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Local edge antimagic chromatic number of comb products involving path graph

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Abstract

Let $G = (V, E)$ be a graph with n vertices and no isolated vertices. A local edge antimagic labeling of G is a bijection $f : V(G) \to \{1, 2, \ldots, n\}$ such that the weights of any two adjacent edges in G are distinct, where the weight of an edge in G is defined as the sum of the labels of its end vertices. Such a labeling induces a proper edge coloring of G, with edge weights serving as the colors. The local edge antimagic chromatic number of G, denoted $\chi'_{lea}(G)$, is the minimum number of colors used across all such labelings. In this paper, we investigate the local edge antimagic chromatic number of comb product graphs, focusing on the case where a path graph is combined with copies of other graphs—specifically paths, cycles, and ladders. The comb product of G and H, with respect to an assigned vertex, is constructed by taking one copy of G and $|V(G)|$ copies of H and identifying the assigned vertex from the i-th copy of H to the i-th vertex of G.

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1. Introduction

Antimagic labeling was introduced by Hartsfield and Ringel in 1990 [11]. This concept was later combined with graph coloring, leading to the introduction of a new term called local antimagic vertex coloring of a graph by Arumugam et al. in 2017 [6]. In the same year, Agustin et al.

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proposed a variation of local antimagic coloring, known as local edge antimagic coloring of a graph [2].

Let $G = (V(G), E(G))$ be a graph without isolated vertex. A bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ is a local edge antimagic labeling of G if for any two adjacent edges $uv, vx \in E(G)$, $w(uv) \neq w(vx)$, where the weight of an edge uv is $w(uv) = f(u) + f(v)$. A local edge antimagic labeling induces a proper edge coloring of G if each edge uv is given the color $w(uv)$. The minimum number of colors taken over all colorings induced by the local edge antimagic labeling of G is called the local edge antimagic chromatic number of G, denoted by $\chi'_{lea}(G)$. It is obvious that for any graph G, $\chi'_{lea}(G) \geq \chi'(G)$, where $\chi'(G)$ is the chromatic index of G. Let $\Delta(G)$ be the maximum degree of G. It is known that Vizing's Theorem gives the bounds for the chromatic index of a graph as cited in Theorem 1.1. From the theorem, we obtain $\chi'_{lea}(G) \geq \chi'(G) \geq \Delta(G)$.

Theorem 1.1. *[7] Every graph G satisfies $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

Agustin et al. have determined the local edge antimagic chromatic number of several special graphs, such as the path, cycle, friendship, ladder, and many more [2]. Moreover, they also found the local edge antimagic chromatic number of graphs resulting from the corona product of any graph and copies of K_1. In another study, Agustin et al. explored the local edge antimagic chromatic number of comb product of graphs [3]. Later on, Hadiputra and Maryati in [10] showed that the lower bound of the local edge antimagic chromatic number of the comb product graph given in [3] is inaccurate.

In 2019, Aisyah et al. studied the local edge antimagic chromatic number of the corona product of paths and cycles, specifically path corona cycle, cycle corona path, path corona path, and cycle corona cycle [4]. In 2022, Rajkumar and Nalliah studied the local edge antimagic chromatic number of wheel related graphs [12]. Recently, Hadiputra and Maryati proved that every graphs admit a local edge antimagic labeling [10]. They also characterized graphs G with $\chi'_{lea}(G) = 1$ and determined the local edge antimagic chromatic number of comb product of $K_{1,m}$ and a graph G with $\chi'_{lea}(G) = \Delta(G)$.

The comb product of graphs was introduced by Accardi, Ghorbal, and Obata [1]. Let G and H be connected graphs and o be a vertex of H. The comb product of G and H, denoted by $G \rhd_o H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and identifying vertex o of the i-th copy of H to the i-th vertex of G, where $i = 1, 2, \ldots, |V(G)|$. For copies of graph, we give the definition of comb product as follows. Let F and G be connected graphs, $H \cong tF$ for $t \geq 1$, and o be a vertex of F. The comb product of G and H is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and identifying vertex o of each F of i-th copy of H to the i-th vertex of G. Comb product graphs has been extensively studied in various aspects, including total coloring [14], rainbow coloring [9], metric dimension [5, 13], and dominating number [8].

In this paper, we study the local edge antimagic chromatic number of comb product of path and copies of some graphs—specifically paths, cycles, and ladder graphs. In addition, we also give the local edge antimagic chromatic number of some copies of graph.

2. Results

In this section, we determine the local edge antimagic chromatic number of the path comb copies of path, path comb copies of even cycle, and path comb copies of ladder. Note that if the edges of the path are deleted from the comb product of a path and tH, where H is any connected graph, we will get the copies of the vertex identification of tH, where the maximum degree of the resulting graph will be reduced by 2. Using this observation, we give the local edge antimagic chromatic number of some copies of graph as a corollary.

Theorem 2.1. *Let P_n be a path of order n and o be a vertex of P_n. For $m, n \geq 3$ and $t \geq 1$, if o is a leaf, then the local edge antimagic chromatic number of $P_m \rhd_o tP_n$ is $\chi'_{lea}(P_m \rhd_o tP_n) = 2 + t$. Otherwise, $\chi'_{lea}(P_m \rhd_o tP_n) = 2 + 2t$.*

Proof. Let $P_n = v_1 v_2 \ldots v_n$ and o be a vertex of P_n.

Case 1. o is a leaf.

W.l.o.g., let $o = v_n$. Let $P_m \rhd_o tP_n$ be a graph with $V(P_m \rhd_o tP_n) = \{x_{i,j,k} : 1 \leq i \leq n-1, 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{0,0,k} : 1 \leq k \leq m\}$ and $E(P_m \rhd_o tP_n) = \{x_{0,0,k}x_{1,j,k} : 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{0,0,k}x_{0,0,k+1} : 1 \leq k \leq m-1\} \cup \{x_{i,j,k}x_{i+1,j,k} : 1 \leq i \leq n-2, 1 \leq j \leq t, 1 \leq k \leq m\}$. Hence, $|V(P_m \rhd_o tP_n)| = tm(n-1) + m$ and $|E(P_m \rhd_o tP_n)| = tm(n-1) + m - 1$. Figure 1 provides an illustration of the graph.

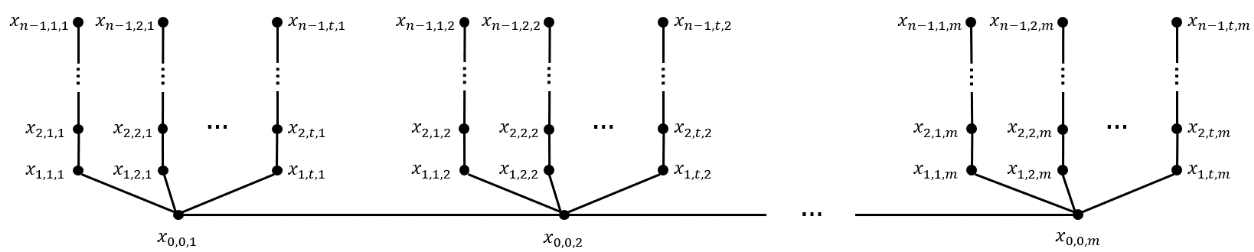

Figure 1. Graph $P_m \rhd_o tP_n$, where o is a leaf of P_n, with the vertex name.

Subcase 1.1. n is odd.

For $m, n \geq 3$ and $t \geq 1$, define $f : V(P_m \rhd_o tP_n) \to \{1, \ldots, tm(n-1) + m\}$ as follows.
For $1 \leq k \leq m$,

$$f(x_{0,0,k}) = \begin{cases} tm(\frac{n-1}{2}) + \frac{k+1}{2}, & \text{if } k \text{ is odd,} \\ tm(\frac{n-1}{2}) + m - \frac{k-2}{2}, & \text{if } k \text{ is even.} \end{cases}$$

For $1 \leq i \leq n-1$, $1 \leq j \leq t$, and $1 \leq k \leq m$,

$$f(x_{i,j,k}) = \begin{cases} tm(\frac{n+i}{2}) - m(j-2) - \frac{k-1}{2}, & \text{if } i \text{ is odd, } k \text{ is odd,} \\ tm(\frac{n+i}{2}) - m(j-1) + \frac{k}{2}, & \text{if } i \text{ is odd, } k \text{ is even,} \\ tm(\frac{n-i-1}{2}) + m(j-1) + \frac{k+1}{2}, & \text{if } i \text{ is even, } k \text{ is odd,} \\ tm(\frac{n-i-1}{2}) + mj - \frac{k-2}{2}, & \text{if } i \text{ is even, } k \text{ is even.} \end{cases}$$

From the labeling f, we obtain the weights of the edges in path P_m for $1 \leq k \leq m-1$ are

$$w(x_{0,0,k}x_{0,0,k+1}) = \begin{cases} tm(n-1) + m + 1, & \text{if } k \text{ is odd,} \\ tm(n-1) + m + 2, & \text{if } k \text{ is even,} \end{cases}$$

the weights of the edges connecting P_m and tP_n for $1 \leq j \leq t$ and $1 \leq k \leq m$ are

$$w(x_{0,0,k}x_{1,j,k}) = tmn - m(j-2) + 1,$$

and the weights of the remaining edges in tP_n for $1 \leq i \leq n-2, 1 \leq j \leq t$, and $1 \leq k \leq m$ are

$$w(x_{i,j,k}x_{i+1,j,k}) = \begin{cases} tm(n-1) + m + 1, & \text{if } i \text{ is odd,} \\ tmn + m + 1, & \text{if } i \text{ is even.} \end{cases}$$

Note that if i is odd, $w(x_{i,j,k}x_{i+1,j,k})$ for $1 \leq k \leq m$ equals $w(x_{0,0,k}x_{0,0,k+1})$ when k is odd. If i is even, then $w(x_{i,j,k}x_{i+1,j,k}) = w(x_{0,0,k}x_{1,1,k})$. Thus, the weights of the edges in tP_n are the same as the weights of some of the edges in P_m or the edges connecting P_m and tP_n, depending on the parity of i.

Since $1 \leq j \leq t$ and $w(x_{0,0,k}x_{1,j,k})$ depends on j, there are t different weights of the edges connecting P_m and tP_n. These weights are different from the weights of the edges in P_m. Consequently, there are $2+t$ distinct edge weights. Therefore, we can conclude that $\chi'_{lea}(P_m \rhd_o tP_n) \leq 2+t$. Since $\Delta(P_m \rhd_o tP_n) = 2+t$, we obtain $\chi'_{lea}(P_m \rhd_o tP_n) \geq 2+t$. In conclusion, $\chi'_{lea}(P_m \rhd_o tP_n) = 2+t$ if o is a leaf of P_n and n is odd.

