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Connected size Ramsey numbers of matchings versus a small path or cycle

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Abstract

Given two graphs G_1, G_2, the connected size Ramsey number $\hat{r}_c(G_1, G_2)$ is defined to be the minimum number of edges of a connected graph G, such that for any red-blue edge colouring of G, there is either a red copy of G_1 or a blue copy of G_2. Concentrating on $\hat{r}_c(nK_2, G_2)$ where nK_2 is a matching, we generalise and improve two previous results as follows. Vito, Nabila, Safitri, and Silaban (J. Phys. Conf. Ser., 2021) obtained the exact values of $\hat{r}_c(nK_2, P_3)$ for $n = 2, 3, 4$. We determine its exact values for all positive integers n. Rahadjeng, Baskoro, and Assiyatun (Proc. Indian Acad. Sci.: Math. Sci., 2017) proved that $\hat{r}_c(nK_2, C_4) \leq 5n - 1$ for $n \geq 4$. We improve the upper bound from $5n - 1$ to $\lfloor (9n - 1)/2 \rfloor$. In addition, we show a result which has the same flavour and has exact values: $\hat{r}_c(nK_2, C_3) = 4n - 1$ for all positive integers n.

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1. Introduction

Graph Ramsey theory is currently among the most active areas in combinatorics. Two of the main parameters in the theory are the Ramsey number and the size Ramsey number, which are defined as follows. Given two graphs G_1 and G_2, we write $G \to (G_1, G_2)$ if for any edge colouring

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of G such that each edge is coloured either red or blue, the graph G always contains either a red copy of G_1 or a blue copy of G_2. The *Ramsey number* $r(G_1, G_2)$ is the smallest possible number of vertices in a graph G satisfying $G \to (G_1, G_2)$. The *size Ramsey number* $\hat{r}(G_1, G_2)$ is the smallest possible number of edges in a graph G satisfying $G \to (G_1, G_2)$. That is to say, $r(G_1, G_2) = \min\{|V(G)| : G \to (G_1, G_2)\}$, and $\hat{r}(G_1, G_2) = \min\{|E(G)| : G \to (G_1, G_2)\}$.

The size Ramsey number was introduced by Erdős, Faudree, Rousseau, and Schelp [4] in 1978. Some variants have also been studied since then. In 2015, Rahadjeng, Baskoro, and Assiyatun [5] initiated the study of such a variant called connected size Ramsey number by adding the condition that G is connected. Formally speaking, the *connected size Ramsey number* $\hat{r}_c(G_1, G_2)$ is the smallest possible number of edges in a connected graph G satisfying $G \to (G_1, G_2)$. It is easy to see that $\hat{r}(G_1, G_2) \le \hat{r}_c(G_1, G_2)$, and equality holds when both G_1 and G_2 are connected graphs. But the latter parameter seems trickier when G_1 or G_2 is disconnected. The previous results are mainly concerned with the connected size Ramsey numbers of a matching versus a sparse graph such as a path, a star, and a cycle.

Let nK_2 be a matching with n edges, and P_m a path with m vertices. Vito, Nabila, Safitri, and Silaban [7] gave an upper bound of $\hat{r}_c(nK_2, P_m)$, and the exact values of $\hat{r}_c(nK_2, P_3)$ for $n = 2, 3, 4$.

Theorem 1.1. *[7] For $n \ge 1$, $m \ge 3$,* $\hat{r}_c(nK_2, P_m) \le \begin{cases} n(m+2)/2 - 1, & \text{if } n \text{ is even}, \\ (n+1)(m+2)/2 - 3, & \text{if } n \text{ is odd}. \end{cases}$
Equality holds for $m = 3$ and $1 \le n \le 4$.

Other related results can be found in [1]. If m is much larger than n, this upper bound cannot be tight. Because Erdős and Faudree [3] constructed a connected graph which implies $\hat{r}_c(nK_2, P_m) \le m + c\sqrt{m}$, where c is a constant depending on n. But for small m, the above upper bound can be tight. Our first result determines the exact values of $\hat{r}_c(nK_2, P_3)$ for all positive integers n, which generalises the equality of Theorem 1.1.

