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Rainbow perfect domination in lattice graphs

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Abstract

Let $0 < n \in \mathbb{Z}$. In the unit distance graph of $\mathbb{Z}^n \subset \mathbb{R}^n$, a perfect dominating set is understood as having induced components not necessarily trivial. A modification of that is proposed: a *rainbow perfect dominating set*, or RPDS, imitates a perfect-distance dominating set via a truncated metric; this has a distance involving at most once each coordinate direction taken as an edge color. Then, lattice-like RPDS s are built with their induced components C having: **(i)** vertex sets $V(C)$ whose convex hulls are n-parallelotopes (resp., both $(n-1)$- and 0-cubes) and **(ii)** each $V(C)$ contained in a corresponding *rainbow sphere* centered at C with radius n (resp., radii 1 and $n-2$).

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1. PRELIMINARIES

Before defining our main concerns in Section 2, we review perfect dominating sets and perfect-distance dominating sets, and sketch our plan.

1.1. Perfect Dominating Sets, (PDS s)

Let $\Gamma = (V, E)$ be a graph and let $S \subset V$. Let $[S]$ be the subgraph of Γ induced by S. The induced components of S, namely the connected components of $[S]$ in Γ, are said to be the *components* of S. Several definitions of perfect dominating sets in graphs are considered in the

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literature [23, 25]. We work with the following one [32] denoted with the short acronym PDS, to make a distinctive difference:

S is a PDS of $\Gamma \Leftrightarrow$ each vertex of $V \setminus S$ has a unique neighbor in S.

This definition (of PDS) differs from that of a 'perfect dominating set' as in [21, 22, 30] (that for us is a stable PDS coinciding with the perfect code of [4] or with the efficient dominating set of [3, 23]), in that $[S]$ is not necessarily trivial.

Let $0 < n \in \mathbb{Z}$. The following graphs are considered. The unit distance graph $\Lambda_n^{\mathbb{R}}$ of $\mathbb{R}^n$ has vertex set $\mathbb{R}^n$ and exactly one edge between each two vertices if and only if their Euclidean distance is 1. Let $\Lambda_n^{\mathbb{Z}}$ be the induced subgraph of $\mathbb{Z}^n$ in $\Lambda_n^{\mathbb{R}}$. If no confusion arises, we write $\Lambda_n = \Lambda_n^{\mathbb{Z}}$ and express the elements $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ with no parentheses or commas, namely as $a_1 \cdots a_n$. This way, we denote: $O = 00 \cdots 0$, $e_1 = 10 \cdots 0$, $e_2 = 010 \cdots 0$, ..., $e_{n-1} = 0 \cdots 010$ and $e_n = 00 \cdots 01$. An *n-cube* is the cartesian graph product $Q_n = K_2 \Box K_2 \Box \cdots \Box K_2$ of precisely n complete graphs K_2. A *grid graph* is the cartesian graph product of two path graphs.

Our definition of a PDS S allows induced components of S in Γ which are not isolated vertices. For example: **(a)** tilings with generalized Lee r-spheres, for fixed r with $1 < r \leq n$ in $\mathbb{Z}$ (e.g., crosses with arms of length one if $r = n$), furnish Λ_n with PDS s whose components are r-cubes [20]; (It is most remarkable that $r = n \Leftrightarrow n \in \{2^r - 1, 3^r - 1; 0 < r \in \mathbb{Z}\}$ [6]); **(b)** *total perfect codes* [1, 26], that is PDS s whose components are $K_2 = P_2$ in the Λ_n s and grid graphs; (these appear as *diameter perfect Lee codes* [19, 24]); **(c)** PDS s in n-cubes [5, 12, 13, 15, 16, 32], where $0 < n \in \mathbb{Z}$, including the perfect codes of [18]; **(d)** PDS s in grid graphs [13, 26].

