)| : i \in [1, \varepsilon]\}$. For $j \in [1, s]$, let E_P^j represents the ordered set $\{e_{ij} : i \in [1, \varepsilon]$ and $|E_P(v_i)| \geqslant j\}$, and E_P the ordered set formed by the union $\bigcup_{j=1}^{s} E_P^j$, preserving the original order of each E_P^j. For instance, considering a tree depicted in Figure 1, we observe $E_P^1 = \{e_{11}, e_{21}, e_{31}, e_{41}\}$, $E_P^2 = \{e_{22}, e_{32}, e_{42}\}$, $E_P^3 = \{e_{33}, e_{43}\}$, $E_P^4 = \{e_{34}\}$, and consequently $E_P = \{e_{11}, e_{21}, e_{31}, e_{41}, e_{22}, e_{32}, e_{42}, e_{33}, e_{43}, e_{34}\}$.

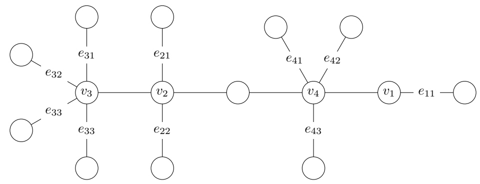

Figure 1. A tree and denotation of its exterior vertices and pendant edges.

For a non-negative integer K, consider the interior paths $P^I_{k_1}, P^I_{k_2}, \dots, P^I_{k_K}$ in T, each having a length of at least four, and ordered such that $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_K$. Similarly, for $L \geqslant 0$, let $P^E_{\ell_1}, P^E_{\ell_2}, \dots, P^E_{\ell_L}$ denote the exterior paths in T, each with a length of at least two, such that for any $1 \leqslant g < h \leqslant L$, the exterior paths $P^E_{\ell_g}$ and $P^E_{\ell_h}$ encompass the exterior vertices v_g and v_h, respectively. Moreover, assume that for some $0 \leqslant p \leqslant L$, exactly p of these exterior paths have length at least four, that is, $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_p \geqslant 5 > \ell_{p+1} \geqslant \ell_{p+2} \geqslant \cdots \geqslant \ell_L \geqslant 3$.

Denote by $x^g_1, x^g_2, \dots, x^g_{k_g}$, $1 \leqslant g \leqslant K$, the vertices of $P^I_{k_g}$ where $x^g_h x^g_{h+1}$ is an edge for each h, and both x^g_1 and $x^g_{k_g}$ are vertices of degree at least three in T. Similarly, denote by $y^g_1, y^g_2, \dots, y^g_{\ell_g}$, $1 \leqslant g \leqslant L$, the vertices of $P^E_{\ell_g}$ where $y^g_h y^g_{h+1}$ is an edge for each h, y^g_1 is a pendant vertex, and $y^g_{\ell_g}$ is a vertex of degree at least three in T. Define an ordered set of special vertices of T as

$$\begin{aligned}\mathcal{S} =& \{x^1_3, x^1_4, \dots, x^1_{k_1-2}, x^2_3, x^2_4, \dots, x^2_{k_2-2}, \dots, x^K_3, x^K_4, \dots, x^K_{k_K-2}\} \\ & \cup \{y^1_2, y^1_3, \dots, y^1_{\ell_1-2}, y^2_2, y^2_3, \dots, y^2_{\ell_2-2}, \dots, y^p_2, y^p_3, \dots, y^p_{\ell_p-2}\} \cup \{y^{p+1}_2, y^{p+2}_2, \dots, y^L_2\}.\end{aligned}$$

Additionally, let $\mathcal{S}_0$ be a subset of $\mathcal{S}$ consisting of the first $t - n_1$ elements of $\mathcal{S}$. By Lemma 2.4, the condition $|\mathcal{S}_0| > 0$ holds if and only if T is a tree of type B with $n_2 \geqslant 2n_1$. Furthermore, by Lemma 2.5, for any tree of type B with $n_2 \geqslant 2n_1$, it follows that $|\mathcal{S}| > t_2 - n_1 = t - n_1 = |\mathcal{S}_0|$, ensuring feasibility.

3. Trees of types B or C with large number of exterior vertices

This section examines the total vertex irregularity strength of non-path trees T of types B or C with many exterior vertices. The main result is presented in Theorem 3.1.

The idea of the proof of Theorem 3.1 is described as follows. Since Lemmas 1.1, 2.2, and 2.3 has provided the lower bound, we only need to show an existence of a vertex irregular total t-labeling of T, where

$$t = \begin{cases} t_2, & \text{if } T \text{ is a tree of type B,} \\ t_3, & \text{if } T \text{ is a tree of type C,} \end{cases}$$

and thus proves the theorem.

The construction algorithm consists of three main steps and incorporates two rounds of labelings, denoted as ϕ and φ. The labeling ϕ is applied in the first step, while φ is utilized in the second and third steps. The latter will then be shown to be a vertex irregular total t-labeling of T.

In the first step, we label pendant edges with integers from $\{1, 2, \dots, t\}$ (if many exterior vertices exist, pendant edges receive labels less than t). We also label interior edges incident to a member of $\mathcal{S}_0$ (if $n_2 \geqslant 2n_1$) with some small integers, and others with t.