Subcase 1.2. n is even.
For $m, n \geq 3$ and $t \geq 1$, define $f : V(P_m \rhd_o tP_n) \to \{1, \ldots, tm(n-1)+m\}$ as follows.
For $1 \leq k \leq m$,

$$f(x_{0,0,k}) = \begin{cases} \frac{tmn}{2} + m - \frac{k-1}{2}, & \text{if } k \text{ is odd,} \\ \frac{tmn+k}{2}, & \text{if } k \text{ is even.} \end{cases}$$

For $1 \leq i \leq n-1$, $1 \leq j \leq t$, and $1 \leq k \leq m$,

$$f(x_{i,j,k}) = \begin{cases} tm(\frac{n-i-1}{2}) + m(j-1) + \frac{k+1}{2}, & \text{if } i \text{ is odd}, k \text{ is odd,} \\ tm(\frac{n-i-1}{2}) + mj - \frac{k-2}{2}, & \text{if } i \text{ is odd}, k \text{ is even,} \\ tm(\frac{n+i}{2}) - m(j-2) - \frac{k-1}{2}, & \text{if } i \text{ is even}, k \text{ is odd,} \\ tm(\frac{n+i}{2}) - m(j-1) + \frac{k}{2}, & \text{if } i \text{ is even}, k \text{ is even.} \end{cases}$$

From the labeling f, we obtain the weights of the edges in path P_m for $1 \leq k \leq m-1$ are

$$w(x_{0,0,k}x_{0,0,k+1}) = \begin{cases} tmn + m + 1, & \text{if } k \text{ is odd,} \\ tmn + m, & \text{if } k \text{ is even,} \end{cases}$$

the weights of the edges connecting P_m and tP_n for $1 \leq j \leq t$ and $1 \leq k \leq m$ are

$$w(x_{0,0,k}x_{1,j,k}) = tm(n-1) + mj + 1,$$

and the weights of the remaining edges in tP_n for $1 \leq i \leq n-2, 1 \leq j \leq t$, and $1 \leq k \leq m$ are

$$w(x_{i,j,k}x_{i+1,j,k}) = \begin{cases} tmn + m + 1, & \text{if } i \text{ is odd,} \\ tm(n-1) + m + 1, & \text{if } i \text{ is even.} \end{cases}$$

Note that if i is odd, $w(x_{i,j,k}x_{i+1,j,k})$ for $1 \leq k \leq m$ equals $w(x_{0,0,k}x_{0,0,k+1})$ when k is odd. If i is even, then $w(x_{i,j,k}x_{i+1,j,k}) = w(x_{0,0,k}x_{1,1,k})$. Thus, the weights of the edges in tP_n are the same as the weights of some of the edges in P_m or the edges connecting P_m and tP_n, depending on the parity of i.

Since $1 \leq j \leq t$ and $w(x_{0,0,k}x_{1,j,k})$ depends on j, there are t different weights of the edges connecting P_m and tP_n. These weights are different from the weights of the edges in P_m. Consequently, there are $2 + t$ distinct edge weights. Therefore, we can conclude that $\chi'_{lea}(P_m \rhd_o tP_n) \leq 2 + t$. Since $\Delta(P_m \rhd_o tP_n) = 2 + t$, we obtain $\chi'_{lea}(P_m \rhd_o tP_n) \geq 2 + t$. In conclusion, $\chi'_{lea}(P_m \rhd_o tP_n) = 2 + t$ if o is a leaf of P_n and n is even.

From Subcases 1.1 and 1.2, we can conclude that $\chi'_{lea}(P_m \rhd_o tP_n) = 2 + t$ if o is a leaf of P_n.

Case 2. o is not a leaf.

Let $o = v_q$, where $q \notin \{1, n\}$, and $P_m \rhd_o tP_n$ be a graph with $V(P_m \rhd_o tP_n) = \{x_{i,j,k} : 1 \leq i \leq n-1, 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{0,0,k} : 1 \leq k \leq m\}$ and $E(P_m \rhd_o tP_n) = \{x_{0,0,k}x_{1,j,k} : 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{0,0,k}x_{q,j,k} : 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{0,0,k}x_{0,0,k+1} : 1 \leq k \leq m-1\} \cup \{x_{i,j,k}x_{i+1,j,k} : 1 \leq i \leq q-2, 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{i,j,k}x_{i+1,j,k} : q \leq i \leq n-2, 1 \leq j \leq t, 1 \leq k \leq m\}$. Hence, $|V(P_m \rhd_o tP_n)| = tm(n-1) + m$ and $|E(P_m \rhd_o tP_n)| = tm(n-1) + m - 1$. Figure 2 provides an illustration of the graph.

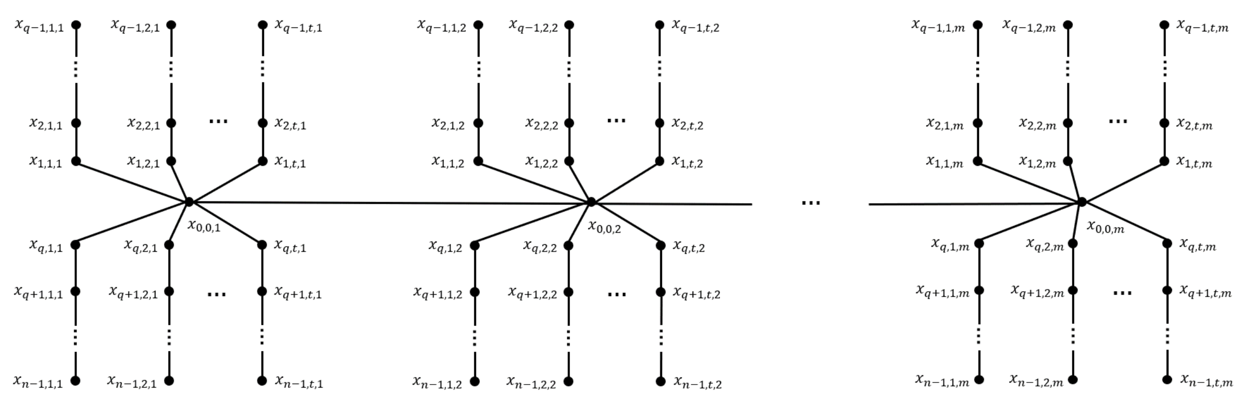

Figure 2. Graph $P_m \rhd_o tP_n$, where o is not a leaf of P_n, with the vertex name.

Subcase 2.1. q is odd.

For $m, n \geq 3$ and $t \geq 1$, define $f : V(P_m \rhd_o tP_n) \to \{1, \ldots, tm(n-1) + m\}$ as follows.

For $1 \leq k \leq m$,

$$f(x_{0,0,k}) = \begin{cases} tm(\frac{q-1}{2}) + \frac{k+1}{2}, & \text{if } k \text{ is odd,} \\ tm(\frac{q-1}{2}) + m - \frac{k-2}{2}, & \text{if } k \text{ is even.} \end{cases}$$

For $1 \le i \le n-1$, $1 \le j \le t$, and $1 \le k \le m$

$$f(x_{i,j,k}) = \begin{cases} tm(n - \frac{q-i}{2}) - m(j-2) - \frac{k-1}{2}, & \text{if } i \le q-1, i \text{ is odd}, k \text{ is odd},\\ tm(n - \frac{q-i}{2}) - m(j-1) + \frac{k}{2}, & \text{if } i \le q-1, i \text{ is odd}, k \text{ is even},\\ tm(\frac{q-i-1}{2}) + m(j-1) + \frac{k+1}{2}, & \text{if } i \le q-1, i \text{ is even}, k \text{ is odd},\\ tm(\frac{q-i-1}{2}) + mj - \frac{k-2}{2}, & \text{if } i \le q-1, i \text{ is even}, k \text{ is even},\\ tm(n - \frac{i+1}{2}) - m(j-2) - \frac{k-1}{2}, & \text{if } i \ge q, i \text{ is odd}, k \text{ is odd},\\ tm(n - \frac{i+1}{2}) - m(j-1) + \frac{k}{2}, & \text{if } i \ge q, i \text{ is odd}, k \text{ is even},\\ tm(\frac{i-2}{2}) + mj + \frac{k+1}{2}, & \text{if } i \ge q, i \text{ is even}, k \text{ is odd},\\ tm(\frac{i-2}{2}) + m(j+1) - \frac{k-2}{2}, & \text{if } i \ge q, i \text{ is even}, k \text{ is even}. \end{cases}$$

From the labeling f, we obtain the the weights of the edges in path P_m for $1 \le k \le m-1$ are

$$w(x_{0,0,k}x_{0,0,k+1}) = \begin{cases} tm(q-1) + m + 1, & \text{if } k \text{ is odd},\\ tm(q-1) + m + 2, & \text{if } k \text{ is even}, \end{cases}$$

the weights of the edges connecting P_m and tP_n for $1 \le j \le t$ and $1 \le k \le m$ are

$$w(x_{0,0,k}x_{1,j,k}) = tmn - m(j-2) + 1,$$

$$w(x_{0,0,k}x_{q,j,k}) = tm(n-1) - m(j-2) + 1,$$

and the weights of the remaining edges in tP_n for $1 \le j \le t$ and $1 \le k \le m$ are as follows.
If $1 \le i \le q-2$,

$$w(x_{i,j,k}x_{i+1,j,k}) = \begin{cases} tm(n-1) + m + 1, & \text{if } i \text{ is odd},\\ tmn + m + 1, & \text{if } i \text{ is even}, \end{cases}$$

and if $q \le i \le n-2$,

$$w(x_{i,j,k}x_{i+1,j,k}) = \begin{cases} tm(n-1) + 2m + 1, & \text{if } i \text{ is odd},\\ tm(n-2) + 2m + 1, & \text{if } i \text{ is even}. \end{cases}$$

Note that for $1 \le i \le q-2$, $w(x_{i,j,k}x_{i+1,j,k})$ equals $w(x_{0,0,k}x_{q,1,k})$ if i is odd and equals $w(x_{0,0,k}x_{1,1,k})$ if i is even. Meanwhile, for $q \le i \le n-2$, $w(x_{i,j,k}x_{i+1,j,k})$ equals $w(x_{0,0,k}x_{1,t,k})$ if i is odd and equals $w(x_{0,0,k}x_{q,t,k})$ if i is even. Thus, the weights of the edges in tP_n are the same as some of the weights of the edges connecting P_m and tP_n.

Since $1 \le j \le t$ and both $w(x_{0,0,k}x_{1,j,k})$ and $w(x_{0,0,k}x_{q,j,k})$ depend on j, there are $2t$ different weights of the edges connecting P_m and tP_n. These weights are different from the weights of the edges in P_m. Consequently, there are $2+2t$ distinct edge weights. Therefore, we can conclude that $\chi'_{lea}(P_m \triangleright_o tP_n) \le 2+2t$. Since $\Delta(P_m \triangleright_o tP_n) = 2+2t$, we obtain $\chi'_{lea}(P_m \triangleright_o tP_n) \ge 2+2t$. In conclusion, $\chi'_{lea}(P_m \triangleright_o tP_n) = 2+2t$ if $o = v_q$ is not a leaf of P_n and q is odd.

Subcase 2.2. q is even.
For $m, n \geq 3$ and $t \geq 1$, define $f : V(P_m \rhd_o tP_n) \rightarrow \{1, \ldots, tm(n-1)+m\}$ as follows.
For $1 \leq k \leq m$,

$$f(x_{0,0,k}) = \begin{cases} tm(n-\frac{q}{2}) + m - \frac{k-1}{2}, & \text{if } k \text{ is odd,} \\ tm(n-\frac{q}{2}) + \frac{k}{2}, & \text{if } k \text{ is even.} \end{cases}$$

For $1 \leq i \leq n-1$, $1 \leq j \leq t$, and $1 \leq k \leq m$

$$f(x_{i,j,k}) = \begin{cases} tm(\frac{q-i-1}{2}) + m(j-1) + \frac{k+1}{2}, & \text{if } i \leq q-1, i \text{ is odd}, k \text{ is odd,} \\ tm(\frac{q-i-1}{2}) + mj - \frac{k-2}{2}, & \text{if } i \leq q-1, i \text{ is odd}, k \text{ is even} \\ tm(n-\frac{q-i}{2}) - m(j-2) - \frac{k-1}{2}, & \text{if } i \leq q-1, i \text{ is even}, k \text{ is odd,} \\ tm(n-\frac{q-i}{2}) - m(j-1) + \frac{k}{2}, & \text{if } i \leq q-1, i \text{ is even}, k \text{ is even,} \\ tm(n-\frac{i-1}{2}) - m(j-1) - \frac{k-1}{2}, & \text{if } i \geq q, i \text{ is odd}, k \text{ is odd,} \\ tm(n-\frac{i-1}{2}) - mj + \frac{k}{2}, & \text{if } i \geq q, i \text{ is odd}, k \text{ is even,} \\ \frac{tmi}{2} + m(j-1) + \frac{k+1}{2}, & \text{if } i \geq q, i \text{ is even}, k \text{ is odd,} \\ \frac{tmi}{2} + mj - \frac{k-2}{2}, & \text{if } i \geq q, i \text{ is even}, k \text{ is even.} \end{cases}$$

