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Characterizing all trees with locating-chromatic number 3

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Abstract

Let c be a proper k-coloring of a connected graph G. Let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be the induced partition of $V(G)$ by c, where S_i is the partition class having all vertices with color i. The color code $c_\Pi(v)$ of vertex v is the ordered k-tuple $(d(v, S_1), d(v, S_2), \ldots, d(v, S_k))$, where $d(v, S_i) = \min\{d(v, x)|x \in S_i\}$, for $1 \leq i \leq k$. If all vertices of G have distinct color codes, then c is called a locating-coloring of G. The locating-chromatic number of G, denoted by $\chi_L(G)$, is the smallest k such that G posses a locating k-coloring. Clearly, any graph of order $n \geq 2$ has locating-chromatic number k, where $2 \leq k \leq n$. Characterizing all graphs with a certain locating-chromatic number is a difficult problem. Up to now, all graphs of order n with locating chromatic number $2, n-1$, or n have been characterized. In this paper, we characterize all trees whose locating-chromatic number is 3. We also give a family of trees with locating-chromatic number 4.

Keywords: Locating-chromatic number, graph, tree.
Mathematics Subject Classification : 05C12.

1. Introduction

Chartrand *et al.* [8] initiated the study on the locating-chromatic number of a graph. This notion is a special case of the partition dimension of a graph, namely the smallest integer k in which there exists a k-partition Π of the graph such that the coordinates of all vertices with respect to Π are distinct. Since then, various results have been obtained by different authors. However,

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determining the locating-chromatic number of any graph in general is classified as an *NP*-hard problem [8]. Furthermore, characterizing all graphs with a certain locating-chromatic number is also a difficult question.

In this paper, we consider only simple connected graphs. Let $G(V, E)$ be a graph. The *distance* $d(u, v)$ from vertex u to vertex v in G is the length of a shortest path from u to v. For $S \subseteq V(G)$, define the *distance* $d(v, S)$ from vertex v to set S as $\min\{d(v, x)|x \in S\}$. Let c be a k-coloring of G and $\Pi = \{S_1, S_2, \cdots, S_k\}$ be a partition of $V(G)$ induced by c, where S_i is the set of vertices receiving color i. The *color code* $c_\Pi(v)$ of v is defined as the ordered k-tuple $(d(v, S_1), d(v, S_2), \ldots, d(v, S_k))$. If all vertices of G have distinct color codes, then c is called a *locating-chromatic k-coloring* of G (*k-locating coloring*, in short). The *locating-chromatic number* $\chi_L(G)$ of graph G is the smallest k such that G has a locating k-coloring.

Chartrand *et al.* [8] determined the locating-chromatic numbers for some well-known classes of graphs, namely paths, cycles, complete multipartite graphs and double stars. The locating-chromatic number of a path P_n is 3, for $n \geq 3$. The locating-chromatic number of a cycle C_n is 3 if n is odd and 4 otherwise. Furthermore, Chartrand *et al.* [9] studied the locating-chromatic number of trees in general. They showed that for any integer $k \in \{3, 4, ..., n-2, n\}$, there exists a tree of order n with locating-chromatic number k. They also showed that no tree of order n exists with locating-chromatic number $n - 1$. Recently, Asmiati *et al.* [1, 3], determined the locating-chromatic number for an amalgamation of stars and firecracker graphs.

Some authors also consider the locating-chromatic number for graphs produced by a graph operation. For instances, Baskoro and Purwasih [4] determined the locating-chromatic number for the corona product of two graphs. Behtoei and Omoomi obtained the locating-chromatic number for the Cartesian product of graphs [6] and for the join product of graphs [7]. In particular, they also obtained the locating chromatic number of the fans, wheels and friendship graphs. In [5], Behtoei and Omoomi also considered the locating chromatic number of Kneser graphs.

Certainly, the only graph with locating-chromatic number n is a multipartite complete graph with n vertices. Furthermore, Chartrand *et al.* [9] also characterized all graphs on n vertices whose locating-chromatic number is $n-1$. In the same paper, they showed that if G is a connected graph of order $n \geq 5$ containing an induced subgraph $F \in \{2K_1 \cup K_2, P_2 \cup P_3, H_1, H_2, H_3, P_2 \cup K_3, P_2, C_5, C_5 + e\}$, then $\chi_L(G) \leq n - 2$. All graphs of order n with locating-chromatic number 3 are still not fully characterized. We know that $P_n, n \geq 3$, is an example of a graph with locating-chromatic number 3. Recently, we characterized all graphs containing a cycle with the locating-chromatic number 3 [2]. In this paper, we will determine all trees with locating-chromatic number 3. Therefore, this paper will complete the characterization of all graphs with locating-chromatic number 3. We also give a family of trees with locating-chromatic number 4.