1.2. Perfect-Distance Dominating Sets

In [2], an extension of the definition of PDS is given as follows. Let $t \geq 1$ and $\Gamma = (V, E)$ be a graph. A set $S \subset V$ is a *t-perfect-distance dominating set* (t-PDDS) in Γ if, for each $v \in V$, there is a unique component C_v of S so that for the graph distance $d(v, C_v)$ from v to C_v it is $d(v, C_v) \leq t$, and there is in C_v a unique vertex w with $d(v, w) = d(v, C_v)$.

We refer to [2] for relations of PDDS s to other domination and coding notions. For $0 < n \in \mathbb{Z}$, the tilings with *generalized Lee spheres* of [20] (see Subsection 1.2 item (a)) provide $\Gamma = \Lambda_n$ with t-PDDS s whose components are r-cubes, for any fixed $t \in \mathbb{Z}$ and $r \in \mathbb{Z}$ such that $t \geq 1$ and $0 \leq r \leq n$.

1.3. Plan of the Paper and Related Motivation

In Section 2, *rainbow perfect dominating sets*, or RPDS s, are defined that generalize PDS s while imitating the definition of PDDS but using a truncated metric [17], pages 40 and 262. This has a *rainbow distance* by coloring the edges of $\Lambda_n^{\mathbb{Z}}$ according to the n coordinates, for $0 < n \in \mathbb{Z}$. With the aim of packing perfectly the resulting *rainbow spheres*, Section 3 takes to the construction of lattice-like RPDS s S whose induced graphs $[S]$ have their components C possessing: **(i)** vertex sets with n-parallelotopes as their convex hulls in $\mathbb{R}^n$ and minimal separating graph distance 3, having a set of representatives that forms a lattice with generating elements precisely along the coordinate directions of $\mathbb{Z}^n$ and **(ii)** each $V(C)$ contained in a corresponding rainbow sphere centered at C with radius n.

It is not clear that similar lattice-like results hold with r-parallelotopes ($0 < r < n$), including lattice-like *rainbow* total perfect codes (case $r = 1$). However, once the concept of lattice-like is generalized in Section 4, we are able to show that a lattice-like RPDS S exists in Λ_n whose $[S]$ has its components C possessing: **(i′)** vertex sets with $(n-1)$- and 0-cubes as their convex hulls in $\mathbb{R}^n$ and **(ii′)** each $V(C)$ contained in a corresponding rainbow sphere centered at C with respective radii 1 and $(n-2)$.

Motivation for this outcome of RPDS s with induced components that are r-cubes of different dimensions r (Theorem 4.2) comes both from the perfect covering codes with spheres of two different radii in Chapter 19 of [11] and from a negative answer to a conjecture [32] claiming that the components of a PDS S in an n-cube Q_n are r-cubes Q_r where r is fixed with $0 \leq r \leq n$. In fact, it was found in [31] that a PDS in Q_n with components that are r-cubes Q_r in Q_{13} of two different dimensions $r = r_1$ and $r = r_2$ exist, specifically with $r_1 = 4$ and $r_2 = 0$. However, this is still the only known counterexample to the conjecture of [32].

2. RAINBOW PERFECT DOMINATING SETS

Let $0 < n \in \mathbb{Z}$. Let $\Gamma = (V, E)$ be a graph edge-colored in $I_n = \{1, \dots, n\}$. A path $P_{\Cap}$ in Γ is a *rainbow path* if no color appears more than once in $P_{\Cap}$ [8, 9, 10, 27, 28, 29]. We consider a truncated metric that generalizes that of [17], pages 40 and 262 and is defined between two vertices u and v in Γ by their *rainbow distance* $\rho(u, v)$, namely: **(i)** the shortest length of a rainbow path $P_{\Cap}$ joining u and v, if such $P_{\Cap}$ exists; **(ii)** $|I_n|+1 = n+1$, otherwise. Notice that ρ is not a well-defined distance like the *graph distance* d of Γ given by the shortest length $d(u, v)$ of a path P between u and v. If K is a component of S and $u \in V$ then we denote $\rho(u, K) = \min\{\rho(u, v); v \in K\}$ and $d(u, K) = \min\{d(u, v); v \in K\}$. Let $1 \leq t \leq n$. A set $S \subseteq V$ is said to be a *t-rainbow perfect dominating set* or t-RPDS in Γ if for each $v \in V$ there are: **(a)** a unique component K_v of S with $\rho(v, K_v) \leq t$ and **(b)** a unique vertex w in K_v with $\rho(v, w) = \rho(v, K_v)$. If in this definition of t-RPDS we replace ρ by d then S becomes a t-PDDS in Γ, as in [24].