From the labeling f, we obtain the the weights of the edges in path P_m for $1 \leq k \leq m-1$ are

$$w(x_{0,0,k}x_{0,0,k+1}) = \begin{cases} tm(2n-q) + m + 1, & \text{if } k \text{ is odd,} \\ tm(2n-q) + m, & \text{if } k \text{ is even,} \end{cases}$$

the weights of the edges connecting P_m and tP_n for $1 \leq j \leq t$ and $1 \leq k \leq m$ are

$$w(x_{0,0,k}x_{1,j,k}) = tm(n-1) + mj + 1,$$

$$w(x_{0,0,k}x_{q,j,k}) = tmn + mj + 1,$$

and the weights of the remaining edges in tP_n for $1 \leq j \leq t$ and $1 \leq k \leq m$ are as follows.
If $1 \leq i \leq q-2$,

$$w(x_{i,j,k}x_{i+1,j,k}) = \begin{cases} tmn + m + 1, & \text{if } i \text{ is odd,} \\ tm(n-1) + m + 1, & \text{if } i \text{ is even,} \end{cases}$$

and if $q \leq i \leq n-2$,

$$w(x_{i,j,k}x_{i+1,j,k}) = \begin{cases} tm(n+1) + 1, & \text{if } i \text{ is odd,} \\ tmn + 1, & \text{if } i \text{ is even.} \end{cases}$$

Note that for $1 \leq i \leq q-2$, $w(x_{i,j,k}x_{i+1,j,k})$ equals $w(x_{0,0,k}x_{q,1,k})$ if i is odd and equals $w(x_{0,0,k}x_{1,1,k})$ if i is even. Meanwhile, for $q \leq i \leq n-2$, $w(x_{i,j,k}x_{i+1,j,k})$ equals $w(x_{0,0,k}x_{q,t,k})$ if i is odd and equals $w(x_{0,0,k}x_{1,t,k})$ if i is even. Thus, the weights of the edges in tP_n are the same as some of the weights of the edges connecting P_m and tP_n.

Since $1 \leq j \leq t$ and both $w(x_{0,0,k}x_{1,j,k})$ and $w(x_{0,0,k}x_{q,j,k})$ depend on j, there are $2t$ different weights of the edges connecting P_m and tP_n. These weights are different from the weights of the edges in P_m. Consequently, there are $2+2t$ distinct edge weights. Therefore, we can conclude that $\chi'_{lea}(P_m \rhd_o tP_n) \leq 2+2t$. Since $\Delta(P_m \rhd_o tP_n) = 2+2t$, we obtain $\chi'_{lea}(P_m \rhd_o tP_n) \geq 2+2t$. In conclusion, $\chi'_{lea}(P_m \rhd_o tP_n) = 2+2t$ if $o = v_q$ is not a leaf of P_n and q is even.

Subcases 2.1 and 2.2 proved that $\chi'_{lea}(P_m \rhd_o tP_n) = 2+2t$ if o is not a leaf of P_n. $\square$

Figures 3 and 4 illustrate examples of local edge antimagic labeling of $P_3 \rhd_o 3P_7$ for o is a leaf of P_7 with $\chi'_{lea}(P_3 \rhd_o 3P_7) = 5$, and o is not a leaf of P_7 with $\chi'_{lea}(P_3 \rhd_o 3P_7) = 8$, respectively.

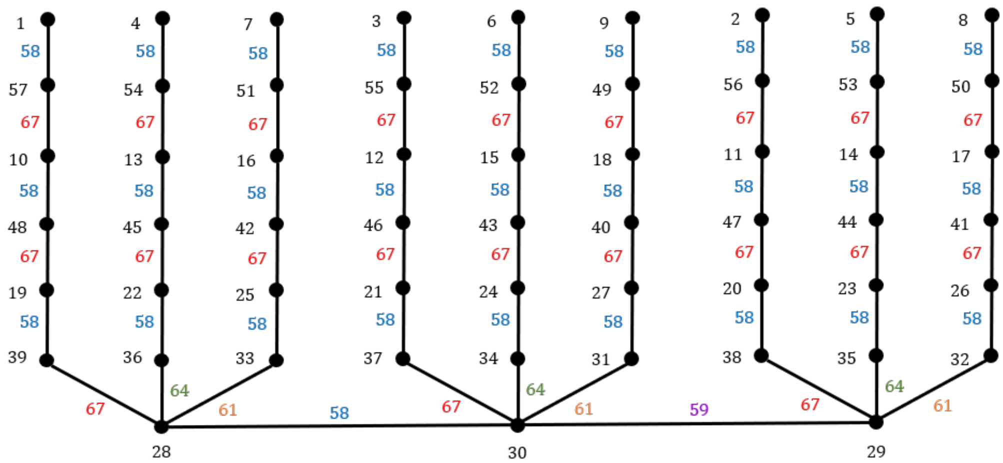

Figure 3. Local edge antimagic labeling of $P_3 \rhd_o 3P_7$, where o is a leaf of P_7, and $\chi'_{lea}(P_3 \rhd_o 3P_7) = 5$.

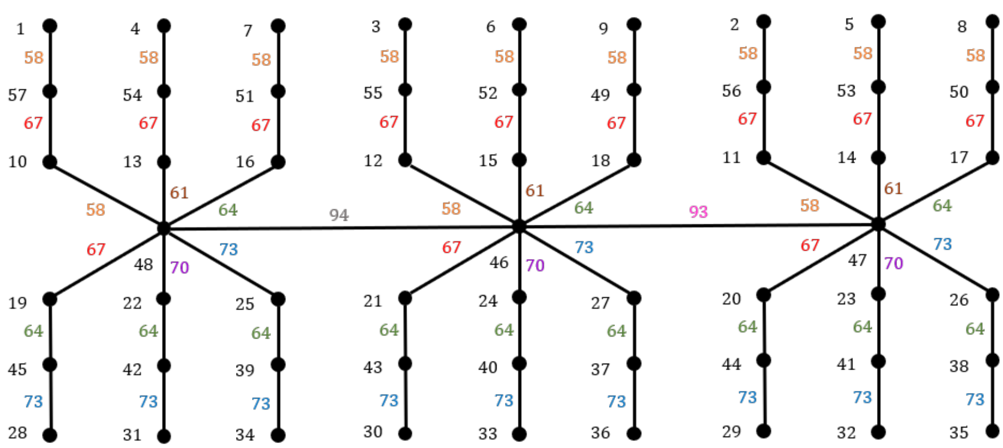

Figure 4. Local edge antimagic labeling of $P_3 \rhd_o 3P_7$, where $o = v_4$ is not a leaf of P_7, and $\chi'_{lea}(P_3 \rhd_o 3P_7) = 8$.

Corollary 2.2 provides a correction to the result presented in [3].

Corollary 2.2. *Let P_n be a path of order n and o be a vertex of P_n. For $m, n \geq 3$, if o is a leaf, then the local edge antimagic chromatic number of $P_m \rhd_o P_n$ is $\chi'_{lea}(P_m \rhd_o P_n) = 3$. Otherwise, $\chi'_{lea}(P_m \rhd_o P_n) = 4$.*

In the proof of Theorem 2.1, Case 2, it appears that the weights of the edges in P_m are different from the weights of other edges in $P_m \rhd_o tP_n$. There are 2 different weights on the edges in P_m. If the edges in P_m are deleted, then the number of different edge weights and the maximum degree of the resulting graph will both be reduced by 2. Therefore, we get Corollary 2.3.

Corollary 2.3. *Let P_n be a path of order n and v be a vertex of degree 2 of P_n. For $n \geq 3$ and $t \geq 1$, the local edge antimagic chromatic number of copies of the graph obtained by identifying the vertex v from each copy of P_n in tP_n is $2t$.*

Theorem 2.4. *Let P_m be a path of order m, C_n be a cycle of order n, and o be a vertex of C_n. For $m, n \geq 3$, n is even, and $t \geq 1$, the local edge antimagic chromatic number of $P_m \rhd_o tC_n$ is $\chi'_{lea}(P_m \rhd_o tC_n) = 2 + 2t$.*

Proof. Let o be a vertex of C_n and $P_m \rhd_o tC_n$ be a graph with vertex set $V(P_m \rhd_o tC_n) = \{x_{0,0,k} : 1 \leq k \leq m\} \cup \{x_{i,j,k} : 1 \leq i \leq n-1, 1 \leq j \leq t, 1 \leq k \leq m\}$ and edge set $E(P_m \rhd_o tC_n) = \{x_{0,0,k}x_{0,0,k+1} : 1 \leq k \leq m-1\} \cup \{x_{0,0,k}x_{1,j,k} : 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{0,0,k}x_{n-1,j,k} : 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{i,j,k}x_{i+1,j,k} : 1 \leq i \leq n-2, 1 \leq j \leq t, 1 \leq k \leq m\}$. Thus, $|V(P_m \rhd_o tC_n)| = tm(n-1)+m$ and $|E(P_m \rhd_o tC_n)| = tmn+m-1$. Figure 5 provides an illustration of the graph.

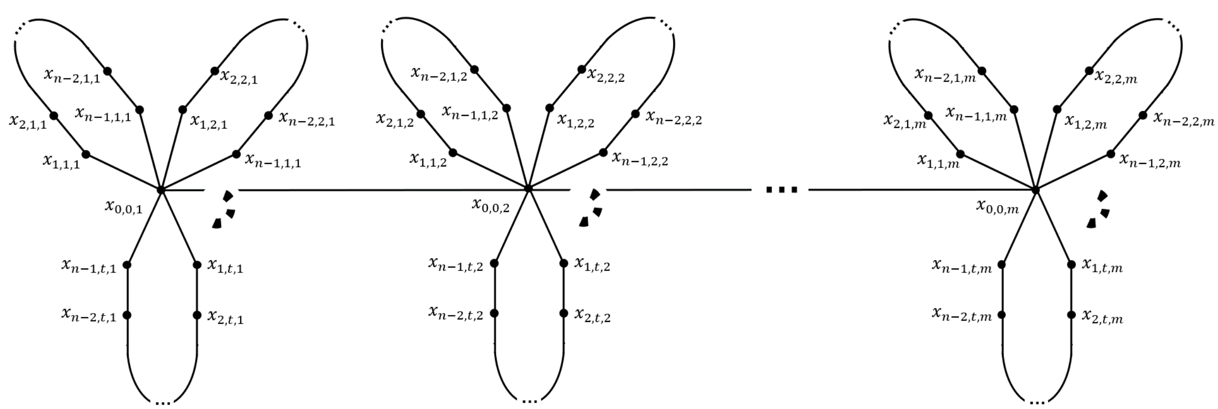

Figure 5. Graph $P_n \rhd_o tC_n$ with the vertex name.

The proof is divided into two cases, based on the divisibility of n by 4.

Case 1. $n \equiv 0 \pmod 4$.

For $m, n \geq 3$ and $t \geq 1$, define $f : V(P_m \rhd_o tC_n) \rightarrow \{1, \ldots, tm(n-1)+m\}$ as follows.