2. Basic Properties

In this section, we give some definitions and basic properties related to graphs with locating-chromatic number 3. Let c be a locating k-coloring on graph $G(V, E)$. Let $\Pi = \{S_1, S_2, \cdots, S_k\}$ be the partition of $V(G)$ induced by c. A vertex $v \in G$ is called a *dominant vertex* if $d(v, S_i) = 1$ if $v \notin S_i$. A path connecting two dominant vertices in G is called a *clear path* if all of its internal vertices are not dominant. Then, we have the following lemma as a direct consequence of the definition of dominant vertices.

Lemma 2.1. *[2] Let G be a graph with $\chi_L(G) = k$. Then, there are at most k dominant vertices in G and all of them must receive different colors.*

Lemma 2.2. *[3] Let G be a graph with $\chi_L(G) = 3$. Then, the length of any clear path in G is odd.*

Lemma 2.3. *[3] Let G be a connected graph with $\chi_L(G) = 3$. If G contains three dominant vertices, then these three dominant vertices must lie in a path.*

3. Characterization

Consider two specific caterpillars $C(2,2,2)$ and $C(2,1,\overbrace{0,\cdots,0}^{t},1,2)$, for any odd t, as depicted in the left side of Figure 1. Let G_1 be the subdivision of $C(2,2,2)$ on six pendant edges in $k_1, k_2, \cdots, k_6$ times respectively, where $k_i \geq 1$. Let G_2 be the subdivision of $C(2,1,\overbrace{0,\cdots,0}^{t},1,2)$ on six pendant edges in $k_1, k_2, \cdots, k_6$ times respectively, where $k_i \geq 1$.

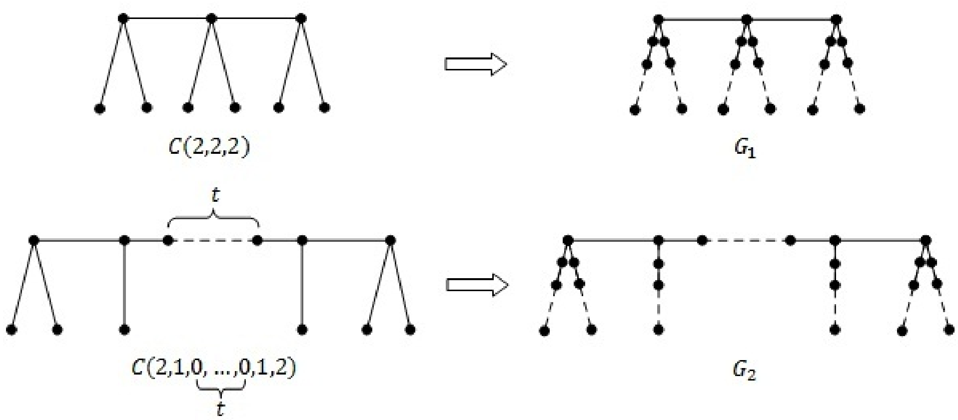

Figure 1. The specific caterpillars and their subdivisions.

Let $\mathcal{T}$ be the class of all trees whose locating-chromatic number is 3. In this section, we characterize all trees which are members of $\mathcal{T}$.

Lemma 3.1. *Let $T \in \mathcal{T}$. The color code of any vertex of T is (c_1, c_2, c_3) such that $\{c_1, c_2, c_3\} = \{0, 1, k\}$ where $k \geq 1$.*

Proof. Let $x \in T$ and without loss of generality assume $c(x) = 1$. Since the neighbor of x must have a different color then the color code of x is either $(0, 1, k)$ or $(0, k, 1)$, where $k \geq 1$. □

For any integer $k \geq 1$, a tree $T \in \mathcal{T}$ is called *k-maximal* if T has all possible color codes with k is the maximum ordinate. In this case, there is a locating coloring of T such that each color class in T has exactly $2k-1$ vertices. For example, graph $C(2,2,2)$ is 2-maximal, since this graph has a locating coloring with each color class having 3 vertices. It can be verified that a path on $6k-3$ vertices is k-maximal. However, not all tree on $6k-3$ vertices are k-maximal.