Let $H = (V, E)$ be a subgraph of Λ_n and let $z \in \mathbb{Z}^n$. Then $H + z$ denotes the graph $H' = (V', E')$ with vertex set $V' = V + z = \{w \in \mathbb{Z}^n; \exists\ v \in V \text{ such that } w = v + z\}$ and such that $uv \in E$ if and only if $(u + z)(v + z) \in E'$. Observe that the subgraph H of Λ_n induced by the set of vertices with entries in $\{0, 1\}$ (and by extension any translation $H' = H + z$ of such an H in Λ_n) constitutes an n-cube Q_n.

Let $i \in I_n$. Each edge of Λ_n parallel to Oe_i is assigned color i. Thus, an edge uv of Λ_n has color i if and only if $u - v \in \{\pm e_i\}$. Considering this for every $i \in I_n$, Λ_n becomes an edge-colored graph having its copies of the n-cube Q_n as its largest properly edge-colored subgraphs.

All 1-RPDS s are PDS s. A PDS is both a 1-RPDS and a 1-PDDS, so that 1-RPDS s and 1-PDDS s coincide as PDS s. However, this is not the case if $t > 1$. The following restriction of a theorem of [32] (Theorem 1 of [2] has a similar proof) is expressed in terms of monochromatic paths in the edge-colored Λ_n with a monochromatic path understood as either one-way infinite or two-way infinite or having length either null or positive.

Theorem 2.1. *If S is a t-RPDS in Λ_n then each component of S is the cartesian product of monochromatic paths of different colors in Λ_n.*

Let J be a cartesian product of finite monochromatic paths of different colors in Λ_n. The *rainbow sphere* $W^{\rho}_{n,J,t}$ of radius t around J in Λ_n is the union of $V(J)$ and the set of those $v \in \mathbb{Z}^n$ with $\rho(v, V(J)) \leq t$. Here, J is said to be the *rainbow center* of $W^{\rho}_{n,J,t}$. The *graph sphere* $W^{d}_{n,J,t}$ is defined similarly and has J as its corresponding *graph center*. If there is no confusion, we drop the initial adjective *rainbow* or *graph*. If J is an r-cube, where $0 \leq r \leq n$, then we write $W^{\rho}_{n,J,t} = W^{\rho}_{n,r,t}$ and $W^{d}_{n,J,t} = W^{d}_{n,r,t}$. It is seen that $W^{\rho}_{n,J,t}$ is a generalized Lee spheres [20]. Figure 1 represents two rainbow spheres (dark gray) in Λ_2 contained in respective graph spheres (two-tone gray), namely $W^{\rho}_{2,0,2}$ (dark gray) $\subset W^{d}_{2,0,2}$ (two-tone gray) and $W^{\rho}_{2,1,2}$ (dark gray) $\subset W^{d}_{2,1,2}$ (two-tone gray).

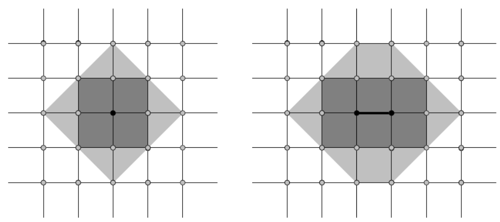

Figure 1. Rainbow spheres contained in respective graph spheres.

A t-RPDS S of Λ_n determines a partition of $\mathbb{Z}^n$ into the spheres $W^{\rho}_{n,K,t}$, with K running over the components of S. A t-PDDS S' of Λ_n determines similarly a partition of $\mathbb{Z}^n$ into the spheres $W^{d}_{n,K',t}$, with K' running over the components of $[S']$.