After the first step, all vertices meet temporary weight lower bounds, but some vertices with degrees two or three may have excessive values. So, in the second step, we define two sets A and B which contain respectively vertices of degrees two and three with large temporary weights, and construct a matching M that covers vertices in $A \cup B$. We then relabel edges in M to ensure suitable temporary weights.

Finally, in the third step, we greedily assign vertices to achieve unique final weights. In fact, at the end of this step, it turns out that the largest label being used is t. Figure 2 shows an example of a tree of type B with its vertex irregular total labeling.

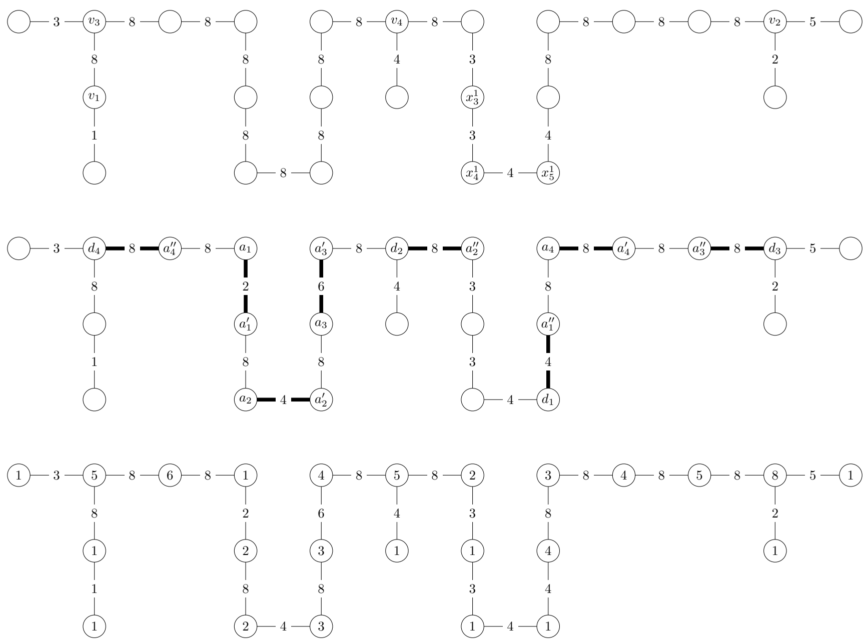

Figure 2. A labeling example for a tree of type B. Top, **Step 1**: Label pendant edges, interior edges incident to a member of $\mathcal{S}_0$ (x_3^1, x_4^1 and x_5^1), and other interior edges. Middle, **Step 2**: Construct a matching M (thick edges) covering vertices in A (in this case, $a = 5$ and b does not exist, so $|A| = 12$ and $|B| = 0$), and relabel edges in M. Bottom, **Step 3**: Greedily label vertices to ensure distinct final weights.

Let T be a non-path tree of types B or C that contains at least $\min\{t-2n_2^*-1, n_1-n_2^*\}$ exterior vertices of degree at least three, where n_2^* is the number of exterior vertices of degree two, and

$$t = \begin{cases} t_2, & \text{if } T \text{ is a tree of type B,} \\ t_3, & \text{if } T \text{ is a tree of type C.} \end{cases}$$

Let us define a labeling ϕ on edges of T as follows. We label pendant edges of T in the following way. For $g \in [1, n_1]$, let

$$\phi(e_{ij}) = \min\{g, t\}, \quad \text{where } e_{ij} \text{ is the } g\text{th element of } E_P \text{ for some } i \text{ and } j. \tag{2}$$

We next label interior edges incident to vertices in $\mathcal{S}$ in the following way. For $g \in [1, K]$, define

$$\phi(x_2^g x_3^g) = \begin{cases} \lfloor \frac{n_1+\sum_{j=1}^{g-1}(k_j-4)+1}{2} \rfloor, & \text{if } x_3^g \in \mathcal{S}_0, \\ t, & \text{otherwise.} \end{cases}$$

For $g \in [1, K]$ and $h \in [3, k_g - 2]$, let

$$\phi(x_h^g x_{h+1}^g) = \begin{cases} \lceil \frac{n_1 + \sum_{j=1}^{g-1}(k_j - 4) + h - 2}{2} \rceil, & \text{if } x_h^g \in \mathcal{S}_0, \text{ for } h \text{ odd}, \\ \lfloor \frac{n_1 + \sum_{j=1}^{g-1}(k_j - 4) + h - 1}{2} \rfloor, & \text{if } x_h^g \in \mathcal{S}_0, \text{ for } h \text{ even}, \\ t & \text{otherwise.} \end{cases}$$

For $g \in [1, p]$ and $h \in [2, \ell_g - 2]$, let

$$\phi(y_h^g y_{h+1}^g) = \begin{cases} \lfloor \frac{n_1 + \sum_{j=1}^{K}(k_j - 4) + \sum_{j=1}^{g-1}(\ell_j - 3) + h}{2} \rfloor, & \text{if } y_h^g \in \mathcal{S}_0, \text{ for } h \text{ even}, \\ \lceil \frac{n_1 + \sum_{j=1}^{K}(k_j - 4) + \sum_{j=1}^{g-1}(\ell_j - 3) + h - 1}{2} \rceil, & \text{if } y_h^g \in \mathcal{S}_0, \text{ for } h \text{ odd}, \\ t, & \text{otherwise.} \end{cases}$$

For $g \in [p + 1, L]$, let

$$\phi(y_2^g y_3^g) = \begin{cases} n_1 + \sum_{j=1}^{K}(k_j - 4) + \sum_{j=1}^{p}(\ell_j - 4), & \text{if } y_2^g \in \mathcal{S}_0, \\ t, & \text{otherwise.} \end{cases}$$

For every edge $e \notin \mathcal{S}$, define $\phi(e) = t$.