Theorem 1.2. *For all positive integers n, we have $\hat{r}_c(nK_2, P_3) = \lfloor (5n-1)/2 \rfloor$.*

Rahadjeng, Baskoro, and Assiyatun [6] proved that $\hat{r}_c(nK_2, C_4) \le 5n - 1$ for $n \ge 4$. This upper bound can be improved from $5n - 1$ to $\lfloor (9n-1)/2 \rfloor$.

Theorem 1.3. *For all positive integers n, we have $\hat{r}_c(nK_2, C_4) \le \lfloor (9n-1)/2 \rfloor$.*

Now we prove the theorem by constructing a graph with $\lfloor (9n-1)/2 \rfloor$ edges. Let $K_{3,3} - e$ be the graph $K_{3,3}$ with one edge deleted. It is easy to check that $K_{3,3} - e \to (2K_2, C_4)$. We use nG to denote n disjoint copies of G. If n is even, then $\frac{n}{2}(K_{3,3} - e) \to (nK_2, C_4)$. The graph $\frac{n}{2}(K_{3,3} - e)$ has $n/2$ components and can be connected by adding $n/2 - 1$ edges. If n is odd, then $\frac{n-1}{2}(K_{3,3} - e) \cup C_4 \to (nK_2, C_4)$. The graph $\frac{n-1}{2}(K_{3,3} - e) \cup C_4$ has $(n+1)/2$ components and can be connected by adding $(n-1)/2$ edges. In both cases, we obtain a connected graph with $\lfloor (9n-1)/2 \rfloor$ edges and hence the upper bound follows.

It seems likely that the determination of $\hat{r}_c(nK_2, C_4)$ for all n is tricky. We believe the upper bound is tight and pose the following conjecture.

Conjecture 1. *For all positive integers n, $\hat{r}_c(nK_2, C_4) = \lfloor (9n-1)/2 \rfloor$.*

Even though solving the above conjecture seems out of our reach, we show a result which has the same flavour and has exact values: $\hat{r}_c(nK_2, C_3) = 4n - 1$.

Theorem 1.4. *For all positive integers n, we have $\hat{r}_c(nK_2, C_3) = 4n - 1$.*

Proofs of Theorem 1.2 and Theorem 1.4 will be presented in Section 2 and Section 3, respectively. To prove the lower bounds, we need to discuss the connectivity of a graph G. If G is not 2-connected, the basic properties of blocks and end blocks are needed, which can be found in Bondy and Murty [2, Chap. 5.2]. Moreover, the following terminology is used frequently in the proofs. We say G has a (G_1, G_2)-*colouring* if there is a red-blue edge colouring of G such that G contains neither a red G_1 nor a blue G_2. Thus, it is equivalent to $G \not\to (G_1, G_2)$.

2. A matching versus P_3

For the upper bound, we know that $C_4 \to (2K_2, P_3)$. If n is even, then $\frac{n}{2}C_4 \to (nK_2, P_3)$. The graph $\frac{n}{2}C_4$ has $n/2$ components and can be connected by adding $n/2 - 1$ edges. If n is odd, then $\frac{n-1}{2}C_4 \cup P_3 \to (nK_2, P_3)$. The graph $\frac{n-1}{2}C_4 \cup P_3$ has $(n+1)/2$ components and can be connected by adding $(n-1)/2$ edges. In both cases, we obtain a connected graph with $\lfloor (5n-1)/2 \rfloor$ edges and hence the upper bound follows.

For the lower bound, we use induction on n. The result is obvious for $n = 1, 2$. Assume that for $k < n$ and any connected graph G with $\lfloor (5k-3)/2 \rfloor$ edges, we have $G \not\to (kK_2, P_3)$. Now consider G to be a connected graph with minimum number of edges such that $G \to (nK_2, P_3)$. Thus, for any proper connected subgraph G' of G, we have $G' \not\to (nK_2, P_3)$. Since $n \geq 3$, G has at least six edges. Suppose to the contrary that G has at most $\lfloor (5n-3)/2 \rfloor$ edges. We will deduce a contradiction and hence $\hat{r}_c(nK_2, P_3) \geq \lfloor (5n-1)/2 \rfloor$.

An edge set E_1 of a connected graph G is called *deletable*, if E_1 satisfies the following conditions:

(a) E_1 can be partitioned into two edge sets E_2 and E_3, where E_2 forms a star and E_3 forms a matching;

(b) any edge of $E(G) \setminus E_1$ is nonadjacent to E_3;

(c) the graph induced by $E(G) \setminus E_1$ is still connected.