For $1 \leq k \leq m$,

$$f(x_{0,0,k}) = \begin{cases} \frac{tmn}{2} + m - \frac{k-1}{2}, & \text{if } k \text{ is odd,} \\ \frac{tmn+k}{2}, & \text{if } k \text{ is even.} \end{cases}$$

For $1 \leq i \leq n-1$, $1 \leq j \leq t$, and $1 \leq k \leq m$,

$$f(x_{i,j,k}) = \begin{cases} tm(\frac{n}{2}-i-1)+m(j-1)+\frac{k+1}{2}, & \text{if } i \leq \frac{n}{2}-1, i \text{ is odd}, k \text{ is odd}, \\ tm(\frac{n}{2}-i-1)+mj-\frac{k-2}{2}, & \text{if } i \leq \frac{n}{2}-1, i \text{ is odd}, k \text{ is even}, \\ tm(\frac{n}{2}+i)-m(j-2)-\frac{k-1}{2}, & \text{if } i \leq \frac{n}{2}-1, i \text{ is even}, k \text{ is odd}, \\ tm(\frac{n}{2}+i)-m(j-1)+\frac{k}{2}, & \text{if } i \leq \frac{n}{2}-1, i \text{ is even}, k \text{ is even}, \\ tm(i-\frac{n}{2})+m(j-1)+\frac{k+1}{2}, & \text{if } i \geq \frac{n}{2}, i \text{ is odd}, k \text{ is odd}, \\ tm(i-\frac{n}{2})+mj-\frac{k-2}{2}, & \text{if } i \geq \frac{n}{2}, i \text{ is odd}, k \text{ is even}, \\ tm(\frac{3n}{2}-i-1)-m(j-2)-\frac{k-1}{2}, & \text{if } i \geq \frac{n}{2}, i \text{ is even}, k \text{ is odd}, \\ tm(\frac{3n}{2}-i-1)-m(j-1)+\frac{k}{2}, & \text{if } i \geq \frac{n}{2}, i \text{ is even}, k \text{ is even}. \end{cases}$$

From the labeling f, we obtain the weights of the edges in path P_m for $1 \leq k \leq m-1$ are

$$w(x_{0,0,k}x_{0,0,k+1}) = \begin{cases} tmn+m+1, & \text{if } k \text{ is odd}, \\ tmn+m, & \text{if } k \text{ is even}, \end{cases}$$

the weights of the edges connecting P_m and tC_n for $1 \leq j \leq t$ and $1 \leq k \leq m$ are

$$w(x_{0,0,k}x_{1,j,k}) = tm(n-2)+mj+1,$$
$$w(x_{0,0,k}x_{n-1,j,k}) = tm(n-1)+mj+1,$$

and the weights of the remaining edges in tC_n for $1 \leq i \leq n-2$, $1 \leq j \leq t$, and $1 \leq k \leq m$ are

$$w(x_{i,j,k}x_{i+1,j,k}) = \begin{cases} tmn+m+1, & \text{if } i \leq \frac{n}{2}-2, i \text{ is odd}, \\ tm(n-2)+m+1, & \text{if } i \leq \frac{n}{2}-2, i \text{ is even}, \\ tm(n-1)+m+1, & \text{if } i = \frac{n}{2}-1, \\ tm(n-2)+m+1, & \text{if } i \geq \frac{n}{2}, i \text{ is odd}, \\ tmn+m+1, & \text{if } i \geq \frac{n}{2}, i \text{ is even}. \end{cases}$$

Note that when $i \leq \frac{n}{2}-2$ and i is odd, or when $i \geq \frac{n}{2}$ and i is even, $w(x_{i,j,k}x_{i+1,j,k})$ for any k equals $w(x_{0,0,k}x_{0,0,k+1})$ when k is odd. Moreover, when $i \leq \frac{n}{2}-2$ and i is even, or when $i \geq \frac{n}{2}$ and i is odd, $w(x_{i,j,k}x_{i+1,j,k}) = w(x_{0,0,k}x_{1,1,k})$. Finally, $w(x_{i,j,k}x_{i+1,j,k}) = w(x_{0,0,k}x_{n-1,1,k})$ when $i = \frac{n}{2}-1$. Thus, the weights of the edges in tC_n are the same as the weights of some of the edges in P_m or the edges connecting P_m and tC_n, depending on the condition of i.

Since $1 \leq j \leq t$ and both $w(x_{0,0,k}x_{1,j,k})$ and $w(x_{0,0,k}x_{n-1,j,k})$ depend on j, there are $2t$ different weights of the edges connecting P_m and tC_n. These weights are different from the weights of the edges in P_m. Consequently, there are $2+2t$ distinct edge weights. Therefore, we can conclude that $\chi'_{lea}(P_m \rhd_o tC_n) \leq 2+2t$. Since $\Delta(P_m \rhd_o tC_n) = 2+2t$, we obtain $\chi'_{lea}(P_m \rhd_o tC_n) \geq 2+2t$. In conclusion, $\chi'_{lea}(P_m \rhd_o tC_n) = 2+2t$ if $n \equiv 0 \pmod 4$.

Case 2. $n \equiv 2 \pmod 4$.
For $m, n \geq 3$ and $t \geq 1$, $f : V(P_m \rhd_o tC_n) \to \{1, \ldots, tm(n-1)+m\}$ is defined as follows.
For $1 \leq k \leq m$,

$$f(x_{0,0,k}) = \begin{cases} tm(\frac{n-2}{2})+\frac{k+1}{2}, & \text{if } k \text{ is odd}, \\ tm(\frac{n-2}{2})+m-\frac{k-2}{2}, & \text{if } k \text{ is even}. \end{cases}$$

For $i \geq 1$, $1 \leq j \leq t$, and $1 \leq k \leq m$,

$$f(x_{i,j,k}) = \begin{cases} tm(\frac{n}{2}+i) - m(j-2) - \frac{k-1}{2}, & \text{if } i \leq \frac{n}{2}-1, i \text{ is odd}, k \text{ is odd},\\ tm(\frac{n}{2}+i) - m(j-1) + \frac{k}{2}, & \text{if } i \leq \frac{n}{2}-1, i \text{ is odd}, k \text{ is even},\\ tm(\frac{n}{2}-i-1) + m(j-1) + \frac{k+1}{2}, & \text{if } i \leq \frac{n}{2}-1, i \text{ is even}, k \text{ is odd},\\ tm(\frac{n}{2}-i-1) + mj - \frac{k-2}{2}, & \text{if } i \leq \frac{n}{2}-1, i \text{ is even}, k \text{ is even},\\ tm(\frac{3n}{2}-i-1) - m(j-2) - \frac{k-1}{2}, & \text{if } i \geq \frac{n}{2}, i \text{ is odd}, k \text{ is odd},\\ tm(\frac{3n}{2}-i-1) - m(j-1) + \frac{k}{2}, & \text{if } i \geq \frac{n}{2}, i \text{ is odd}, k \text{ is even},\\ tm(i-\frac{n}{2}) + m(j-1) + \frac{k+1}{2}, & \text{if } i \geq \frac{n}{2}, i \text{ is even}, k \text{ is odd},\\ tm(i-\frac{n}{2}) + mj - \frac{k-2}{2}, & \text{if } i \geq \frac{n}{2}, i \text{ is even}, k \text{ is even}. \end{cases}$$

From the labeling f, we obtain the weights of the edges in path P_m for $1 \leq k \leq m-1$ are

$$w(x_{0,0,k}x_{0,0,k+1}) = \begin{cases} tm(n-2)+m+1, & \text{if } k \text{ is odd},\\ tm(n-2)+m+2, & \text{if } k \text{ is even}, \end{cases}$$

the weights of the edges connecting P_m and tC_n for $1 \leq j \leq t$ and $1 \leq k \leq m$ are

$$w(x_{0,0,k}x_{1,j,k}) = tmn - m(j-2) + 1,$$

$$w(x_{0,0,k}x_{n-1,j,k}) = tm(n-1) - m(j-2) + 1,$$

and the weights of the remaining edges in tC_n for $1 \leq i \leq n-2$, $1 \leq j \leq t$, and $1 \leq k \leq m$ are

$$w(x_{i,j,k}x_{i+1,j,k}) = \begin{cases} tm(n-2)+m+1, & \text{if } i \leq \frac{n}{2}-2, i \text{ is odd},\\ tmn+m+1, & \text{if } i \leq \frac{n}{2}-2, i \text{ is even},\\ tm(n-1)+m+1, & \text{if } i = \frac{n}{2}-1,\\ tmn+m+1, & \text{if } i \geq \frac{n}{2}, i \text{ is odd},\\ tm(n-2)+m+1, & \text{if } i \geq \frac{n}{2}, i \text{ is even}. \end{cases}$$

Note that when $i \leq \frac{n}{2}-2$ and i is odd, or when $i \geq \frac{n}{2}$ and i is even, $w(x_{i,j,k}x_{i+1,j,k})$ for any k equals $w(x_{0,0,k}x_{0,0,k+1})$ when k is odd. Moreover, when $i \leq \frac{n}{2}-2$ and i is even, or when $i \geq \frac{n}{2}$ and i is odd, $w(x_{i,j,k}x_{i+1,j,k}) = w(x_{0,0,k}x_{1,1,k})$. Finally, $w(x_{i,j,k}x_{i+1,j,k}) = w(x_{0,0,k}x_{n-1,1,k})$ when $i = \frac{n}{2}-1$. Thus, the weights of the edges in tC_n are the same as the weights of some of the edges in P_m or the edges connecting P_m and tC_n, depending on the condition of i.

Since $1 \leq j \leq t$ and both $w(x_{0,0,k}x_{1,j,k})$ and $w(x_{0,0,k}x_{n-1,j,k})$ depend on j, there are $2t$ different weights of the edges connecting P_m and tC_n. These weights are different from the weights of the edges in P_m. Consequently, there are $2+2t$ distinct edge weights. Therefore, we can conclude that $\chi'_{lea}(P_m \triangleright_o tC_n) \leq 2+2t$. Since $\Delta(P_m \triangleright_o tC_n) = 2+2t$, we obtain $\chi'_{lea}(P_m \triangleright_o tC_n) \geq 2+2t$. In conclusion, $\chi'_{lea}(P_m \triangleright_o tC_n) = 2+2t$ if $n \equiv 2 \pmod 4$.

Cases 1 and 2 proved that $\chi'_{lea}(P_m \triangleright_o tC_n) = 2+2t$ if n is even. □

The following two figures provide examples of local edge antimagic labeling of $P_3 \rhd_o 2C_n$, where n is even. Figure 6 shows the labeling for $P_3 \rhd_o 2C_6$ with $\chi'_{lea}(P_3 \rhd_o 2C_6) = 6$ and Figure 7 illustrates the labeling for $P_3 \rhd_o 2C_8$ with $\chi'_{lea}(P_3 \rhd_o 2C_8) = 6$.

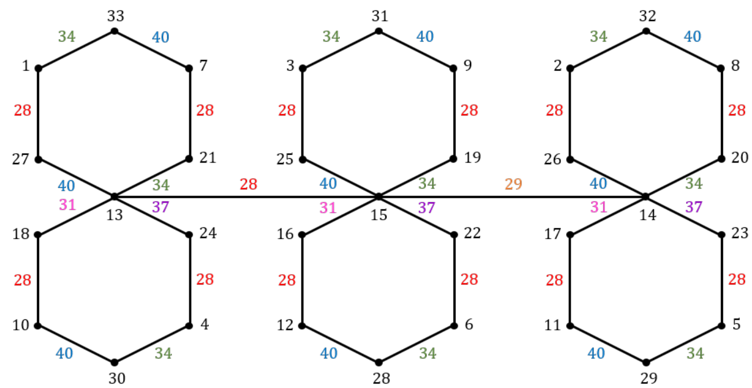

Figure 6. Local edge antimagic labeling of $P_3 \rhd_o 2C_6$, where $\chi'_{lea}(P_3 \rhd_o 2C_6) = 6$.

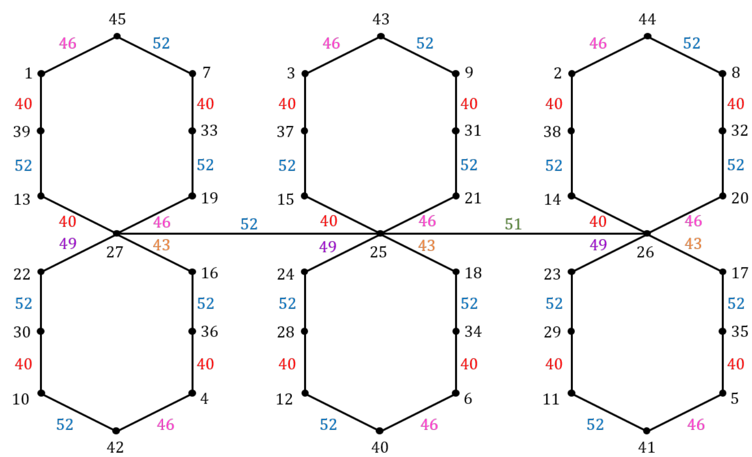

Figure 7. Local edge antimagic labeling of $P_3 \rhd_o 2C_8$, where $\chi'_{lea}(P_3 \rhd_o 2C_8) = 6$.