Clearly, all pendant edges and all interior edges incident to vertices not belonging to $\mathcal{S}_0$ are labeled at most t. Moreover, as $|\mathcal{S}_0| = t - n_1$, the labels of interior edges incident to a vertex in $\mathcal{S}_0$ are not greater than

$$n_1 + \sum_{j=1}^{K}(k_j - 4) + \sum_{j=1}^{p}(\ell_j - 4) \leqslant n_1 + |\mathcal{S}_0| = t,$$

thus $\phi(e) \leqslant t$ for every edge $e \in E$.

Since T has at least $\min\{t - n_2^* - 1, n_1\}$ exterior vertices, it can be verified that for $1 \in [i, \Delta]$ and $j \in [1, n_i]$,

$$\sum_{z=1}^{i-1} n_z + j + 1 - t \leqslant \omega_\phi(v_{ij}^\phi) \leqslant it. \tag{3}$$

Let a be the smallest positive integer (if any) such that $\omega_\phi(v_{2a}^\phi) \geqslant n_1 + a + 1$, and b the smallest positive integer (if any) such that $\omega_\phi(v_{3b}^\phi) \geqslant n_1 + n_2 + b + 1$. Therefore,

$$\begin{aligned} &\omega_\phi(v_{2j}^\phi) \leqslant n_1 + j \quad \text{for } j \in [1, a - 1], \\ &\omega_\phi(v_{3j}^\phi) \leqslant n_1 + n_2 + j \quad \text{for } j \in [1, b - 1]. \end{aligned} \tag{4}$$

Next, let $A = \{v_{2a}^\phi, v_{2(a+1)}^\phi, \ldots, v_{2n_2}^\phi\}$ and $B = \{v_{3b}^\phi, v_{3(b+1)}^\phi, \ldots, v_{3n_3}^\phi\}$. Define M as a matching obtained from a maximal matching of T that includes all non-pendant vertices by removing each edge xy where $x, y \in V \backslash (A \cup B)$. Thus, M is a matching of T that covers all vertices in $A \cup B$. Suppose that

$$M = \{a_h a_h' : h \in [1, p_1]\} \cup \{a_i'' d_i : i \in [1, p_2]\} \cup \{b_j b_j' : j \in [1, q_1]\}$$

$$\cup \{b''_k d'_k : k \in [1, q_2]\} \cup \{a'''_l b'''_l : l \in [1, r]\},$$

where $p_1, p_2, q_1, q_2, r \geqslant 0$ such that the following properties hold:

(a) $\{a_h, a'_h : h \in [1, p_1]\} \cup \{a''_i : i \in [1, p_2]\} \cup \{a'''_l : l \in [1, r]\} = A$.
(b) $\{b_j, b'_j : j \in [1, q_1]\} \cup \{b''_k : k \in [1, q_2]\} \cup \{b'''_l : l \in [1, r]\} = B$.
(c) $(\{d_i : i \in [1, p_2]\} \cup \{d'_k : k \in [1, q_2]\}) \subseteq (V \backslash (A \cup B))$.
(d) $\psi(a_h a'_h) \geqslant \psi(a_{h+1} a'_{h+1})$ for each h, $\psi(a''_i d_i) \geqslant \psi(a''_{i+1} d_{i+1})$ for each i, $\psi(b_j b'_j) \geqslant \psi(b_{j+1} b'_{j+1})$ for each j, $\psi(b''_k d'_k) \geqslant \psi(b''_{k+1} d'_{k+1})$ for each k, and $\psi(a'''_l b'''_l) \geqslant \psi(a'''_{l+1} b'''_{l+1})$ for each l, where $\psi(uv) = \min\left\{\frac{\omega_\phi(u)}{\deg(u)+1}, \frac{\omega_\phi(v)}{\deg(v)+1}\right\}$.
(e) $\omega_\phi(a_h) \geqslant \omega_\phi(a'_h)$ for each h and $\omega_\phi(b_j) \geqslant \omega_\phi(b'_j)$ for each j.

Note that $n_2 - a + 1 = 2p_1 + p_2 + r$ and $n_3 - b + 1 = 2q_1 + q_2 + r$.

Theorem 3.1. *Let T be a tree of type B or C other than a path. If T contains at least $\min\{t - 2n_2^* - 1, n_1 - n_2^*\}$ exterior vertices of degree at least three, where n_2^* is the number of exterior vertices of degree two, and*

$$t = \begin{cases} t_2, & \text{if } T \text{ is a tree of type B}, \\ t_3, & \text{if } T \text{ is a tree of type C}, \end{cases}$$

then $\operatorname{tvs}(T) = t$.