A t-RPDS S in Λ_n such that the components of S are all isomorphic to a fixed finite graph H is said to be a t-RPDS$[H]$. Let S be a t-RPDS$[H]$ and let K be a component of S with K isomorphic to H. Then S is said to be *lattice-like* ([2]) if: there is a lattice $L \subseteq \mathbb{Z}^n$ such that K' is a component of S if and only if $K' = K + z$ with $z \in L$. A set $S \subset \mathbb{Z}^n$ is *periodic* [2] if there are integers $p_1, \ldots, p_n$ such that $v \in S$ implies $v \pm p_i e_i \in S$ for all $i = 1, \ldots, n$. Notice that each lattice-like t-RPDS$[H]$ S is periodic [2]. Thus, a suitable restriction of such an S yields a t-RPDS$[H]$ in a cartesian product of n cycles $C_{k_1p_1} \Box C_{k_2p_2} \Box \cdots \Box C_{k_np_n}$ with $0 < k_i \in \mathbb{Z}$, for $i = 1, 2, \ldots, n$. This observation is easily adapted to the generalizations of a t-RPDS$[H]$ below, up to Section 4. In fact, the second parts of the statements of Theorems 3.1, 4.1 and 4.2 use them. However, we prove just the existence of those RPDS s in the Λ_n s, leaving the covering and projection (onto cartesian products of cycles) parts of the proofs to the reader.

Let H be a cartesian product of finite monochromatic paths of different colors in Λ_n. If just r elements of I_n color the edges of H, then we say that H is an *r-box*. In this case, H is a cartesian product $\Pi_{i=1}^{n} P_i$ where P_i is a finite path, for $1 \leq i \leq n$, with exactly r paths P_i having positive length. Clearly, the convex hull of an r-box is an r-parallelotope and any r-cube in Λ_n is an r-box, for $0 \leq r \leq n$..

A *constellation* of a lattice L in $\mathbb{Z}^n$ is a subset $T \subseteq \mathbb{Z}^n$ that contains exactly one vertex from each class mod L so that T is in fact a complete system of coset representatives of L in Λ_n. (Compare with *fundamental region*, [7], pg. 26). We still say that a partition of $\mathbb{Z}^n$ into constellations of L is a *tiling* of $\mathbb{Z}^n$ and that those constellations are its *tiles*.

3. TOP RADIUS AND BOX DIMENSION

A particular case of t-RPDS$[H]$ is that in which H is an n-box in Λ_n. For each such n-box H we show that there is a lattice-like RPDS$[H]$ in Λ_n. (In [6], n-boxes of unit volume in Λ_n are shown to determine 1-PDDS$[H]$ s if and only if either $n = 2^r - 1$ or $n = 3^r - 1$).

Theorem 3.1. *For each n-box $H = \Pi_{i=1}^n P_i$ in Λ_n, where P_i is a path of color i and length ℓ_i $(i = 1, \ldots, n)$, there exists a lattice-like n-RPDS$[H]$ S of Λ_n. This S covers an n-RPDS$[H]$ in any cartesian product of cycles $C_{(\ell_1+3)k_1} \Box C_{(\ell_2+3)k_2} \Box \cdots \Box C_{(\ell_n+3)k_n}$ with $1 < k_i \in \mathbb{Z}$ $(i = 1, \ldots, n)$. The minimum graph distance between the induced components of S is 3.*