Lemma 3.2. *Let $T \in \mathcal{T}$. Every vertex x of T has degree at most 4.*

Proof. To the contrary, assume there is a vertex x with $d(x) \geq 5$. Let a_1, a_2, a_3, a_4, a_5 be the neighbors of x. Let c be a locating 3-coloring of T. Assume $c(x) = 1$ and so $c(a_i)$ is either 2 or 3, for any $i \in \{1,2,3,4,5\}$. If there are $i \neq j$ such that $c(a_i) \neq c(a_j)$ then there are at least three vertices a_i with the same color, say color 2. Thus, two of these vertices will have the same color code, a contradiction. Now, assume that the colors of all vertices a_i are the same, say $c(a_i) = 2$, for all $i \in \{1,2,\cdots,5\}$. Let $r = \min\{d(a_i, S_3)|\ i = 1,2,\cdots,5\}$, where S_3 is the partition class consisting of all vertices whose color is 3. Then, the possible color codes for vertices a_i are $(1,0,r), (1,0,r+1)$, or $(1,0,r+2)$. Therefore, we will have two vertices a_i with the same color code, a contradiction. □

From now on, let $T \in \mathcal{T}$. By Lemma 2.1, T has at most three dominant vertices. Clearly, if T is either a path P_3, or P_4, a double star $S_{1,2}$ or $S_{2,2}$, then T has a locating coloring such that T has only one or two dominant vertices. If T is not isomorphic to one of them, then T must have exactly three dominant vertices. Let x, y, z be their dominant vertices. Up to isomorphism, assume that $c(x) = 1$, $c(y) = 2$ and $c(z) = 3$. By Lemma 2.3, there are two clear paths in T: one connecting vertices x to y, and the other one connecting y to z. Let the two paths be ${}_xP_y := (x = u_0, u_1, u_2, \cdots, u_{r-1}, u_r = y)$ and ${}_yP_z := (y = v_0, v_1, v_2, \cdots, v_{s-1}, v_s = z)$ with r, s odd. Then, $c(u_i) = 1$ for even i and 2 for odd i; and $c(v_i) = 2$ for even i and 3 for odd i. Otherwise, there would be the fourth dominant vertex in T. Since x is a dominant vertex in T, then $d(x) \geq 2$. Therefore, there must be a neighbor of x (other than u_1), say a with $c(a) = 3$. Similarly, there must be a vertex b, a neighbor of z (other than v_{s-1}), with $c(b) = 1$. So, we have a path P, where $P = (a, x, u_1, u_2, \cdots, u_{r-1}, u_r = y, v_1, v_2, \cdots, v_{s-1}, v_s = z, b)$, with r, s odd. If $r, s > 1$ then define $u^* = u_{\lfloor \frac{r}{2} \rfloor}$, $u^{**} = u_{\lfloor \frac{r+1}{2} \rfloor}$, $v^* = v_{\lfloor \frac{s}{2} \rfloor}$, and $v^{**} = v_{\lfloor \frac{s+1}{2} \rfloor}$.

Lemma 3.3. *If $r = s = 1$ then $1 \leq d(a) \leq 2$, $2 \leq d(x) \leq 3$, $2 \leq d(y) \leq 4$, $2 \leq d(z) \leq 3$, and $1 \leq d(b) \leq 2$. Furthermore, every vertex $w \in V(T)\backslash P$ has degree at most 2 and is connected by a unique shortest path to one of $\{a, x, y, z, b\}$.*

Proof. For a contradiction, assume $d(a) \geq 3$ then two neighbors of a other than x will receive color 1. However, this implies that these neighbors will have the same color code, a contradiction. Therefore, $d(a) \leq 2$. Similarly, we also conclude that $d(b) \leq 2$. Next, since x is a dominant vertex, then $d(x) \geq 2$. Now, assume that $d(x) \geq 4$. Then, two of the neighbors of x will have the same color codes, a contradiction. Therefore, $2 \leq d(x) \leq 3$. Similarly, we have that $2 \leq d(z) \leq 3$. Since y is a dominant vertex and by Lemma 3.2 we have that $2 \leq d(y) \leq 4$.