Proof. Let T be a tree of types B or C that is not a path and has the property given in the statement. As we mentioned earlier, it remains to prove that $\operatorname{tvs}(T) \leqslant t$, where

$$t = \begin{cases} t_2, & \text{if } T \text{ is a tree of type B}, \\ t_3, & \text{if } T \text{ is a tree of type C}. \end{cases}$$

We construct a total labeling φ on vertices and edges of T as follows. Label edges of T in the following way.

$$\begin{aligned}
\varphi(a_h a'_h) &= \min\{n_1 + a + 2h - 2 - (\omega_\phi(a_h) - \phi(a_h a'_h)), \phi(a_h a'_h)\} \quad \text{for } h \in [1, p_1], \\
\varphi(a''_i d_i) &= \min\{n_1 + a + 2p_1 + r + i - 2 - (\omega_\phi(a''_i) - \phi(a''_i d_i)), \phi(a''_i d_i)\} \quad \text{for } i \in [1, p_2], \\
\varphi(b_j b'_j) &= \min\{n_1 + n_2 + b + 2j - 2 - (\omega_\phi(b_j) - \phi(b_j b'_j)), \phi(b_j b'_j)\} \quad \text{for } j \in [1, q_1], \\
\varphi(b''_k d'_k) &= \min\{n_1 + n_2 + b + 2q_1 + r + k - 2 - (\omega_\phi(b''_k) - \phi(b''_k d'_k)), \phi(b''_k d'_k)\} \quad \text{for } k \in [1, q_2], \\
\varphi(a'''_l b'''_l) &= \min\{n_1 + a + 2p_1 + l - 2 - (\omega_\phi(a'''_l) - \phi(a'''_l b'''_l)), \\
&\qquad n_1 + n_2 + b + 2q_1 + l - 2 - (\omega_\phi(b'''_l) - \phi(a'''_l b'''_l)), \phi(a'''_l b'''_l)\} \quad \text{for } l \in [1, r], \\
\varphi(e) &= \phi(e) \quad \text{for all the remaining edges } e \text{ of } T.
\end{aligned}$$

Obviously, for every edge $e \in E$,

$$\varphi(e) \leqslant \phi(e) \leqslant t. \tag{5}$$

Moreover, the temporary weights of vertices under the labeling φ are the following.

$$
\begin{aligned}
\omega_\varphi(a_h) &= \varphi(a_h a'_h) + (\omega_\phi(a_h) - \phi(a_h a'_h)) \\
&\leqslant n_1 + a + 2h - 2 - (\omega_\phi(a_h) - \phi(a_h a'_h)) + (\omega_\phi(a_h) - \phi(a_h a'_h)) \\
&= n_1 + a + 2h - 2 \quad \text{for } h \in [1, p_1], \\
\omega_\varphi(a'_h) &= \varphi(a_h a'_h) + (\omega_\phi(a'_h) - \phi(a_h a'_h)) \\
&\leqslant n_1 + a + 2h - 2 - (\omega_\phi(a_h) - \phi(a_h a'_h)) + (\omega_\phi(a'_h) - \phi(a_h a'_h)) \\
&= n_1 + a + 2h - 2 + \omega_\phi(a'_h) - \omega_\phi(a_h) \\
&\leqslant n_1 + a + 2h - 2 \quad \text{for } h \in [1, p_1], \\
\omega_\varphi(a''_i) &= \varphi(a''_i d_i) + (\omega_\phi(a''_i) - \phi(a''_i d_i)) \\
&\leqslant n_1 + a + 2p_1 + r + i - 2 - (\omega_\phi(a''_i) - \phi(a''_i d_i)) + (\omega_\phi(a''_i) - \phi(a''_i d_i)) \\
&= n_1 + a + 2p_1 + r + i - 2 \quad \text{for } i \in [1, p_2], \\
\omega_\varphi(d_i) &= \varphi(a''_i d_i) + (\omega_\phi(d_i) - \phi(a''_i d_i)) \\
&\leqslant n_1 + a + 2p_1 + r + i - 2 - (\omega_\phi(a''_i) - \phi(a''_i d_i)) + (\omega_\phi(d_i) - \phi(a''_i d_i)) \\
&= n_1 + a + 2p_1 + r + i - 2 + \omega_\phi(d_i) - \omega_\phi(a''_i) \quad \text{for } i \in [1, p_2], \\
\omega_\varphi(b_j) &= \varphi(b_j b'_j) + (\omega_\phi(b_j) - \phi(b_j b'_j)) \\
&\leqslant n_1 + n_2 + b + 2j - 2 - (\omega_\phi(b_j) - \phi(b_j b'_j)) + (\omega_\phi(b_j) - \phi(b_j b'_j)) \\
&= n_1 + n_2 + b + 2j - 2 \quad \text{for } j \in [1, q_1], \\
\omega_\varphi(b'_j) &= \varphi(b_j b'_j) + (\omega_\phi(b'_j) - \phi(b_j b'_j)) \\
&\leqslant n_1 + n_2 + b + 2j - 2 - (\omega_\phi(b_j) - \phi(b_j b'_j)) + (\omega_\phi(b'_j) - \phi(b_j b'_j)) \\
&= n_1 + n_2 + b + 2j - 2 + \omega_\phi(b'_j) - \omega_\phi(b_j) \\
&\leqslant n_1 + n_2 + b + 2j - 2 \quad \text{for } j \in [1, q_1], \\
\omega_\varphi(b''_k) &= \varphi(b''_k d'_k) + (\omega_\phi(b''_k) - \phi(b''_k d'_k)) \\
&\leqslant n_1 + n_2 + b + 2q_1 + r + k - 2 - (\omega_\phi(b''_k) - \phi(b''_k d'_k)) + (\omega_\phi(b''_k) - \phi(b''_k d'_k)) \\
&= n_1 + n_2 + b + 2q_1 + r + k - 2 \quad \text{for } k \in [1, q_2], \\
\omega_\varphi(d'_k) &= \varphi(b''_k d'_k) + (\omega_\phi(d'_k) - \phi(b''_k d'_k)) \\
&\leqslant n_1 + n_2 + b + 2q_1 + r + k - 2 - (\omega_\phi(b''_k) - \phi(b''_k d'_k)) + (\omega_\phi(d'_k) - \phi(b''_k d'_k)) \\
&= n_1 + n_2 + b + 2q_1 + r + k - 2 + \omega_\phi(d'_k) - \omega_\phi(b''_k) \quad \text{for } k \in [1, q_2], \\
\omega_\varphi(a'''_l) &= \varphi(a'''_l b'''_l) + (\omega_\phi(a'''_l) - \phi(a'''_l b'''_l)) \\
&\leqslant n_1 + a + 2p_1 + l - 2 - (\omega_\phi(a'''_l) - \phi(a'''_l b'''_l)) + (\omega_\phi(a'''_l) - \phi(a'''_l b'''_l)) \\
&= n_1 + a + 2p_1 + l - 2 \quad \text{for } l \in [1, r], \\
\omega_\varphi(b'''_l) &= \varphi(a'''_l b'''_l) + (\omega_\phi(b'''_l) - \phi(a'''_l b'''_l)) \\
&\leqslant n_1 + n_2 + b + 2q_1 + l - 2 - (\omega_\phi(b'''_l) - \phi(a'''_l b'''_l)) + (\omega_\phi(b'''_l) - \phi(a'''_l b'''_l)) \\
&= n_1 + n_2 + b + 2q_1 + l - 2 \quad \text{for } l \in [1, r], \\
\omega_\varphi(v) &= \omega_\phi(v) \quad \text{for all the remaining vertices } v \text{ of } T.
\end{aligned}
\tag{6}
$$