Corollary 2.5 provides a correction to the result presented in [3].

Corollary 2.5. *Let P_m be a path of order m, C_n be a cycle of order n, and o be a vertex of C_n. For $m, n \geq 3$ and n is even, the local edge antimagic chromatic number of $P_m \rhd_o C_n$ is $\chi'_{lea}(P_m \rhd_o C_n) = 4$.*

Ladder graph L_n is defined as the Cartesian product of P_n and P_2. This graph has $2n$ vertices and $3n - 2$ edges.

Theorem 2.6. *Let P_m be a path of order m, L_n be a ladder graph of order $2n$, and o be a vertex of L_n. For $m, n \geq 3$ and $t \geq 1$, if o is a vertex of degree 2, then the local edge antimagic chromatic number of $P_m \rhd_o tL_n$ is $\chi'_{lea}(P_m \rhd_o tL_n) = 2 + 2t$. Otherwise, $\chi'_{lea}(P_m \rhd_o tL_n) = 2 + 3t$.*

Proof. Let L_n be a graph with vertex set $V(L_n) = \{v_i : 1 \le i \le 2n\}$ and edge set $E(L_n) = \{v_iv_{i+1} : 1 \le i \le n-1\} \cup \{v_iv_{i+1} : n+1 \le i \le 2n-1\} \cup \{v_iv_{2n+1-i} : 1 \le i \le n\}$. Therefore, vertices v_1, v_n, v_{n+1}, and v_{2n} have degree 2, while the remaining vertices have degree 3. Let o be a vertex in L_n.

Case 1. o is a vertex of degree 2.
Let $P_m \rhd_o tL_n$ be a graph with vertex set $V(P_m \rhd_o tL_n) = \{x_{i,j,k} : 1 \le i \le 2n-1, 1 \le j \le t, 1 \le k \le m\} \cup \{x_{0,0,k} : 1 \le k \le m\}$ and edge set $E(P_m \rhd_o tL_n) = \{x_{0,0,k}x_{0,0,k+1} : 1 \le k \le m-1\} \cup \{x_{i,j,k}x_{i+1,j,k} : 1 \le i \le n-2, 1 \le j \le t, 1 \le k \le m\} \cup \{x_{i,j,k}x_{i+1,j,k} : n \le i \le 2n-2, 1 \le j \le t, 1 \le k \le m\} \cup \{x_{0,0,k}x_{n-1,j,k} : 1 \le j \le t, 1 \le k \le m\} \cup \{x_{0,0,k}x_{n,j,k} : 1 \le j \le t, 1 \le k \le m\} \cup \{x_{i,j,k}x_{2n-i,j,k} : 1 \le i \le n-1, 1 \le j \le t, 1 \le k \le m\}$. Therefore, $|V(P_m \rhd_o tL_n)| = tm(2n-1) + m$ and $|E(P_m \rhd_o tL_n)| = tm(3n-2) + m - 1$. Figure 8 provides an illustration of the graph.

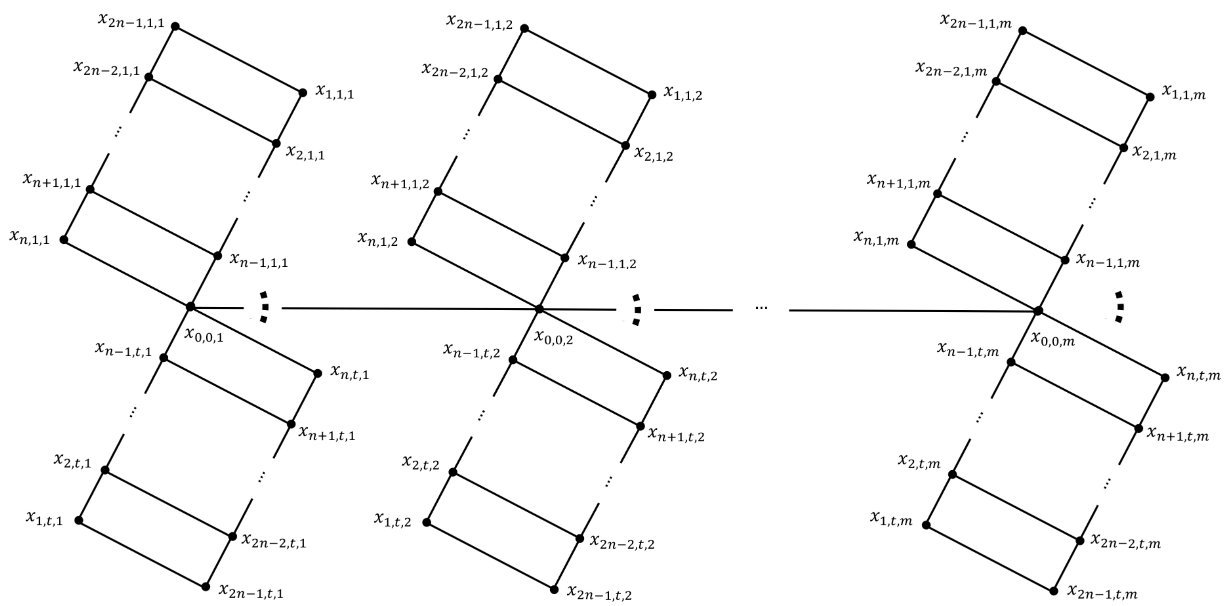

Figure 8. Graph $P_m \rhd_o tL_n$, where o is a vertex of degree 2 in L_n, with the vertex name.

Subcase 1.1. n is odd.
For $m, n \ge 3$ and $t \ge 1$, define $f : V(P_m \rhd_o tL_n) \to \{1, .., tm(2n-1) + m\}$ as follows.
For $1 \le k \le m$,

$$f(x_{0,0,k}) = \begin{cases} tmn + m - \frac{k-1}{2}, & \text{if } k \text{ is odd,} \\ tmn + \frac{k}{2}, & \text{if } k \text{ is even.} \end{cases}$$

For $1 \leq i \leq 2n-1$, $1 \leq j \leq t$, and $1 \leq k \leq m$,

$$f(x_{i,j,k}) = \begin{cases} tm(2n-i) - m(j-2) - \frac{k-1}{2}, & \text{if } i \leq n-1, i \text{ is odd}, k \text{ is odd}, \\ tm(2n-i) - m(j-1) + \frac{k}{2}, & \text{if } i \leq n-1, i \text{ is odd}, k \text{ is even}, \\ tm(i-1) + m(j-1) + \frac{k+1}{2}, & \text{if } i \leq n-1, i \text{ is even}, k \text{ is odd}, \\ tm(i-1) + mj - \frac{k-2}{2}, & \text{if } i \leq n-1, i \text{ is even}, k \text{ is even}, \\ tm(2n-i-1) + m(j-1) + \frac{k+1}{2}, & \text{if } i \geq n, i \text{ is odd}, k \text{ is odd}, \\ tm(2n-i-1) + mj - \frac{k-2}{2}, & \text{if } i \geq n, i \text{ is odd}, k \text{ is even}, \\ tmi - m(j-2) - \frac{k-1}{2}, & \text{if } i \geq n, i \text{ is even}, k \text{ is odd}, \\ tmi - m(j-1) + \frac{k}{2}, & \text{if } i \geq n, i \text{ is even}, k \text{ is even}. \end{cases}$$

From the labeling f, we obtain the weights of edges in path P_m for $1 \leq k \leq m-1$ are

$$w(x_{0,0,k}x_{0,0,k+1}) = \begin{cases} 2tmn + m + 1, & \text{if } k \text{ is odd}, \\ 2tmn + m, & \text{if } k \text{ is even}, \end{cases}$$

the weights of the edges connecting P_m and tL_n for $1 \leq j \leq t$ and $1 \leq k \leq m$ are

$$w(x_{0,0,k}x_{n-1,j,k}) = 2tm(n-1) + mj + 1,$$

$$w(x_{0,0,k}x_{n,j,k}) = tm(2n-1) + mj + 1,$$

the weights of the edges in the paths P_2 of the ladder graphs for $1 \leq i \leq n-1$, $1 \leq j \leq t$, and $1 \leq k \leq m$ are

$$w(x_{i,j,k}x_{2n-i,j,k}) = tm(2n-1) + m + 1,$$

and the weights of the remaining edges in tL_n for $1 \leq i \leq 2n-2$, $i \neq n-1$, $1 \leq j \leq t$, and $1 \leq k \leq m$ are

$$w(x_{i,j,k}x_{i+1,j,k}) = \begin{cases} 2tmn + m + 1, & \text{if } i \text{ is odd}, \\ 2tm(n-1) + m + 1, & \text{if } i \text{ is even}. \end{cases}$$

Note that $w(x_{i,j,k}x_{i+1,j,k})$ when i is odd equals $w(x_{0,0,k}x_{0,0,k+1})$ when k is odd. On the other hand, when i is even, $w(x_{i,j,k}x_{i+1,j,k})$ equals $w(x_{0,0,k}x_{n-1,1,k})$. Thus, the weights of the edges in tL_n are the same as the weights of some of the edges in P_m or the weights of the edges connecting P_m and tL_n, depending on the parity of i. In addition, since $w(x_{i,j,k}x_{2n-i,j,k})$ equals $w(x_{0,0,k}x_{n,1,k})$, we can conclude that the weights of the edges in the paths P_2 of the ladder graphs are the same as the weights of some of the edges connecting P_m and tL_n.

Since $1 \leq j \leq t$ and both $w(x_{0,0,k}x_{n-1,j,k})$ and $w(x_{0,0,k}x_{n,j,k})$ depend on j, there are $2t$ different edge weights. These weights are different from the weights of the edges in P_m. Consequently, there are $2+2t$ distinct edge weights. Therefore, we can conclude that $\chi'_{lea}(P_m \rhd_o tL_n) \leq 2+2t$. Since $\Delta(P_m \rhd_o tL_n) = 2+2t$, we get $\chi'_{lea}(P_m \rhd_o tL_n) \geq 2+2t$. In conclusion, $\chi'_{lea}(P_m \rhd_o tL_n) = 2+2t$ if o is a vertex of degree 2 in L_n and n is odd.