Let $w \in V(T)\backslash P$. Since T is a tree, then there exists a unique shortest path L connecting w to a vertex of P. If $d(w) \geq 3$ then there are two neighbors of w, say w_1 and w_2, which are not in L. Since $\chi_L(T) = 3$ and x, y and z are the dominant vertices of T then the color codes of w_1

and w_2 will be the same, a contradiction. Therefore, every vertex $V(T)\backslash P$ must have degree at most 2. The path L which connects w to P is unique (since T is a tree) and goes through one of $\{a, x, y, z, b\}$. □

Lemma 3.4. *If $r = 1$ and $s > 1$ then $1 \leq d(a) \leq 2$, $2 \leq d(x) \leq 3$, $2 \leq d(y) \leq 3$, $2 \leq d(v^*) \leq 3$, $2 \leq d(v^{**}) \leq 3$, $d(z) = 2$, and $1 \leq d(b) \leq 2$. All the other internal vertices v_i in P have degree 2. Furthermore, every vertex $w \in V(T)\backslash P$ has degree at most 2 and is connected by a unique shortest path to one of $\{a, x, y, b, v^*, v^{**}\}$.*

Proof. To show $1 \leq d(a) \leq 2$, $2 \leq d(x) \leq 3$, $2 \leq d(y) \leq 3$, and $d(b) \leq 2$, we use a similar argument as in Lemma 3.3. Next, since v^* and v^{**} are internal vertices in P, then $d(v^*), d(v^{**}) \geq 2$. Assume $d(v^*) \geq 4$. Since v^* is not a dominant vertex then its two neighbors not in P will receive the same color. This implies that their color codes are the same, a contradiction. Therefore, $d(v^*) \leq 3$. Similarly, we have $d(v^{**}) \leq 3$. If z has the third neighbor z_1 then the color code of z_1 will be the same as the color code of either v_{s-1} or b. Therefore, $d(z) = 2$. Now, let v_i be any internal vertex in ${}_yP_z$ other than v^* or v^{**}. Assume $d(v_i) \geq 3$. Since all the neighbors of v_i are not dominant vertices, they will receive the same color. Thus, two of them will have the same color code, a contradiction. Therefore, $d(v_i) = 2$ for any v_i other than v^* and v^{**}.

Let $w \in V(T)\backslash P$. Since T is a tree, then there exists a unique shortest path L connecting w to a vertex of P. If $d(w) \geq 3$ then there are two neighbors of w, say w_1, w_2, which are not in L. Since $\chi_L(T) = 3$ and x, y and z are the dominant vertices of T then the color codes of w_1 and w_2 will be the same, a contradiction. Therefore, every vertex $V(T)\backslash P$ has degree at most 2. The path L which connects w to P is unique (since T is a tree) and goes through one of $\{a, x, y, b, v^*, v^{**}\}$. □

Lemma 3.5. *If $r > 1$ and $s > 1$ then $1 \leq d(a) \leq 2$, $d(x) = d(y) = d(z) = 2$, $2 \leq d(u^*) \leq 3$, $2 \leq d(u^{**}) \leq 3$, $2 \leq d(v^*) \leq 3$, $2 \leq d(v^{**}) \leq 3$, and $d(b) \leq 2$. All the other internal vertices in P have degree 2. Furthermore, every vertex $w \in V(T)\backslash P$ has degree at most 2 and is connected by a unique shortest path to one of $\{a, b, u^*, u^{**}, v^*, v^{**}\}$.*

Proof. The proof is similar as in the proof of Lemma 3.4. □

Theorem 3.1. *If $T \in \mathcal{T}$ and T has maximum number of vertices of degree higher than 2, then T is isomorphic to either G_1 or G_2.*

Proof. Let $T \in \mathcal{T}$. By Lemma 2.1, T contains at most three dominant vertices. If T is either a path P_3, or P_4, a double star $S_{1,2}$ or $S_{2,2}$, then T has a locating coloring such that T has only one or two dominant vertices. If T is not isomorphic to one of these four graphs above, then T will have a locating coloring with exactly three dominant vertices. Let x, y, z be such vertices. Then by Lemma 2.3, there are two clear paths, namely: ${}_xP_y = (x = u_0, u_1, u_2, \cdots, u_{r-1}, u_r = y)$, ${}_yP_z = (y = v_0, v_1, v_2, \cdots, v_{s-1}, v_s = z)$, with r, s odd.

If $r = s = 1$ then by Lemma 3.3, T will have maximum number of vertices of degree higher than 2 if there are two paths attached to y and one path attached to each vertex of a, x, z, and b, as depicted in Figure 2(i). Now, define a coloring $c : V(T) \rightarrow \{1, 2, 3\}$ such that:

1. $c(a) = 3, c(x) = 1$, $c(y) = 2$, $c(z) = 3$, $c(b) = 1$;

2. The colors of vertices of the path L_a attached to a are 1 and 3 alternately;
3. The colors of vertices of the path L_x attached to x are 2 and 1 alternately;
4. The colors of vertices of the first path L_y^1 attached to y are 1 and 2 alternately;
5. The colors of vertices of the second path L_y^2 attached to y are 3 and 2 alternately;
6. The colors of vertices of the path L_z attached to z are 2 and 3 alternately;
7. The colors of vertices of the path L_b attached to b are 3 and 1 alternately.