Proof. Assume S is an n-RPDS$[H]$ in Λ_n. As already commented, S determines a partition of $\mathbb{Z}^n$ into the spheres $W^\rho_{n,K,n}$ with K running over the components (isomorphic to H) of S. These spheres conform a tiling which is associated to a lattice L_S to be set now. In each such $W^\rho_{n,K,n}$ let $b_1 b_2 \cdots b_n$ be the vertex $a_1 a_2 \cdots a_n$ for which $a_1 + a_2 + \cdots + a_n$ is minimal. We say that this $b_1 b_2 \cdots b_n$ is the *anchor* of $W^\rho_{n,K,n}$. Then the anchors of the spheres $W^\rho_{n,K,n}$ form the lattice $L = L_S$. Without loss of generality we can assume that O is the anchor of a $W^\rho_{n,H_0,n}$ whose center H_0 is a component of S isomorphic to H. In $W^\rho_{n,H_0,n}$ let $c_0 c_1 \cdots c_n$ be the vertex $a_1 a_2 \cdots a_n$ in $W^\rho_{n,H_0,n}$ for which $a_1 + a_2 + \cdots a_n$ is maximal. Then L_S has generating set $\{(1+c_1)e_1, (1+c_2)e_2, \ldots, (1+c_n)e_n\}$ and is formed by all linear combinations of its elements. This insures that S exists and is lattice-like via L_S. Remaining details of the proof are left to the reader, who must check that $\ell_i = c_i - 2$, for $i = 1, \ldots, n$. □

Theorem 3.1 can be proved alternatively by the additive-group epimorphism technique [2, 1, 24] modified in Section 5 as Proposition 5.1. Figure 3 in Section 4 below suggests in light-gray color at least two induced components of a t-RPDS$[H]$ in Λ_n as in Theorem 3.1, where $t = 3$, $H = Q_0$ and $n = 3$.

The *Voronoi diagram* of $\mathbb{Z}^n$ in $\mathbb{R}^n$ has its composing *Voronoi regions* ([7], pg. 26) as the unit-volume n-dimensional cubes, cartesian product $\Pi_{i=1}^n [a_i - \frac{1}{2}, a_i + \frac{1}{2}]$. Let H_0 and L be as in the proof of Theorem 3.1. Consider the vertices v of a sphere $W^\rho_{n,H_0,n} + \ell$, $(\ell \in L)$. The union of the Voronoi regions of those v is called the *Voronoi box* $B^\rho_{n,H_0,n,\ell}$ of $W^\rho_{n,H_0,n} + \ell$. Then, $\mathbb{R}^n$ admits a *Voronoi partition* $\mathcal{V}$ into constellations (in a way similar to that of [7], pg.26) each contained in a corresponding Voronoi box $B^\rho_{n,H_0,n,\ell}$ but containing its anchor in $L = L_S$ and just one point from each pair of antipodal points in its boundary (equidistant from the barycenter of $B^\rho_{n,H_0,n,\ell}$ along a straight line). As a result, $L = L_S$ is a set of representatives of $\mathcal{V}$, but $\mathcal{V}$ is not uniquely defined.

Corollary 3.1. *For each n-box H, $\mathbb{R}^n$ admits a Voronoi partition into constellations associated to the Voronoi boxes $B^\rho_{n,H_0,n,\ell}$ where both H_0 and L are as in the proof of* Theorem 3.1*, and ℓ runs over L.*

4. SMALLER RADII AND BOX DIMENSIONS

Existing results of lattice-like t-RPDS s in Λ_n with $t < n$ concern solely $t = 1$ (that is for PDS s). In fact, constructions in [2, 6, 20, 24] lead to lattice-like 1-RPDS. However, it seems that there are not many of these 1-RPDS s. For example, [14] shows that there is only one lattice-like 1-RPDS$[Q_2]$ and no 1-RPDS$[Q_2]$ which is not lattice-like. In contrast with the existence of a lattice-like 2-PDDS$[P_2]$ in Λ_3 arising from a Minkowsky tiling cited in [2], we may combine the conjecture in Subsection 1.3 with the related conjecture that there are no lattice-like t-RPDS in Λ_n, for $1 < t < n$.

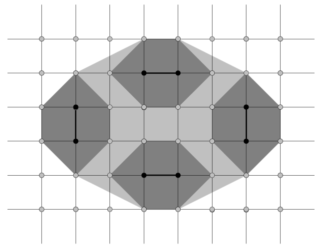

Figure 2. Constellation of a lattice L_S for a 1-RPDS$[Q_1; 4]$ S in Λ_3.