Subcase 1.2. n is even.
For $m, n \geq 3$ and $t \geq 1$, define $f : V(P_m \triangleright_o tL_n) \to \{1, .., tm(2n-1)+m\}$ as follows.
For $1 \leq k \leq m$,

$$f(x_{0,0,k}) = \begin{cases} tm(n-1) + \frac{k+1}{2}, & \text{if } k \text{ is odd,} \\ tm(n-1) + m - \frac{k-2}{2}, & \text{if } k \text{ is even.} \end{cases}$$

For $1 \leq i \leq 2n-1$, $1 \leq j \leq t$, and $1 \leq k \leq m$,

$$f(x_{i,j,k}) = \begin{cases} tm(2n-i) - m(j-2) - \frac{k-1}{2}, & \text{if } i \leq n-1, i \text{ is odd}, k \text{ is odd,} \\ tm(2n-i) - m(j-1) + \frac{k}{2}, & \text{if } i \leq n-1, i \text{ is odd}, k \text{ is even,} \\ tm(i-1) + m(j-1) + \frac{k+1}{2}, & \text{if } i \leq n-1, i \text{ is even}, k \text{ is odd,} \\ tm(i-1) + mj - \frac{k-2}{2}, & \text{if } i \leq n-1, i \text{ is even}, k \text{ is even,} \\ tm(2n-i-1) + m(j-1) + \frac{k+1}{2}, & \text{if } i \geq n, i \text{ is odd}, k \text{ is odd,} \\ tm(2n-i-1) + mj - \frac{k-2}{2}, & \text{if } i \geq n, i \text{ is odd}, k \text{ is even,} \\ tmi - m(j-2) - \frac{k-1}{2}, & \text{if } i \geq n, i \text{ is even}, k \text{ is odd,} \\ tmi - m(j-1) + \frac{k}{2}, & \text{if } i \geq n, i \text{ is even}, k \text{ is even.} \end{cases}$$

From the labeling f, we obtain the weights of edges in path P_m for $1 \leq k \leq m-1$ are

$$w(x_{0,0,k}x_{0,0,k+1}) = \begin{cases} 2tm(n-1) + m + 1, & \text{if } k \text{ is odd,} \\ 2tm(n-1) + m + 2, & \text{if } k \text{ is even,} \end{cases}$$

the weights of the edges connecting P_m and tL_n for $1 \leq j \leq t$ and $1 \leq k \leq m$ are

$$w(x_{0,0,k}x_{n-1,j,k}) = 2tmn - m(j-2) + 1,$$

$$w(x_{0,0,k}x_{n,j,k}) = tm(2n-1) - m(j-2) + 1,$$

the weights of the edges in the paths P_2 of the ladder graphs for $1 \leq i \leq n-1$, $1 \leq j \leq t$, and $1 \leq k \leq m$ are

$$w(x_{i,j,k}x_{2n-i,j,k}) = tm(2n-1) + m + 1,$$

and the weights of the remaining edges in tL_n for $1 \leq i \leq 2n-2$, $i \neq n-1$, $1 \leq j \leq t$, and $1 \leq k \leq m$ are

$$w(x_{i,j,k}x_{i+1,j,k}) = \begin{cases} 2tmn + m + 1, & \text{if } i \text{ is odd,} \\ 2tm(n-1) + m + 1, & \text{if } i \text{ is even.} \end{cases}$$

Note that $w(x_{i,j,k}x_{i+1,j,k})$ when i is odd equals $w(x_{0,0,k}x_{n-1,1,k})$. On the other hand, when i is even, $w(x_{i,j,k}x_{i+1,j,k})$ equals $w(x_{0,0,k}x_{0,0,k+1})$ when k is odd. Thus, the weights of the edges in tL_n are the same as the weights of some of the weights of the edges connecting P_m and tL_n or the edges in P_m, depending on the parity of i. In addition, since $w(x_{i,j,k}x_{2n-i,j,k})$ equals $w(x_{0,0,k}x_{n,1,k})$, we can conclude that the weights of the edges in the paths P_2 of the ladder graphs are the same as the weights of some of the edges connecting P_m and tL_n.

Since $1 \leq j \leq t$ and both $w(x_{0,0,k}x_{n-1,j,k})$ and $w(x_{0,0,k}x_{n,j,k})$ depend on j, there are $2t$ different edge weights. These weights are different from the weights of the edges in P_m. Consequently, there are $2+2t$ distinct edge weights. Therefore, we can conclude that $\chi'_{lea}(P_m \rhd_o tL_n) \leq 2+2t$. Since $\Delta(P_m \rhd_o tL_n) = 2+2t$, we get $\chi'_{lea}(P_m \rhd_o tL_n) \geq 2+2t$. In conclusion, $\chi'_{lea}(P_m \rhd_o tL_n) = 2+2t$ if o is a vertex of degree 2 in L_n and n is even.

Subcases 1.1 and 1.2 proved that $\chi'_{lea}(P_m \rhd_o tL_n) = 2+2t$ if o is a vertex of degree 2 in L_n.

Case 2. o is a vertex of degree 3.

Let $o = v_q$ be a vertex of L_n, where $q \in \{2, \ldots, n-1\}$. Note that if $q \in \{n+2, \ldots, 2n-1\}$, the comb product of P_m and tL_n, with respect to the vertex o, is isomorphic to the comb product when $q \in \{2, \ldots, n-1\}$.

Subcase 2.1. o is adjacent to any vertex of degree 2 in L_n.

W.l.o.g., let $o = v_2$. Let $P_m \rhd_o tL_n$ be a graph with vertex set $V(P_m \rhd_o tL_n) = \{x_{i,j,k} : 1 \leq i \leq 2n-1, 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{0,0,k} : 1 \leq k \leq m\}$ and edge set $E(P_m \rhd_o tL_n) = \{x_{0,0,k}x_{0,0,k+1} : 1 \leq k \leq m-1\} \cup \{x_{i,j,k}x_{i+1,j,k} : 2 \leq i \leq n-2, 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{i,j,k}x_{i+1,j,k} : n \leq i \leq 2n-2, 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{0,0,k}x_{1,j,k} : 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{0,0,k}x_{2,j,k} : 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{0,0,k}x_{2n-2,j,k} : 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{1,j,k}x_{2n-1,j,k} : 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{i,j,k}x_{2n-i-1,j,k} : 2 \leq i \leq n-1, 1 \leq j \leq t, 1 \leq k \leq m\}$. Therefore, $|V(P_m \rhd_o tL_n)| = tm(2n-1)+m$ and $|E(P_m \rhd_o tL_n)| = tm(3n-2)+m-1$.

For $m, n \geq 3$ and $t \geq 1$, define $f : V(P_m \rhd_o tL_n) \to \{1, .., tm(2n-1)+m\}$ as follows.
For $1 \leq k \leq m$,

$$f(x_{0,0,k}) = \begin{cases} tm(2n-2)+m-\frac{k-1}{2}, & \text{if } k \text{ is odd,} \\ tm(2n-2)+\frac{k}{2}, & \text{if } k \text{ is even.} \end{cases}$$

For $1 \leq j \leq t$ and $1 \leq k \leq m$,

$$f(x_{1,j,k}) = \begin{cases} tm(i-1)+m(j-1)+\frac{k+1}{2}, & \text{if } k \text{ is odd,} \\ tm(i-1)+mj-\frac{k-2}{2}, & \text{if } k \text{ is even,} \end{cases}$$

$$f(x_{2n-1,j,k}) = \begin{cases} tmi-m(j-2)-\frac{k-1}{2}, & \text{if } k \text{ is odd,} \\ tmi-m(j-1)+\frac{k}{2}, & \text{if } k \text{ is even.} \end{cases}$$

if $2 \leq i \leq n-1$,

$$f(x_{i,j,k}) = \begin{cases} tm(2n-i)-m(j-1)-\frac{k-1}{2}, & \text{if } i \text{ is odd}, k \text{ is odd,} \\ tm(2n-i)-mj+\frac{k}{2}, & \text{if } i \text{ is odd}, k \text{ is even,} \\ tmi+m(j-1)+\frac{k+1}{2}, & \text{if } i \text{ is even}, k \text{ is odd,} \\ tmi+mj-\frac{k-2}{2}, & \text{if } i \text{ is even}, k \text{ is even,} \end{cases}$$

and if $n \leq i \leq 2n-2$,

$$f(x_{i,j,k}) = \begin{cases} tm(i+1)-m(j-1)-\frac{k-1}{2}, & \text{if } i \text{ is odd}, k \text{ is odd,} \\ tm(i+1)-mj+\frac{k}{2}, & \text{if } i \text{ is odd}, k \text{ is even,} \\ tm(2n-i-1)+m(j-1)+\frac{k+1}{2}, & \text{if } i \text{ is even}, k \text{ is odd,} \\ tm(2n-i-1)+mj-\frac{k-2}{2}, & \text{if } i \text{ is even}, k \text{ is even.} \end{cases}$$

From the labeling f, we obtain the weights of edges in path P_m for $1 \leq k \leq m$ are

$$w(x_{0,0,k}x_{0,0,k+1}) = \begin{cases} 2tm(2n-2)+m+1, & \text{if } k \text{ is odd,} \\ 2tm(2n-2)+m, & \text{if } k \text{ is even,} \end{cases}$$

the weights of the edges connecting P_m and tL_n for $1 \leq j \leq t$ and $1 \leq k \leq m$ are

$$w(x_{0,0,k}x_{1,j,k}) = 2tm(n-1)+mj+1,$$

$$w(x_{0,0,k}x_{2,j,k}) = 2tmn+mj+1,$$

$$w(x_{0,0,k}x_{2n-2,j,k}) = tm(2n-1)+mj+1,$$

and the weights of the edges in the paths P_2 of the ladder graphs for $1 \leq j \leq t$ and $1 \leq k \leq m$ are

$$w(x_{1,j,k}x_{2n-1,j,k}) = tm(2n-1)+m+1,$$

and for $2 \leq i \leq n-1$, $1 \leq j \leq t$, and $1 \leq k \leq m$ are

$$w(x_{i,j,k}x_{2n-i-1,j,k}) = 2tmn+1.$$

Finally, the weights of the remaining edges in L_n for $1 \leq j \leq t$ and $1 \leq k \leq m$ are as follows. If $2 \leq i \leq n-2$,

$$w(x_{i,j,k}x_{i+1,j,k}) = \begin{cases} tm(2n+1)+1, & \text{if } i \text{ is odd,} \\ tm(2n-1)+1, & \text{if } i \text{ is even,} \end{cases}$$

and if $n \leq i \leq 2n-2$,

$$w(x_{i,j,k}x_{i+1,j,k}) = \begin{cases} tm(2n-1)+1, & \text{if } i \leq 2n-3, i \text{ is odd,} \\ tm(2n+1)+1, & \text{if } i \leq 2n-3, i \text{ is even,} \\ 2tmn+m+1, & \text{if } i = 2n-2. \end{cases}$$

Note that for $1 \leq j \leq t$ and $1 \leq k \leq m$,

$$w(x_{i,j,k}x_{i+1,j,k}) = \begin{cases} w(x_{0,0,k}x_{2,t,k}), & \text{if } 2 \leq i \leq n-2, i \text{ is odd,} \\ w(x_{0,0,k}x_{1,t,k}), & \text{if } 2 \leq i \leq n-2, i \text{ is even,} \\ w(x_{0,0,k}x_{1,t,k}), & \text{if } n \leq i \leq 2n-3, i \text{ is odd,} \\ w(x_{0,0,k}x_{2,t,k}), & \text{if } n \leq i \leq 2n-3, i \text{ is even,} \\ w(x_{0,0,k}x_{2,1,k}), & \text{if } i = 2n-2. \end{cases}$$

Moreover, $w(x_{1,j,k}x_{2n-1,j,k}) = w(x_{0,0,k}x_{2n-2,1,k})$ and $w(x_{i,j,k}x_{2n-i-1,j,k}) = w(x_{0,0,k}x_{2n-2,t,k})$. Thus, the weights of the edges in tL_n, as well as the weights of the edges in the paths P_2 of the ladder graphs, are the same as the weights of some of the edges connecting P_m and tL_n.

Since $1 \leq j \leq t$ and all $w(x_{0,0,k}x_{1,j,k})$, $w(x_{0,0,k}x_{2,j,k})$, and $w(x_{0,0,k}x_{2n-2,j,k})$ depend on j, there are $3t$ different edge weights. These weights are different from the weights of the edges in P_m. Consequently, there are $2+3t$ distinct edge weights. Therefore, we can conclude that $\chi'_{lea}(P_m \rhd_o tL_n) \leq 2+3t$. Since $\Delta(P_m \rhd_o tL_n) = 2+3t$, we get $\chi'_{lea}(P_m \rhd_o tL_n) \geq 2+3t$. In conclusion, $\chi'_{lea}(P_m \rhd_o tL_n) = 2+3t$ if o is a vertex of degree 3 in L_n and is adjacent to a vertex of degree 2 in L_n.