Thus from (2), (4) and (6) we can conclude that

$$\omega_\varphi(v_{ij}^\varphi) \leqslant \sum_{z=1}^{i-1} n_z + j \quad \text{for } i \in [1,3] \text{ and } j \in [1,n_i]. \tag{7}$$

Also, from (3) we have

$$\omega_\varphi(v_{ij}^\varphi) \geqslant \sum_{z=1}^{i-1} n_z + j + 1 - t \quad \text{for } i \in [1,\Delta] \text{ and } j \in [1,n_i]. \tag{8}$$

We next label vertices of T. Label vertices of degrees one, two, and three in the following way.

$$\varphi(v_{ij}^\varphi) = \sum_{z=1}^{i-1} n_z + j + 1 - \omega_\varphi(v_{ij}^\varphi) \quad \text{for } i \in [1,3] \text{ and } j \in [1,n_i]. \tag{9}$$

Thus from (7), (8) and (9),

$$1 \leqslant \varphi(v_{ij}^\varphi) \leqslant t \quad \text{for } i \in [1,3] \text{ and } j \in [1,n_i]. \tag{10}$$

Let $w_1, w_2, \ldots, w_N$, where $N = |V| - n_1 - n_2 - n_3$, be vertices of degree at least four in T, with $\omega_\varphi(w_i) \leqslant \omega_\varphi(w_{i+1})$ for each i. We define $\varphi(w_1) = \max\{1, n_1 + n_2 + n_3 + 2 - \omega_\varphi(w_1)\}$, which leads to $wt(w_1) = \varphi(w_1) + \omega_\varphi(w_1)$. For $i \in [2, N]$, we define recursively

$$\varphi(w_i) = \max\{1, wt(w_{i-1}) + 1 - \omega_\varphi(w_i)\}.$$

Under the labeling φ, the weights of vertices with degrees one, two, and three form the sets $\{2, 3, \ldots, n_1{+}1\}$, $\{n_1{+}2, n_1{+}3, \ldots, n_1{+}n_2{+}1\}$, and $\{n_1{+}n_2{+}2, n_1{+}n_2{+}3, \ldots, n_1{+}n_2{+}n_3{+}1\}$, respectively. The weights of other vertices are given by $n_1 + n_2 + n_3 + 2 \leqslant wt(w_1) < wt(w_2) < \cdots < wt(w_N)$, ensuring all weights are distinct. We show that labels in φ do not exceed t. From (5) and (10), it remains to prove that $\varphi(w_i) \leqslant t$ for $i \in [1, N]$. Using the tree property $n_1 = 2 + \sum_{j=3}^{\Delta}(j-2)n_j$, for $j \in [4, \Delta]$,

$$n_j = \frac{n_1 - n_3 - 2 - \sum_{i=4, i\neq j}^{\Delta}(i-2)n_i}{j-2} < t_1 \leqslant t.$$