Note that for a deletable edge set E_1, the graph $G - E_1$ may have some isolated vertices, but all edges of $G - E_1$ belong to the same connected component. We have the following property of a deletable edge set.

Claim 2.1. *Every deletable edge set has size at most two.*

Proof. Let E_1 be a deletable edge set. If $|E_1| \geq 3$, then the graph induced by $E(G) \setminus E_1$ has at most $\lfloor (5n-3)/2 \rfloor - 3$ edges and hence an $((n-1)K_2, P_3)$-colouring by induction. We then colour all edges of E_2 red and all edges of E_3 blue. This is a (nK_2, P_3)-colouring of G, a contradiction. □

A *non-cut vertex* of a connected graph is a vertex whose deletion still results in a connected graph. That is, every vertex of a nontrivial connected graph is either a cut vertex or a non-cut vertex. Since E_3 in the definition of a deletable edge set can be empty, the edges incident to a non-cut vertex form a deletable edge set. We have the following direct corollary.

Claim 2.2. *Every non-cut vertex has degree at most two.*

If G is a 2-connected graph, by Claim 2.2, G is a cycle. Beginning from any edge of G, we may colour all edges consecutively along the cycle. We alternately colour two edges red and one edge blue, until all edges of G have been coloured. Obviously G contains no blue P_3. From $(5n-3)/2 \leq 3(n-1)$ and the colouring of G we see that G contains no red matching with n edges. Thus, $G \not\to (nK_2, P_3)$.

Now we assume that G is connected but not 2-connected. Recall that a *block* of a graph is a subgraph that is nonseparable and is maximal with respect to this property. An *end block* is a block that contains exactly one cut vertex of G. We have the following observation.

Claim 2.3. *Every end block is either a K_2 or a cycle.*

Proof. Let B be an end block with at least three vertices, and let v be the single cut vertex of G that is contained in B. Since B is 2-connected, the subgraph $B - v$ is still connected. By Claim 2.2, every non-cut vertex has degree two. It follows that $B - v$ is either a path or a cycle. We see that v has two neighbours in B. If not, v has at least three neighbours in B, each of which has degree one in $B - v$. Since a path has two vertices of degree one and a cycle has no vertex of degree one, $B - v$ is neither a path nor a cycle, a contradiction. Hence, every vertex of B has two neighbours in B. Since B is 2-connected, it must be a cycle. □

Since G is not 2-connected, there is at least one cut vertex. Choose any cut vertex as a *root*, denoted by r. For a vertex u of G, if any path from u to r must pass through a cut vertex v, then u is called a *(vertex) descendant* of v. For any edge e of G, if both ends of e are descendants of v, then e is called an *edge descendant* of v. For a cut vertex v, the block containing v but no other descendant of v is called a *parent block* of v. It is obvious that every cut vertex has a unique parent block, except that the root r has no parent block. If v is a cut vertex but every descendant of v is not a cut vertex of G, we call v an *end-cut*. It is obvious that G has at least one end-cut. We have the following property of end-cuts.

Claim 2.4. *Every end-cut is contained in a unique end block, which is K_2. Moreover, if an end-cut is not the root of G, its parent block is also K_2.*

Proof. Let v be an end-cut. If v is not the root r, by the definition of end-cut, every block containing v is an end block, except for its parent block. If we delete v and all descendants of v from G, the induced subgraph is still connected, denoted by G'. This is because, any vertex of G' is not a descendant of v. For any two vertices of G', there is a path joining them in G without passing through v. So the path still exists in G' and hence G' is connected. In the following, regardless of whether v is the root or not, we first colour all edges incident to v red, then give a colouring of all edge descendants of v. After that, we find a colouring of G' by the inductive hypothesis. We prove that this edge colouring of G is an (nK_2, P_3)-colouring under certain conditions.