If S is a periodic t-RPDS$[H]$ in Λ_n and is not lattice-like, then for some positive integer m there is a tiling of Λ_n with tiles that are the vertex set of a connected subgraph H^* induced in Λ_n by the union of both: **(a)** m disjoint copies $H^1, \dots, H^m$ of H that intervene as components of S and **(b)** the set formed by the vertices $v \in \mathbb{Z}^n$ for which $\rho(v, H^j) \le t$, for some $j \in I_m$. If so, by taking m as small as possible, we say that S is a t-RPDS$[H; m]$.

For example, Section 5 of [2] shows the existence of a 1-RPDS$[Q_1; 4]$ S which is not lattice-like. However, there exists a lattice L_S based on such S with each of its constellations containing two copies of Q_2 in color 1 (of edge Oe_1) and two copies of Q_1 in color 2 (of edge Oe_2), all four copies of Q_2 being components of S. This is represented in Figure 2, where the rainbow 1-spheres of such four components (in thick trace) are shaded dark gray and the remaining area completing their convex hull is in light gray.

Here we can take a fixed vertex v_T in each resulting tile T so that all the vertices v_T constitute the lattice L_S. Thus, even for a non-lattice-like t-RPDS we can recover a lattice formed by selected vertices v_T in the corresponding tiles T associated to S. However, when describing S as a t-RPDS$[H; m]$, we can say that S is a *lattice-like* t-RPDS$[H; m]$ as there is indication between brackets of the components of S in a resulting typical tile T in which to fix a sole distinguished vertex v_T so that all such distinguished vertices constitute a lattice L_S and the resulting tiling is effectively a lattice-like tiling. We generalize this situation as follows.

A t-RPDS S in Λ_n with the components of S isomorphic to two different fixed finite graphs H_0 and H_1 is said to be a t-RPDS$[H_0, H_1]$. Even though such an S cannot be lattice-like, it may happen that there exists a lattice L_S such that for some positive integers m_0 and m_1 there exists a constellation of L_S in Λ_n given by the union of two disjoint subgraphs H_0^* and H_1^*, where H_i^* $(i = 0, 1)$ is induced in Λ_n by the disjoint union of: **(a)** m_i copies $H_i^1, \dots, H_i^{m_i}$ of H_i that intervene as components of S and **(b)** the sets of vertices $v \in \mathbb{Z}^n$ for which $0 < \rho(v, H_i^j) \le t$, for $j \in I_{m_i}$. In this case, S is said to be a *lattice-like* t-RPDS$[H_0, H_1; m_0, m_1]$. We can take a fixed vertex v_T in each resulting tile T so that all the vertices v_T form the lattice L_S. We obtain a 1-RPDS$[H_0, H_1; m_0, m_1]$ in the following statement, where $H_0 = Q_2$, $H_1 = Q_0$, $m_0 = 2$, $m_1 = 2$,

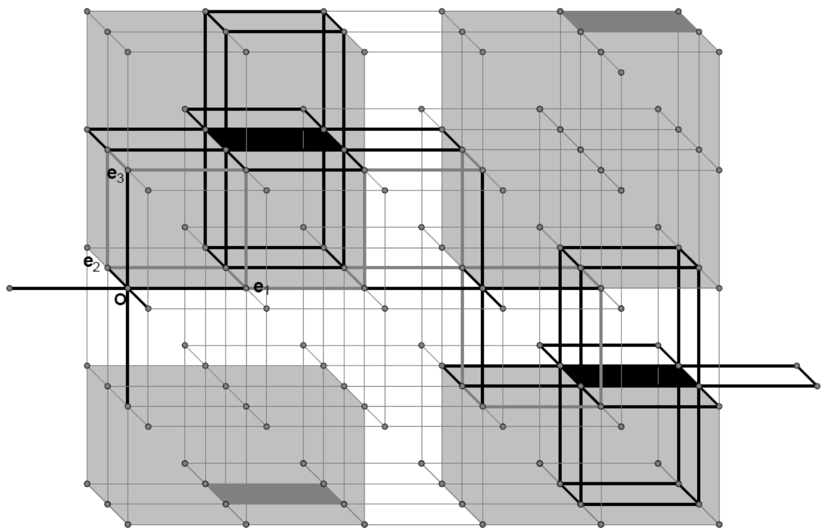

Figure 3. Elements of a constellation for a 1-RPDS$[Q_2, Q_0; 2, 2]$ in Λ_3.

with a constructive proof of it in Section 6 by means of Proposition 5.1.