This together with (8) implies that the maximum label contributing to the corresponding final weights is at most t, i.e., $\varphi(w_i) \leqslant t$ for $i \in [1, N]$. Hence $\mathrm{tvs}(T) = t$. □

4. Trees of type A

This section showcases the total vertex irregularity strength for all trees of type A with maximum degree three or more, as shown in Theorem 4.1. The idea of the proof of Theorem 4.1 resembles that of Theorem 3.1, but requires relabeling pendant edges. Figure 3 provides an example of a vertex irregular total labeling of a tree of type A.

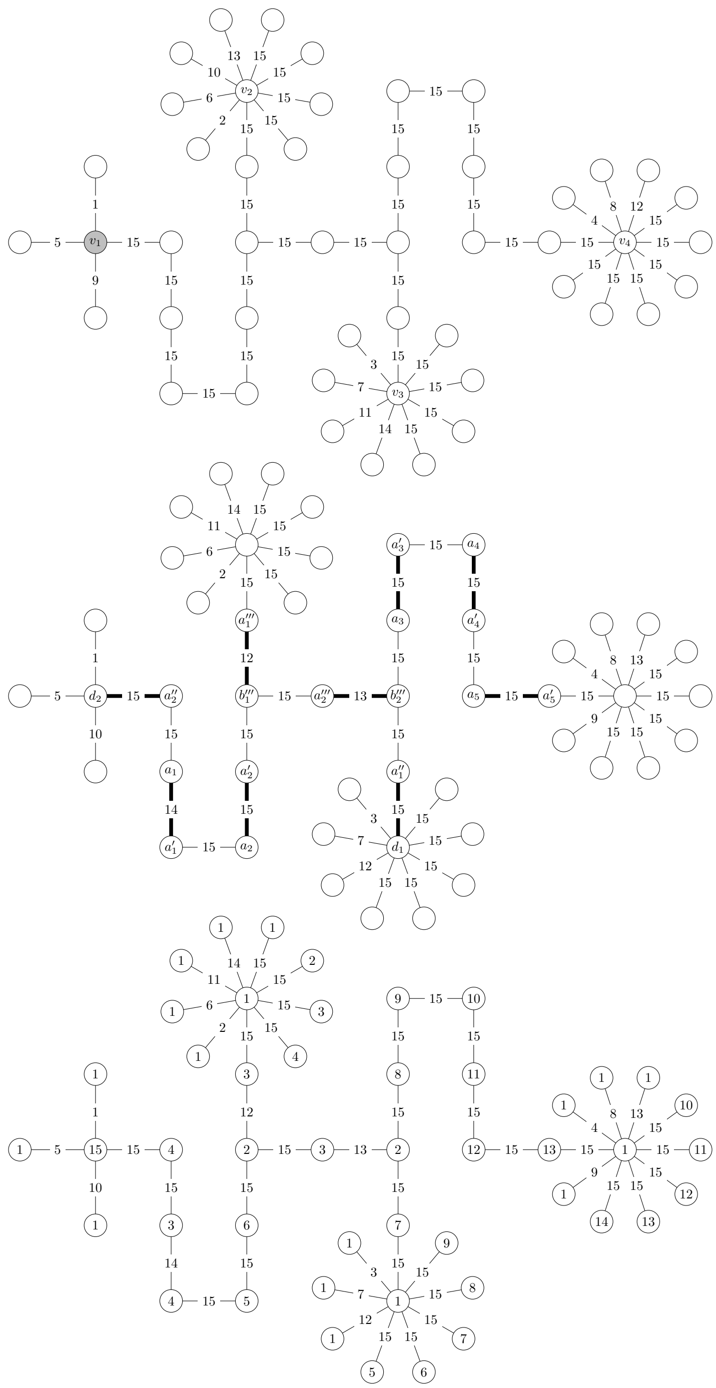

Figure 3. A labeling example for a tree of type A. Top, **Step 1**: Label pendant and interior edges. In this case, relabeling pendant edges is needed due to an exterior vertex (colored lightgray) with small temporary weight (v_{i_0} exists with $\rho = 1$). Middle, **Step 2**: Relabel pendant edges, construct a matching M (thick edges) covering vertices in $A \cup B$ (in this case, $a = 1$ and $b = 1$, so $|A| = 14$ and $|B| = 2$), and relabel edges in M. Bottom, **Step 3**: Greedily label vertices to ensure distinct final weights.