Subcase 2.2. o is not adjacent to any vertex of degree 2 in L_n.
Let $o = v_q$ be a vertex of L_n, where $q \in \{3,\dots,n-2\}$, and $P_m \rhd_o tL_n$ be a graph with vertex set $V(P_m \rhd_o tL_n) = \{x_{i,j,k} : 1 \leq i \leq 2n-1, 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{0,0,k} : 1 \leq k \leq m\}$ and edge set $E(P_m \rhd_o tL_n) = \{x_{0,0,k}x_{0,0,k+1} : 1 \leq k \leq m-1\} \cup \{x_{i,j,k}x_{i+1,j,k} : 1 \leq i \leq q-2, 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{i,j,k}x_{i+1,j,k} : q \leq i \leq n-2, 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{i,j,k}x_{i+1,j,k} : n \leq i \leq 2n-2, 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{0,0,k}x_{q-1,j,k} : 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{0,0,k}x_{q,j,k} : 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{0,0,k}x_{2n-q,j,k} : 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{i,j,k}x_{2n-i,j,k} : 1 \leq i \leq q-1, 1 \leq j \leq t, 1 \leq k \leq m\} \cup \{x_{i,j,k}x_{2n-i-1,j,k} : q \leq i \leq n-1, 1 \leq j \leq t, 1 \leq k \leq m\}$. Therefore, $|V(P_m \rhd_o tL_n)| = tm(2n-1)+m$ and $|E(P_m \rhd_o tL_n)| = tm(3n-2)+m-1$.

Subsubcase 2.2.1. q is odd.
For $m,n \geq 3$ and $t \geq 1$, define $f : V(P_m \rhd_o tL_n) \to \{1,..,tm(2n-1)+m\}$ as follows.
For $1 \leq k \leq m$,

$$f(x_{0,0,k}) = \begin{cases} tm(2n-q)+m-\frac{k-1}{2}, & \text{if } k \text{ is odd},\\ tm(2n-q)+\frac{k}{2}, & \text{if } k \text{ is even}.\end{cases}$$

For $1 \leq j \leq t$ and $1 \leq k \leq m$, if $1 \leq i \leq q-1$,

$$f(x_{i,j,k}) = \begin{cases} tm(2n-i)-m(j-2)-\frac{k-1}{2}, & \text{if } i \text{ is odd}, k \text{ is odd},\\ tm(2n-i)-m(j-1)+\frac{k}{2}, & \text{if } i \text{ is odd}, k \text{ is even},\\ tm(i-1)+m(j-1)+\frac{k+1}{2}, & \text{if } i \text{ is even}, k \text{ is odd},\\ tm(i-1)+mj-\frac{k-2}{2}, & \text{if } i \text{ is even}, k \text{ is even},\end{cases}$$

if $q \leq i \leq n-1$,

$$f(x_{i,j,k}) = \begin{cases} tmi+m(j-1)+\frac{k+1}{2}, & \text{if } i \text{ is odd}, k \text{ is odd},\\ tmi+mj-\frac{k-2}{2}, & \text{if } i \text{ is odd}, k \text{ is even},\\ tm(2n-i)-m(j-1)-\frac{k-1}{2}, & \text{if } i \text{ is even}, k \text{ is odd},\\ tm(2n-i)-mj+\frac{k}{2}, & \text{if } i \text{ is even}, k \text{ is even},\end{cases}$$

if $n \leq i \leq 2n-q$,

$$f(x_{i,j,k}) = \begin{cases} tm(2n-i-1)+m(j-1)+\frac{k+1}{2}, & \text{if } i \text{ is odd}, k \text{ is odd},\\ tm(2n-i-1)+mj-\frac{k-2}{2}, & \text{if } i \text{ is odd}, k \text{ is even},\\ tm(i+1)-m(j-1)-\frac{k-1}{2}, & \text{if } i \text{ is even}, k \text{ is odd},\\ tm(i+1)-mj+\frac{k}{2}, & \text{if } i \text{ is even}, k \text{ is even},\end{cases}$$

and if $2n - q + 1 \leq i \leq 2n - 1$,

$$f(x_{i,j,k}) = \begin{cases} tm(2n-i-1) + m(j-1) + \frac{k+1}{2}, & \text{if } i \text{ is odd}, k \text{ is odd},\\ tm(2n-i-1) + mj - \frac{k-2}{2}, & \text{if } i \text{ is odd}, k \text{ is even},\\ tmi - m(j-2) - \frac{k-1}{2}, & \text{if } i \text{ is even}, k \text{ is odd},\\ tmi - m(j-1) + \frac{k}{2}, & \text{if } i \text{ is even}, k \text{ is even}. \end{cases}$$

From the labeling f, we obtain the weights of the edges in path P_m for $1 \leq k \leq m$ are

$$w(x_{0,0,k}x_{0,0,k+1}) = \begin{cases} 2tm(2n-q) + m + 1, & \text{if } k \text{ is odd},\\ 2tm(2n-q) + m, & \text{if } k \text{ is even}, \end{cases}$$

the weights of the edges connecting P_m and tL_n for $1 \leq j \leq t$ and $1 \leq k \leq m$ are

$$w(x_{0,0,k}x_{q-1,j,k}) = 2tm(n-1) + mj + 1,$$

$$w(x_{0,0,k}x_{q,j,k}) = 2tmn + mj + 1,$$

$$w(x_{0,0,k}x_{2n-q,j,k}) = tm(2n-1) + mj + 1,$$

the weights of the edges in the paths P_2 of the ladder graphs for $1 \leq i \leq q-1$, $1 \leq j \leq t$, and $1 \leq k \leq m$ are

$$w(x_{i,j,k}x_{2n-i,j,k}) = tm(2n-1) + m + 1,$$

and for $q \leq i \leq n-1$, $1 \leq j \leq t$, and $1 \leq k \leq m$ are

$$w(x_{i,j,k}x_{2n-i-1,j,k}) = 2tmn + 1.$$

Finally, the weights of the remaining edges in tL_n for $1 \leq j \leq t$ and $1 \leq k \leq m$ are as follows. If $1 \leq i \leq q-2$,

$$w(x_{i,j,k}x_{i+1,j,k}) = \begin{cases} 2tmn + m + 1, & \text{if } i \text{ is odd},\\ 2tm(n-1) + m + 1, & \text{if } i \text{ is even}, \end{cases}$$

if $q \leq i \leq n-2$,

$$w(x_{i,j,k}x_{i+1,j,k}) = \begin{cases} tm(2n-1) + 1, & \text{if } i \text{ is odd},\\ tm(2n+1) + 1, & \text{if } i \text{ is even}. \end{cases}$$

and if $n \leq i \leq 2n-2$,

$$w(x_{i,j,k}x_{i+1,j,k}) = \begin{cases} tm(2n+1) + 1, & \text{if } i \leq 2n-q-1, i \text{ is odd},\\ tm(2n-1) + 1, & \text{if } i \leq 2n-q-1, i \text{ is even},\\ 2tmn + m + 1, & \text{if } i = 2n-q,\\ 2tmn + m + 1, & \text{if } i \geq 2n-q+1, i \text{ is odd},\\ 2tm(n-1) + m + 1, & \text{if } i \geq 2n-q+1, i \text{ is even}. \end{cases}$$

Note that for $1 \leq j \leq t$ and $1 \leq k \leq m$,

$$w(x_{i,j,k}x_{i+1,j,k}) = \begin{cases} w(x_{0,0,k}x_{q,1,k}), & \text{if } 1 \leq i \leq q-2, i \text{ is odd},\\ w(x_{0,0,k}x_{q-1,1,k}), & \text{if } 1 \leq i \leq q-2, i \text{ is even},\\ w(x_{0,0,k}x_{q-1,t,k}), & \text{if } q \leq i \leq n-2, i \text{ is odd},\\ w(x_{0,0,k}x_{q,t,k}), & \text{if } q \leq i \leq n-2, i \text{ is even},\\ w(x_{0,0,k}x_{q,t,k}), & \text{if } n \leq i \leq 2n-q-1, i \text{ is odd},\\ w(x_{0,0,k}x_{q-1,t,k}), & \text{if } n \leq i \leq 2n-q-1, i \text{ is even},\\ w(x_{0,0,k}x_{q,1,k}), & \text{if } i = 2n-q,\\ w(x_{0,0,k}x_{q,1,k}), & \text{if } 2n-q+1 \leq i \leq 2n-2, i \text{ is odd},\\ w(x_{0,0,k}x_{q-1,1,k}), & \text{if } 2n-q+1 \leq i \leq 2n-2, i \text{ is even}. \end{cases}$$

Moreover, $w(x_{i,j,k}x_{2n-i,j,k}) = w(x_{0,0,k}x_{2n-q,1,k})$ and $w(x_{i,j,k}x_{2n-i-1,j,k}) = w(x_{0,0,k}x_{2n-q,t,k})$. Thus, the weights of the edges in tL_n, as well as the weights of the edges in the paths P_2 of the ladder graphs, are the same as the weights of some of the edges connecting P_m and tL_n.

Since $1 \leq j \leq t$ and all $w(x_{0,0,k}x_{q-1,j,k})$, $w(x_{0,0,k}x_{q,j,k})$, and $w(x_{0,0,k}x_{2n-q,j,k})$ depend on j, there are $3t$ different edge weights. These weights are different from the weights of the edges in P_m. Consequently, there are $2+3t$ distinct edge weights. Therefore, we can conclude that $\chi'_{lea}(P_m \rhd_o tL_n) \leq 2+3t$. Since $\Delta(P_m \rhd_o tL_n) = 2+3t$, we get $\chi'_{lea}(P_m \rhd_o tL_n) \geq 2+3t$. In conclusion, $\chi'_{lea}(P_m \rhd_o tL_n) = 2+3t$ if $o = v_q$ is a vertex of degree 3 in L_n, where q is odd, and o is not adjacent to any vertex of degree 2 in L_n.

Subsubcase 2.2.2. q is even.

For $m, n \geq 3$ and $t \geq 1$, define $f : V(P_m \rhd_o tL_n) \to \{1, .., tm(2n-1)+m\}$ as follows.

For $1 \leq k \leq m$,

$$f(x_{0,0,k}) = \begin{cases} tm(2n-q)+m-\frac{k-1}{2}, & \text{if } k \text{ is odd},\\ tm(2n-q)+\frac{k}{2}, & \text{if } k \text{ is even}. \end{cases}$$

For $1 \leq j \leq t$ and $1 \leq k \leq m$, if $1 \leq i \leq q-1$,

$$f(x_{i,j,k}) = \begin{cases} tm(i-1)+m(j-1)+\frac{k+1}{2}, & \text{if } i \text{ is odd}, k \text{ is odd},\\ tm(i-1)+mj-\frac{k-2}{2}, & \text{if } i \text{ is odd}, k \text{ is even},\\ tm(2n-i)-m(j-2)-\frac{k-1}{2}, & \text{if } i \text{ is even}, k \text{ is odd},\\ tm(2n-i)-m(j-1)+\frac{k}{2}, & \text{if } i \text{ is even}, k \text{ is even}, \end{cases}$$

if $q \leq i \leq n-1$,

$$f(x_{i,j,k}) = \begin{cases} tm(2n-i)-m(j-1)-\frac{k-1}{2}, & \text{if } i \text{ is odd}, k \text{ is odd},\\ tm(2n-i)-mj+\frac{k}{2}, & \text{if } i \text{ is odd}, k \text{ is even},\\ tmi+m(j-1)+\frac{k+1}{2}, & \text{if } i \text{ is even}, k \text{ is odd},\\ tmi+mj-\frac{k-2}{2}, & \text{if } i \text{ is even}, k \text{ is even}, \end{cases}$$

if $n \le i \le 2n-q$,

$$f(x_{i,j,k}) = \begin{cases} tm(i+1) - m(j-1) - \frac{k-1}{2}, & \text{if } i \text{ is odd}, k \text{ is odd},\\ tm(i+1) - mj + \frac{k}{2}, & \text{if } i \text{ is odd}, k \text{ is even},\\ tm(2n-i-1) + m(j-1) + \frac{k+1}{2}, & \text{if } i \text{ is even}, k \text{ is odd},\\ tm(2n-i-1) + mj - \frac{k-2}{2}, & \text{if } i \text{ is even}, k \text{ is even}, \end{cases}$$

and if $2n-q+1 \le i \le 2n-1$,

$$f(x_{i,j,k}) = \begin{cases} tmi - m(j-2) - \frac{k-1}{2}, & \text{if } i \text{ is odd}, k \text{ is odd},\\ tmi - m(j-1) + \frac{k}{2}, & \text{if } i \text{ is odd}, k \text{ is even},\\ tm(2n-i-1) + m(j-1) + \frac{k+1}{2}, & \text{if } i \text{ is even}, k \text{ is odd},\\ tm(2n-i-1) + mj - \frac{k-2}{2}, & \text{if } i \text{ is even}, k \text{ is even}. \end{cases}$$