By Claim 2.3, every end block is either a K_2 or a cycle. Assume that v has t_1 neighbours in its parent block. Note that if v is the root of G, then $t_1 = 0$. Assume v is contained in t_2 blocks which are K_2, and in t blocks which are cycles. Let $p_1+2, p_2+2, \ldots, p_t+2$ be the cycle lengths of these t cycles. If we remove v from G, the cycles become t disjoint paths with lengths $p_1, p_2, \ldots, p_t$ respectively. We colour all edges incident to v red. For each path with length p_i, where $1 \leq i \leq t$, we colour all edges from one leaf to the other leaf consecutively along the path, alternately with one edge blue and two edges red. Now we have coloured $x := t_1+t_2+2t+p_1+p_2+\cdots+p_t$ edges, no blue P_3 appears, and the maximum red matching has $y := 1+\lfloor (p_1+1)/3 \rfloor + \cdots + \lfloor (p_t+1)/3 \rfloor$ edges.

If v is the root, then we have already coloured all edges of G. So $x \leq \lfloor (5n-3)/2 \rfloor$, and we need to check that $y \leq n-1$. Since G has at least six edges, we have $6t_2 + 7t + p_1 \geq 11$. Thus, $n \geq (2x+3)/5 \geq (2t_2 + 4t + 2p_1 + \cdots + 2p_t + 3)/5 \geq (t + p_1 + \cdots + p_t + 4)/3 > y$. This implies that $G \not\to (nK_2, P_3)$.

If v is not the root, recall that G' is formed by the remaining edges and is connected. We can use the inductive hypothesis. The graph G' has at most $\lfloor (5n-3)/2 \rfloor - x$ edges, which is $\lfloor (5(n-2x/5)-3)/2 \rfloor \leq \lfloor (5(n-\lfloor 2x/5 \rfloor)-3)/2 \rfloor$. So G' has an $((n-\lfloor 2x/5 \rfloor)K_2, P_3)$-colouring. It is not difficult to check that G has no blue P_3, and the maximum red matching has at most $y + n - \lfloor 2x/5 \rfloor - 1$ edges. It is left to show that under what conditions $y + n - \lfloor 2x/5 \rfloor - 1$ is less than n. So we deduce a contradiction.

Since v has at least one neighbour in its parent block, it follows that $t_1 \geq 1$. If $t \geq 1$, then $1 + (t-1)/3 \leq (4t+1)/5$, and $\lfloor (p_1+1)/3 \rfloor \leq (2p_1+1)/5$. Thus,

$$\begin{aligned}
y &= 1 + \lfloor (p_1+1)/3 \rfloor + \lfloor (p_2+1)/3 \rfloor + \cdots + \lfloor (p_t+1)/3 \rfloor \\
&\leq 1 + \lfloor (p_1+1)/3 \rfloor + (p_2+1)/3 + \cdots + (p_t+1)/3 \\
&\leq 1 + \lfloor (p_1+1)/3 \rfloor + (t-1)/3 + 2(p_2 + \cdots + p_n)/5 \\
&\leq (4t+1)/5 + (2p_1+1)/5 + 2(p_2 + \cdots + p_n)/5 \\
&\leq 2(t_1 + t_2 + 2t + p_1 + p_2 + \cdots + p_t)/5 = 2x/5.
\end{aligned}$$

If $t = 0$ and $t_1 + t_2 \geq 3$, then $y = 1 < 6/5 \leq 2x/5$. In both cases, we have $y \leq 2x/5$ and hence $y + n - \lfloor 2x/5 \rfloor - 1 < n$.

Now we consider the remainder case when $t = 0$ and $t_1 + t_2 \leq 2$. Since v is an end-cut, we have $t_2 \geq 1$. Thus, $t_1 = t_2 = 1$. It follows from $t = 0$ and $t_2 = 1$ that v is contained in a unique end block which is K_2. It follows from $t_1 = 1$ that v has only one neighbour in its parent block. Since each block is either a K_2 or a 2-connected subgraph, the parent block of v must be K_2. □

Let v be an end-cut. By Claim 2.4, it cannot be the root of G. Let u be the other end of its parent block, and v^+ the descendant of v. If u is contained in an end block which is an edge uu^+, then uv, uu^+, vv^+ form a deletable edge set. If u is contained in an end block which is not an edge, by Claim 2.3, the end block is a cycle. Let u^+ be a neighbour of u on the cycle. Then uv, uu^+, vv^+ form a deletable edge set. If u has at least two end-cuts as its descendants, let w be another end-cut and uw, ww^+ edge descendants of u. Then uv, vv^+, uw, ww^+ form a deletable edge set. If u has only one end-cut as its descendant, which is v, then all edges incident to u and the edge vv^+ form a deletable edge set. By Claim 2.1, each of the above cases leads to a contradiction. This completes the proof of the lower bound.