Let T be a tree of type A, distinct from a path. We map α from E_P to $\{1, 2, \ldots, t_1\}$ as follows.

$$\alpha(e_{ij}) = \min\{g, t_1\}, \quad \text{where } e_{ij} \text{ is the } g\text{th element of } E_P \text{ for some } i \text{ and } j.$$

Select an exterior vertex v_{i_0} (if exists) with degree at least three, where $i_0 \in \{1, 2, \ldots, \varepsilon - 1\}$ such that i_0 is minimal and

$$\rho = \sum_{z=1}^{\deg(v_{i_0})-1} n_z + 2 - t_1 - \sum_{j=1}^{|E_P(v_{i_0})|} \alpha(e_{i_0 j}) - (\deg(v_{i_0}) - |E_P(v_{i_0})|)t_1$$

is positive. Define $e_{i_0|E_P(v_{i_0})|}$ as the g_0th element of E_p. Note that v_{i_0} exists only if $\varepsilon < t_1 - n_2^* - 1$, and when it does not exist, set $\rho = 0$ and $g_0 = 1$. As $n_2 \leqslant t_1$, we observe that $g_0 + \rho \leqslant t_1$. Let ϕ be an edge labeling of T defined in the following way.

$$\begin{aligned} \phi(e_{ij}) &= \alpha(e_{ij}) = \min\{g, t_1\} = g \quad \text{for } g \in [1, g_0 - 1], \\ \phi(e_{ij}) &= \min\{g + \rho, t_1\} \quad \text{for } g \in [g_0, n_1 - \rho], \\ \phi(e_{ij}) &= g + g_0 + \rho - n_1 - 1 \quad \text{for } g \in [n_1 - \rho + 1, n_1], \\ \phi(e) &= t_1 \quad \text{for every edge } e \in E \backslash E_P. \end{aligned} \tag{11}$$

Thus every edge of T has label no larger than t_1 under ϕ.

We next show that for $i \in [1, \Delta]$ and $j \in [1, n_i]$,

$$\sum_{z=1}^{\deg(v_{ij}^{\phi})-1} n_z + j + 1 - t_1 \leqslant \omega_\phi(v_{ij}^{\phi}) \leqslant i t_1. \tag{12}$$

The upper bound arises as edges have labels up to t_1. Next, we prove the lower bound, considering only vertices of degree i with temporary weights under ϕ less than it_1. Thus, these must be exterior vertices, and according to ϕ given in (11), their number is k, where $k \leqslant t_1 - 1$. For $i \in [1, 2]$ and $j \in [1, n_i]$, it is clear that

$$\omega_\phi(v_{ij}^{\phi}) \geqslant \sum_{z=1}^{\deg(v_{ij}^{\phi})-1} n_z + j + 1 - t_1.$$

So, it remains to prove that for $i \in [3, \Delta]$ and $j \in [1, n_i]$,

$$\omega_\phi(v_{ij}^{\phi}) \geqslant \sum_{z=1}^{\deg(v_{ij}^{\phi})-1} n_z + j + 1 - t_1. \tag{13}$$

If the exterior vertex v_{i_0} does not exist, then the lower bound (13) holds for $i \in [3, \Delta]$ and $j \in [1, n_i]$. If v_{i_0} exists with degree d where $d \geqslant 3$, then the lower bound (13) holds for $i \in [3, \Delta] \backslash \{d\}$ and $j \in [1, n_i]$. With i_0 minimal, we have $v_{d1}^{\phi} = v_{i_0}$, and so

$$\omega_\phi(v_{d1}^{\phi}) = \omega_\phi(v_{i_0})$$

$$
\begin{aligned}
&= \sum_{j=1}^{|E_P(v_{i_0})|-1} \phi(e_{i_0j}) + \phi(e_{i_0|E_P(v_{i_0})|}) + (\deg(v_{i_0}) - |E_P(v_{i_0})|)t_1 \\
&= \sum_{j=1}^{|E_P(v_{i_0})|-1} \alpha(e_{i_0j}) + g_0 + \rho + (\deg(v_{i_0}) - |E_P(v_{i_0})|)t_1 \\
&= \sum_{j=1}^{|E_P(v_{i_0})|-1} \alpha(e_{i_0j}) + g_0 + (\deg(v_{i_0}) - |E_P(v_{i_0})|)t_1 + \sum_{z=1}^{\deg(v_{i_0})-1} n_z + 2 - t_1 \\
&\quad - \sum_{j=1}^{|E_P(v_{i_0})|-1} \alpha(e_{i_0j}) - \alpha(e_{i_0|E_P(v_{i_0})|}) - (\deg(v_{i_0}) - |E_P(v_{i_0})|)t_1 \\
&= \sum_{z=1}^{\deg(v_{i_0})-1} n_z + 2 - t_1 \\
&= \sum_{z=1}^{\deg(v_{d1}^{\phi})-1} n_z + 2 - t_1.
\end{aligned}
$$

Moreover, based on ϕ, for $1 \leqslant j < j' \leqslant k$, at least one pair $e_j, e_{j'}$ of pendant edges incident respectively to v_{ij}^{ϕ} and $v_{ij'}^{\phi}$ gets distinct labels. This confirms the lower bound (13) for $i = d$ and $j \in [1, n_d]$, thus validating the inequality (12).

Theorem 4.1. *Let T be a tree of type A other than a path. Then* $\mathrm{tvs}(T) = t_1$.

Proof. Let T be a non-path tree of type A. From Lemmas 1.1 and 2.1, we have $\mathrm{tvs}(T) \geqslant t_1$.