From the labeling f, we obtain the weights of edges in path P_m for $1 \le k \le m$ are

$$w(x_{0,0,k}x_{0,0,k+1}) = \begin{cases} 2tm(2n-q) + m + 1, & \text{if } k \text{ is odd},\\ 2tm(2n-q) + m, & \text{if } k \text{ is even}, \end{cases}$$

the weights of the edges connecting P_m and tL_n for $1 \le j \le t$ and $1 \le k \le m$ are

$$w(x_{0,0,k}x_{q-1,j,k}) = 2tm(n-1) + mj + 1,$$

$$w(x_{0,0,k}x_{q,j,k}) = 2tmn + mj + 1,$$

$$w(x_{0,0,k}x_{2n-q,j,k}) = tm(2n-1) + mj + 1,$$

and the weights of the edges in the paths P_2 of the ladder graphs for $1 \le i \le q-1$, $1 \le j \le t$, and $1 \le k \le m$ are

$$w(x_{i,j,k}x_{2n-i,j,k}) = tm(2n-1) + m + 1,$$

and for $q \le i \le n-1$, $1 \le j \le t$, and $1 \le k \le m$ are

$$w(x_{i,j,k}x_{2n-i-1,j,k}) = 2tmn + 1.$$

Finally, the weights of the remaining edges in L_n for $1 \le j \le t$ and $1 \le k \le m$ are as follows.
If $1 \le i \le q-2$,

$$w(x_{i,j,k}x_{i+1,j,k}) = \begin{cases} 2tm(n-1) + m + 1, & \text{if } i \text{ is odd},\\ 2tmn + m + 1, & \text{if } i \text{ is even}, \end{cases}$$

if $q \le i \le n-2$,

$$w(x_{i,j,k}x_{i+1,j,k}) = \begin{cases} tm(2n+1) + 1, & \text{if } i \text{ is odd},\\ tm(2n-1) + 1, & \text{if } i \text{ is even}, \end{cases}$$

and if $n \leq i \leq 2n-2$,

$$w(x_{i,j,k}x_{i+1,j,k}) = \begin{cases} tm(2n-1)+1, & \text{if } i \leq 2n-q-1, i \text{ is odd,} \\ tm(2n+1)+1, & \text{if } i \leq 2n-q-1, i \text{ is even,} \\ 2tmn+m+1, & \text{if } i = 2n-q, \\ 2tm(n-1)+m+1, & \text{if } i \geq 2n-q+1, i \text{ is odd,} \\ 2tmn+m+1, & \text{if } i \geq 2n-q+1, i \text{ is even.} \end{cases}$$

Note that for $1 \leq j \leq t$ and $1 \leq k \leq m$,

$$w(x_{i,j,k}x_{i+1,j,k}) = \begin{cases} w(x_{0,0,k}x_{q-1,1,k}), & \text{if } 1 \leq i \leq q-2, i \text{ is odd,} \\ w(x_{0,0,k}x_{q,1,k}), & \text{if } 1 \leq i \leq q-2, i \text{ is even,} \\ w(x_{0,0,k}x_{q,t,k}), & \text{if } q \leq i \leq n-2, i \text{ is odd,} \\ w(x_{0,0,k}x_{q-1,t,k}), & \text{if } q \leq i \leq n-2, i \text{ is even,} \\ w(x_{0,0,k}x_{q-1,t,k}), & \text{if } n \leq i \leq 2n-q-1, i \text{ is odd,} \\ w(x_{0,0,k}x_{q,t,k}), & \text{if } n \leq i \leq 2n-q-1, i \text{ is even,} \\ w(x_{0,0,k}x_{q,1,k}), & \text{if } i = 2n-q, \\ w(x_{0,0,k}x_{q-1,1,k}), & \text{if } 2n-q+1 \leq i \leq 2n-2, i \text{ is odd,} \\ w(x_{0,0,k}x_{q,1,k}), & \text{if } 2n-q+1 \leq i \leq 2n-2, i \text{ is even.} \end{cases}$$

Moreover, $w(x_{i,j,k}x_{2n-i,j,k}) = w(x_{0,0,k}x_{2n-q,1,k})$ and $w(x_{i,j,k}x_{2n-i-1,j,k}) = w(x_{0,0,k}x_{2n-q,t,k})$. Thus, the weights of the edges in tL_n, as well as the weights of the edges in the paths P_2 of the ladder graphs, are the same as the weights of some of the edges connecting P_m and tL_n.

Since $1 \leq j \leq t$ and all $w(x_{0,0,k}x_{q-1,j,k})$, $w(x_{0,0,k}x_{q,j,k})$, and $w(x_{0,0,k}x_{2n-q,j,k})$ depend on j, there are $3t$ different edge weights. These weights are different from the weights of the edges in P_m. Consequently, there are $2+3t$ distinct edge weights. Therefore, we can conclude that $\chi'_{lea}(P_m \rhd_o tL_n) \leq 2+3t$. Since $\Delta(P_m \rhd_o tL_n) = 2+3t$, we get $\chi'_{lea}(P_m \rhd_o tL_n) \geq 2+3t$. In conclusion, $\chi'_{lea}(P_m \rhd_o tL_n) = 2+3t$ if $o = v_q$ is a vertex of degree 3 in L_n, where q is even, and o is not adjacent to any vertex of degree 2 in L_n

Subsubcases 2.2.1 and 2.2.2 proved that $\chi'_{lea}(P_m \rhd_o tL_n) = 2+3t$ if o is a vertex of degree 3 in L_n and is not adjacent to any vertex of degree 2 in L_n.

From Subcases 2.1 and 2.2, it can be concluded that $\chi'_{lea}(P_m \rhd_o tL_n) = 2+3t$ if o is a vertex of degree 3 in L_n. □

Examples of local edge antimagic labeling of $P_4 \rhd_o 2L_5$ are presented in Figures 9 and 10. In Figure 9, the graph is constructed by identifying a vertex of degree 2 in L_5 and $\chi'_{lea}(P_4 \rhd_o 2L_5) = 6$. Meanwhile, in Figure 10, the graph is obtained by identifying a vertex of degree 3 in L_5 and $\chi'_{lea}(P_4 \rhd_o 2L_5) = 8$.

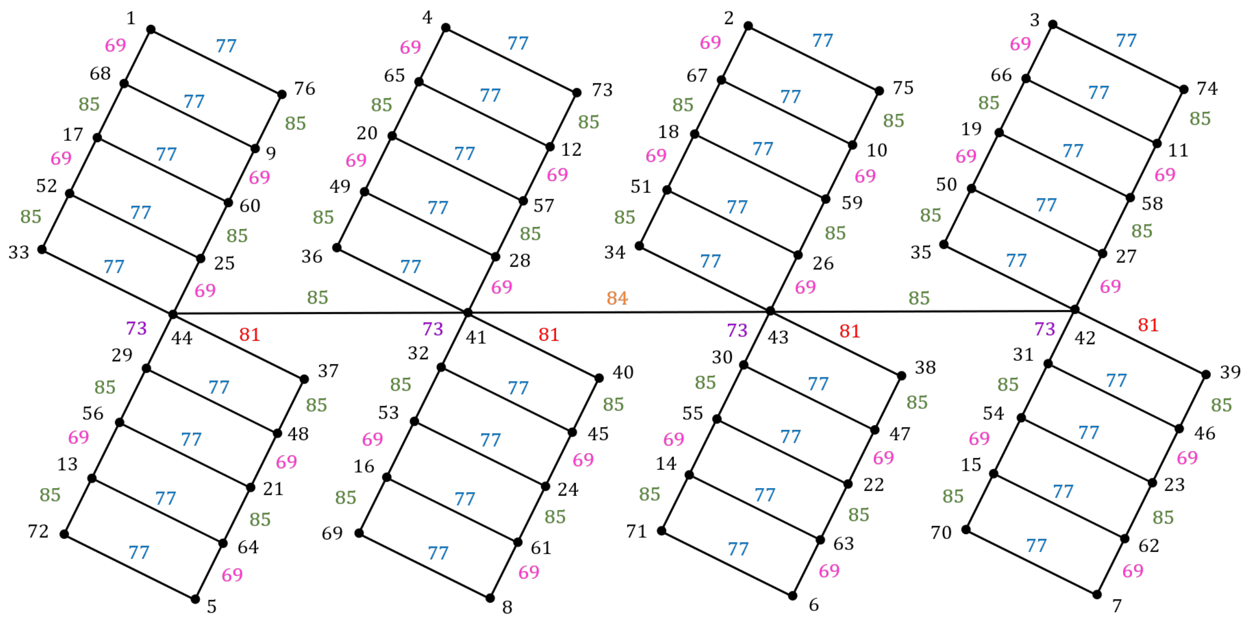

Figure 9. Local edge antimagic labeling of $P_4 \rhd_o 2L_5$ where o is a vertex with degree 2 and $\chi'_{lea}(P_4 \rhd_o 2L_5) = 6$.

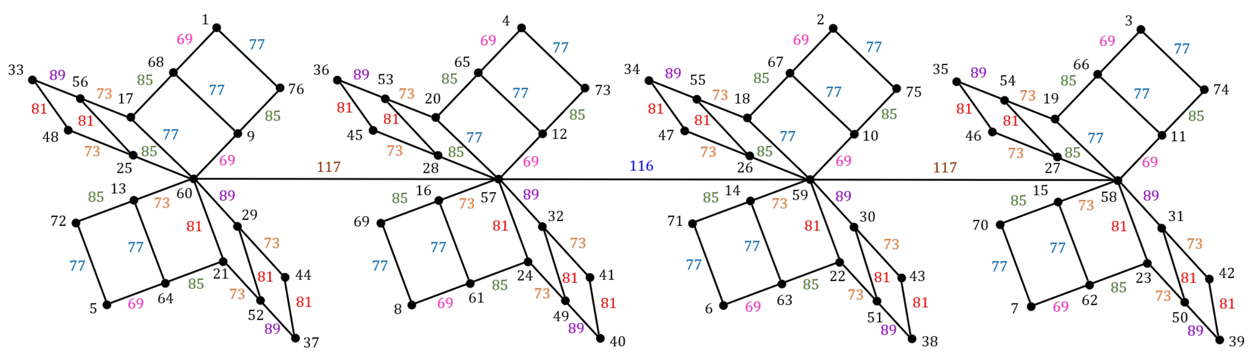

Figure 10. Local edge antimagic labeling of $P_4 \rhd_o 2L_5$, where o is a vertex with degree 3 and $\chi'_{lea}(P_4 \rhd_o 2L_5) = 8$.

In the proof of Theorem 2.6, Case 2, it appears that the weights of the edges in P_m are different from the weights of other edges in $P_m \rhd_o tL_n$. There are 2 different weights on the edges in P_m. If the edges in P_m are deleted, then the number of different edge weights and the maximum degree of the resulting graph will both be reduced by 2. Therefore, we get Corollary 2.7.

Corollary 2.7. *Let L_n be a path of order n and v be a vertex of degree 3 of L_n. For $n \geq 3$ and $t \geq 1$, the local edge antimagic chromatic number of copies of the graph obtained by identifying the vertex v from each copy of L_n in tL_n is $3t$.*

3. Conclusions

In this study, we investigated the local edge antimagic chromatic number of comb product of path with copies of path, path with copies of even cycle, and path with copies of ladder. In addition, we have also found the local edge antimagic chromatic number of copies of graph obtained by identifying a particular vertex of degree 2 from some copies of path and copies of the graph obtained by identifying a particular vertex of degree 3 from some copies of ladder. The results found leads us to the following open problem.

Open Problem 1. *Determine the local edge antimagic chromatic number of $P_m \rhd_o tC_n$ where $m, n \geq 3$, n is odd, and $t \geq 1$.*

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