The color codes of all vertices of L_a are $(1, \text{even}, 0)$ or $(0, \text{odd}, 1)$. The color codes of all vertices of L_x are $(0, 1, \text{odd})$ or $(1, 0, \text{even})$. The color codes of all vertices of L_y^1 are $(0, 1, \text{even})$ or $(1, 0, \text{odd})$. The color codes of all vertices of L_y^2 are $(\text{odd}, 0, 1)$ or $(\text{even}, 1, 0)$. The color codes of all vertices of L_z are $(\text{even}, 0, 1)$ or $(\text{odd}, 1, 0)$. The color codes of all vertices of L_b are $(0, \text{even}, 1)$ or $(1, \text{odd}, 0)$. Therefore, all the color codes are different. Thus, c is a locating-coloring on T. Since 3 is the smallest possible number of colors then $\chi_L(T) = 3$. In this case, T is isomorphic to G_1.

If $r = 1$ and $s > 1$ then by Lemma 3.4, T will have maximum number of vertices of degree higher than 2 if there is one path attached to vertices a, x, y, v^*, v^{**} and b each as depicted in Figure 2(ii). Now, define a coloring $c : V(T) \rightarrow \{1, 2, 3\}$ such that:

1. $c(a) = 3, c(x) = 1, c(y) = 2, c(z) = 3, c(b) = 1;$
2. The colors of vertices of the path L_a attached to a are 1 and 3 alternately;
3. The colors of vertices of the path L_x attached to x are 2 and 1 alternately;
4. The colors of vertices of the path L_y attached to y are 1 and 2 alternately;
5. The colors of internal vertices $v_i s$ are 3 and 2 alternately;
6. The colors of vertices of the path L_{v^*} attached to v^* are 2 and 3 alternately;
7. The colors of vertices of the path $L_{v^{**}}$ attached to v^{**} are 3 and 2 alternately;
8. The colors of vertices of the path L_b attached to b are 3 and 1 alternately.

Then, it can be verified that the color codes of all vertices in T are distinct. Therefore, c is a locating-coloring on T. Since 3 is the smallest possible number of colors then $\chi_L(T) = 3$. In this case, T is isomorphic to G_2.

If $r > 1$ and $s > 1$ then by Lemma 3.5, T will have maximum number of vertices of degree higher than 2 if there is one path attached to $a, u^*, u^{**}, v^*, v^{**}$ and b each, as depicted in Figure 1(iii). By defining a similar coloring c we can obtain $\chi_L(T) = 3$. In this case, T is isomorphic to G_2. □

Theorem 3.2. *A tree T has the locating-chromatic number 3 if and only if T is either a path P_3 or P_4, a double star $S_{1,2}$ or $S_{2,2}$ or a subtree containing a path P of either G_1 or G_2.*

Proof. If T is either P_3, P_4, $S_{1,2}$ or $S_{2,2}$ then clearly it has locating-chromatic number 3. Now, let T^* be a subtree of either G_1 or G_2, and it contains a path P of length at least 4, as illustrated in Figure 2. Then, by using the coloring c in Theorem 3.1 restricted to the subtree T^*, we obtain that all the color codes are different. Therefore, $\chi_L(T^*) = 3$.

Conversely, let T be a tree with locating-chromatic number 3. If the diameter of T is ≤ 3 then T must be either P_3, P_4, $S_{1,2}$ or $S_{2,2}$. Now, if the diameter of T is ≥ 4 then by Lemma 2.1 T has at