Let φ be a total labeling on vertices and edges of T as defined in Theorem 3.1, with ϕ now given in (11) and $t = t_1$. We conclude that φ is a vertex irregular total t_1-labeling of T, hence $\mathrm{tvs}(T) = t_1$. □

5. Concluding remarks

We showed that the total vertex irregularity strength for all type A trees with maximum degree at least three is t_1. Additionally, we found that if T is a type B or C tree with maximum degree at least three and at least $\min\{t - 2n_2^* - 1, n_1 - n_2^*\}$ exterior vertices of degree three or higher, where n_2^* counts exterior vertices of degree two, and

$$
t = \begin{cases} t_2, & \text{if } T \text{ is a type } B \text{ tree,} \\ t_3, & \text{if } T \text{ is a type } C \text{ tree,} \end{cases}
$$

then $\mathrm{tvs}(T) = t$. We also believe that the following conjecture is valid.

Conjecture 1. *Let T be a tree with maximum degree at least three containing at most $\min\{t - 2n_2^* - 2, n_1 - n_2^* - 1\}$ exterior vertices of degree at least three, where n_2^* is the number of exterior* *vertices of degree two, and*

$$t = \begin{cases} t_2, & \text{if } T \text{ is a tree of type } B, \\ t_3, & \text{if } T \text{ is a tree of type } C \text{ and } n_2 \neq 3n_3 - n_1 - 1. \end{cases}$$

Then $\operatorname{tvs}(T) = t$*.*

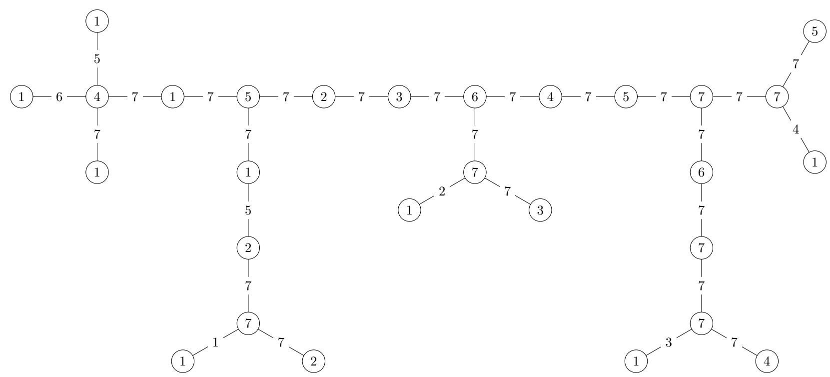

Figure 4. A tree $T \in \mathcal{T}$ with $\operatorname{tvs}(T) = t_3$.

Let $\mathcal{T}$ be the set of all type C trees with $n_2 = 3n_3 - n_1 - 1$, having at most $\min\{t_3 - 2n_2^* - 2, n_1 - n_2^* - 1\}$ exterior vertices of degree three or more. As shown in [14], an infinite number of trees T in $\mathcal{T}$ exhibits $\operatorname{tvs}(T) = t_3 + 1$. On the other hand, a tree T in Figure 4 is a part of $\mathcal{T}$ and $\operatorname{tvs}(T) = t_3$. These facts lead to the following problem.

Problem 1. *Let T be a tree of type C containing at most $\min\{t_3 - 2n_2^* - 2, n_1 - n_2^* - 1\}$ exterior vertices of degree at least three, where n_2^* is the number of exterior vertices of degree two. Find the necessary and sufficient conditions for which $\operatorname{tvs}(T) = t_3$.*

For a graph $G = (V, E)$ of order n, the *modular total vertex irregularity strength* of G, denoted by $\operatorname{mtvs}(G)$, is the minimum k for which there exists a mapping $\varphi : V \cup E \to \{1, 2, \ldots, k\}$ such that the *modular total vertex weights*, $wt(v) = \varphi(v) + \sum_{uv \in E} \varphi(uv) \pmod{n}$, are all distinct. This concept was established by Ali et al. [1] as a modification of the vertex irregular total labeling. The subsequent property was proved in [1].

Theorem 5.1 ([1])**.** *Let G be a graph. If vertex weights under a vertex irregular total $\operatorname{tvs}(G)$-labeling of G constitute a set of consecutive integers, then $\operatorname{mtvs}(G) = \operatorname{tvs}(G)$.*

Notice that in the proofs of Theorems 3.1 and 4.1, the weights of vertices with degrees one, two, and three form the set of consecutive integers $\{2, 3, \ldots, n_1 + n_2 + n_3 + 1\}$. This, combined with Theorem 5.1, yields the following corollary.

Corollary 5.1. *Let T be a tree with maximum degree three. If T is of type A then $\operatorname{mtvs}(T) = t_1$. Moreover, if T is of types B or C containing at least $\min\{t - 2n_2^* - 1, n_1 - n_2^*\}$ exterior vertices of degree three, where n_2^* is the number of exterior vertices of degree two, and*

$$t = \begin{cases} t_2, & \text{if } T \text{ is a tree of type } B, \\ t_3, & \text{if } T \text{ is a tree of type } C, \end{cases}$$

then $\mathrm{mtvs}(T) = t$.

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