3. A matching versus C_3

Now we prove Theorem 1.4. The graph nC_3 has n components and can be connected by adding $n-1$ edges. Denote this connected graph by H. It follows from $nC_3 \to (nK_2, C_3)$ that $H \to (nK_2, C_3)$. Thus, $\hat{r}_c(nK_2, C_3) \leq 4n-1$.

Set $\mathcal{G} = \{G : |E(G)| \leq 4n-2, \text{and } G \to (nK_2, C_3)\}$. We will prove that $\mathcal{G}$ is an empty set and hence the lower bound follows. Suppose not, choose a graph G from $\mathcal{G}$ with minimum order and minimum size subjective to its order. Thus, for any proper connected subgraph G' of G with at most $4k-2$ edges, we have $G' \not\to (kK_2, C_3)$. We present the proof through a sequence of claims.

Claim 3.1. *The minimum degree of G is at least two.*

Proof. Suppose that G has a vertex v of degree one. Then $G - v$ has an (nK_2, C_3)-colouring. It can be extended to an (nK_2, C_3)-colouring of G by colouring the edge incident to v blue, which contradicts our assumption that $G \to (nK_2, C_3)$. Thus, $\delta(G) \geq 2$. □

Claim 3.2. *The graph G has no cut edge.*

Proof. Suppose that G has a cut edge e. Then $G - e$ has two connected components X and Y. Let k_1, k_2 be the integers such that $4k_1 - 5 \leq |E(X)| \leq 4k_1 - 2$ and $4k_2 - 5 \leq |E(Y)| \leq 4k_2 - 2$ respectively. Then X has a (k_1K_2, C_3)-colouring and Y has a (k_2K_2, C_3)-colouring. It can be extended to an $((k_1 + k_2 - 1)K_2, C_3)$-colouring of G by colouring e blue. So the maximum red matching has at most $k_1 + k_2 - 2$ edges. From $(4k_1 - 5) + (4k_2 - 5) + 1 \leq |E(X)| + |E(Y)| + 1 \leq 4n - 2$ we deduce that $k_1 + k_2 - 2 \leq n - 1/4$, a contradiction which implies our claim. □

Claim 3.3. *The graph G is 2-connected.*

Proof. If G is not 2-connected, let v be a cut vertex of G, and $B_1, B_2, \ldots, B_\ell$ the blocks containing v, where $\ell \geq 2$. If v has only one neighbour in B_i for some i with $1 \leq i \leq \ell$, then B_i is not 2-connected. Since any block is either 2-connected or a K_2, B_i has to be K_2. Hence, B_i is a cut edge [8, Chap. 4.1.18], which contradicts Claim 3.2. Thus, for each B_i with $1 \leq i \leq \ell$, v has at least two neighbours in B_i.

The vertex set $V(G)$ can be partitioned into two parts X and Y as follows. If any path from u to v has to pass through a vertex of B_1 other than v, then $u \in X$; otherwise $u \in Y$. Let G_1 and G_2 be the subgraphs induced by $X \cup \{v\}$ and Y respectively. It is obvious that G_1 contains B_1 and G_2 contains $B_2 \cup \cdots \cup B_\ell$. And they share only one vertex, which is v. Let k_1, k_2 be the integers such that $4k_1 - 5 \leq |E(G_1)| \leq 4k_1 - 2$ and $4k_2 - 5 \leq |E(G_2)| \leq 4k_2 - 2$ respectively. Then G_1 has a (k_1K_2, C_3)-colouring and G_2 has a (k_2K_2, C_3)-colouring. Combining the two colourings we have an $((k_1 + k_2 - 1)K_2, C_3)$-colouring of G. So the maximum red matching has at most $k_1 + k_2 - 2$ edges. From $(4k_1 - 5) + (4k_2 - 5) \leq |E(G_1)| + |E(G_2)| \leq 4n - 2$ we deduce that $k_1 + k_2 - 2 \leq n$. If $k_1 + k_2 - 2 < n$, then this colouring is an (nK_2, C_3)-colouring of G. If $k_1 + k_2 - 2 = n$, then we have $|E(G_1)| = 4k_1 - 5$ and $|E(G_2)| = 4k_2 - 5$. We obtain an (nK_2, C_3)-colouring of G as follows. If $\ell = 2$, then both $B_1 - v$ and $B_2 - v$ are connected. Hence, both $G_1 - v$ and $G_2 - v$ are connected, which implies that they have a $((k_1 - 1)K_2, C_3)$-colouring and a $((k_2 - 1)K_2, C_3)$-colouring, respectively. It can be extended to an $((k_1 + k_2 - 2)K_2, C_3)$-colouring of G by colouring all edges incident to v red. Thus, $G \not\to (nK_2, C_3)$. If $\ell \geq 3$, then for each i with $2 \leq i \leq \ell$, we delete an edge vv_i from B_i. Both $G_1 - v$ and $G_2 - \{vv_2, \ldots, vv_\ell\}$ are connected. So they have a $((k_1 - 1)K_2, C_3)$-colouring and a $((k_2 - 1)K_2, C_3)$-colouring, respectively. It can be extended to an $((k_1 + k_2 - 2)K_2, C_3)$-colouring of G by colouring the remaining edges red. Again, $G \not\to (nK_2, C_3)$. □