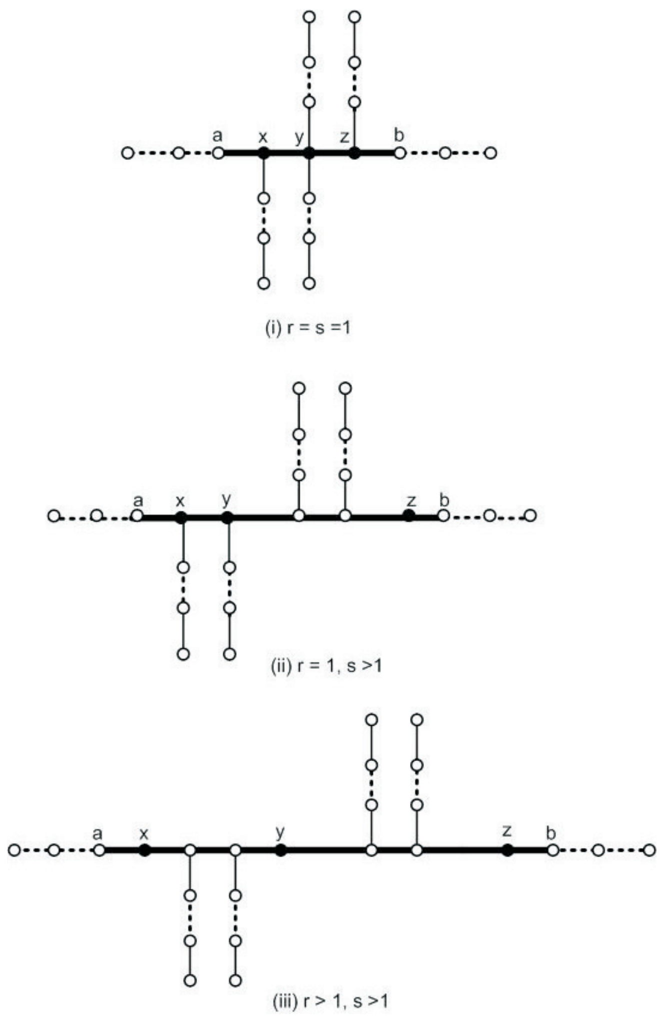

Figure 2. A tree $T \in \mathcal{T}$ with maximum number of vertices of degree higher than 2 and it contains a path $P = \{a, x, u_1, \cdots, u_r = y, v_1, \cdots, v_s = z, b\}$.

most 3 dominant vertices. Clearly, if T is not isomorphic to one of these four graphs above, then T will have a locating coloring such that T has exactly three dominant vertices. By Lemma 2.3, the three dominant vertices must lie in a path. This path must be of length at least 4. By Theorem 3.1, we conclude that T must be a subtree of one of the trees in Figure 2. As a consequence, T is a subtree of either G_1 or G_2. □

In the following theorems, we will give an infinite number of trees with locating-chromatic number 4 constructed from the trees with locating-chromatic number 3.

Theorem 3.3. *Let T' be a tree constructed from either G_1 or G_2 by attaching a path of arbitrary length to each vertex. Then, $\chi_L(T') = 4$.*

Proof. Define a coloring $c' : V(T') \to \{1, 2, 3, 4\}$ such that:

$$c'(u) = c(u), \text{ for any } u \in T,$$

where c is a coloring on T used in Theorem 3.1, and define the values of c' on any path $L := (w = w_0, w_1, \cdots, w_t)$ attached to a vertex w as follows:

$$c'(w_i) = c(w) \text{ for even } i, \text{ and } c'(w_i) = 4 \text{ for odd } i.$$

We will show that c' is a locating-coloring. Let u, v be any two vertices of T' with $c'(u) = c'(v)$. If u and v are in T then the color codes are distinct, since their color codes are derived from the previous color codes (under c) by adding the fourth ordinate with entry 1. If $u \in T$ and $v \notin T$ then $d(v, S) > d(u, S)$, where S is either S_1, S_2 or S_3, with S_i being the set of vertices receiving color i under c'. Now, let $u \notin T$ and $v \notin T$. If u and v are in the same path attached to vertex w then $d(u, S) \neq d(v, S)$ with S being either S_1, S_2 or S_3. Now, let u be in a path L_1 attached to w' and v be in a path L_2 attached to w''. If $c'(w') = c'(w'')$ then the color codes of u and v are different since the color codes of w' and w'' are different. If $c'(w') \neq c'(w'')$ then $d(u, S) = 1 < d(v, S)$ with S being the partition class containing vertex w'. Therefore, all vertices in T' have distinct color codes. Thus, c' is a locating-coloring on T'. Since 4 is the smallest possible number of colors (by Theorem 3.1) then $\chi_L(T') = 4$. □

Theorem 3.4. *Let T' be a tree constructed in Theorem 3.3. Every subtree of T' which is not a subtree of G_1 or G_2 has locating-chromatic number 4.*

Proof. A direct consequence of Theorem 3.3. □

To conclude this paper, we present an open problem related to the locating-chormatic number of graphs.

Problem 1. *Characterize all graphs of order $n \geq 4$ with locating-chromatic number 4.*

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