Claim 3.4. *The maximum degree of G is at most three.*

Proof. For any vertex v of G, by Claim 3.3, $G - v$ is still connected. If $d(v) \geq 4$, $G - v$ has at most $4(n-1) - 2$ edges and hence an $((n-1)K_2, C_3)$-colouring by the choice of G. It can be extended to an (nK_2, C_3)-colouring of G by colouring all edges incident to v red, a contradiction. Thus, the maximum degree of G is at most three. □

Claim 3.5. *The graph G is 3-regular.*

Proof. By Claim 3.1 and Claim 3.4, $2 \leq d(v) \leq 3$ for any vertex v of G. If G is 2-regular, by Claim 3.3, G is a cycle. If G is a triangle, then $n \geq 2$. We colour all edges of G red, which is a (nK_2, C_3)-colouring. If G is not a triangle, then we colour all edges of G blue, which is a (K_2, C_3)-colouring. Thus, G cannot be 2-regular.

If G is not 3-regular, then G has some vertices with degree two and some with degree three. There exist two adjacent vertices with degrees two and three respectively, denoted by v_1 and v_2. Since $G - v_2$ is connected, and v_1 has only one neighbour in $G - v_2$, it follows that $G - \{v_1, v_2\}$ is connected. This graph has at most $4(n-1) - 2$ edges and hence an $((n-1)K_2, C_3)$-colouring. It can be extended to an (nK_2, C_3)-colouring of G by colouring all edges incident to v_2 red and the remaining edge incident to v_1 blue, a contradiction which implies our claim. □

Claim 3.6. *Each edge of G is contained in at least one triangle.*

Proof. Suppose that G has an edge e which is not contained in any triangle. By Claim 3.2, $G - e$ is connected. It follows from the choice of G that $G - e$ has an (nK_2, C_3)-colouring. It can be extended to an (nK_2, C_3)-colouring of G by colouring e blue, a contradiction. □

Consider a triangle $v_1v_2v_3$. By Claim 3.5, each of v_1, v_2, v_3 has another neighbour, denoted by v_4, v_5, v_6 respectively. If v_4, v_5, v_6 are the same vertex, then v_1, v_2, v_3, v_4 forms a K_4. Since G is a 3-regular 2-connected graph, the whole graph G is a K_4 and $n \geq 2$. We colour the triangle $v_1v_2v_3$ red, and the other three edges blue. This is a $(2K_2, C_3)$-colouring of G, a contradiction. Thus, at least two of v_4, v_5, v_6 are distinct, say, v_4 and v_5 are two distinct vertices. The vertex v_3 cannot be adjacent to both v_4 and v_5, since otherwise $d(v_3) \geq 4$. Without loss of generality, assume that v_3 is not adjacent to v_4. Moreover, v_4 is not adjacent to v_2, since otherwise $d(v_2) \geq 4$. By Claim 3.6, v_1v_4 is contained in a triangle, denoted by $v_1v_4v_6$. Since v_6 is different from v_2, v_3, we have $d(v_1) \geq 4$, a final contradiction.

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