Theorem 4.1. *There exists a lattice-like* 1-RPDS$[Q_2, Q_0; 2, 2]$ S *in* Λ_3*. This* S *covers a* 1-RPDS $[Q_2, Q_0; 2, 2]$ *of any cartesian product* $C_{6k_1}\square C_{6k_2}\square\, C_{3k_3}$ *with* $0 < k_i$*, for* $i = 1, 2, 3$*. The minimum graph distance between the induced components* Q_2 *(resp.,* Q_0*) of* S *is* 3*.*

This is represented in Figure 3, where the components of S in one of the constellations of L_S formed by two copies of Q_2 and two copies of Q_0, are blackened and the edges in the rainbow 1-spheres having them as centers are shown in dark trace; the other edges induced in the union of these four components are in dark-gray trace. For better reference, the rainbow 3-spheres of the 3-RPDS$[Q_0]$ resulting from Theorem 3.1 are shaded in light gray. Also, dark gray was used to indicate two other copies of Q_2 appearing in the figure that are components of S. Notice that vertices O, e_1, e_2, e_3 are indicated in the figure. The minimum distance between the induced components Q_{n-1} (resp., Q_0) of S is 3.

More generally, let $1 \leq t_i \leq n$, for $i = 0, 1$. A set $S \subset V$ is said to be a (t_0, t_1)-RPDS$[H_0, H_1]$ in Λ_n if for each $v \in V$ there is: **(i)** a unique index $i \in \{0, 1\}$ and a unique component K_v^i of S such that the distance $\rho(v, K_v^i)$ from v to C_K^i satisfies $\rho(v, K_v^i) \leq t_i$ and **(ii)** there is a unique vertex w in K_v^i such that $\rho(v, w) = \rho(v, K_v^i)$.

Even though such an S cannot be lattice-like, it may happen that there exists a lattice L_S such that for some positive integers m_0 and m_1 there exists a constellation of L_S in Λ_n given by the union of two disjoint subgraphs H_0^* and H_1^*, where H_i^* $(i = 0, 1)$ is induced in Λ_n by the disjoint union of: **(a)** m_i disjoint copies $H_i^1, \ldots, H_i^{m_i}$ of H_i that intervene as components of S and **(b)** the sets of vertices $v \in \mathbb{Z}^n$ for which $0 < \rho(v, H_i^j) \leq t_i$, for $j \in I_{m_i}$. In this case, S is said to be a *lattice-like* (t_0, t_1)-RPDS$[H_0, H_1; m_0, m_1]$. We can take a fixed vertex v_T in each resulting tile T so that all the vertices v_T form a lattice L_S. A family of lattice-like (t_0, t_1)-RPDS$[H_0, H_1; m_0, m_1]$ s in the lattices Λ_n that extend the 1-RPDS$[Q_2, Q_0; 2, 2]$ of Theorem 4.1 is obtained as follows, where

$H_0 = Q_{n-1}$, $H_1 = Q_0$, $m_0 = m_1 = 2$, $t_0 = 1$, $t_1 = n-2$, with a constructive proof of it in Section 7.

Theorem 4.2. *There exists a lattice-like* $(1, n-2)$-RPDS$[Q_{n-1}, Q_0; 2, 2]$ S *in* Λ_n. *This* S *covers a* $(1, n-2)$-RPDS$[Q_{n-1}, Q_0; 2, 2]$ *in any cartesian product* $C_{6k_1}\square \dots \square C_{6k_{n-1}}\square C_{3k_n}$ *with* $0 < k_i$, *for* $i = 1, \dots, n$.

5. ADDITIVE-GROUP EPIMORPHISMS

All the constructions of RPDS s mentioned in this paper can be confirmed by means of the additive-group epimorphism technique presented in this section. In fact, we use a modification of Corollary 2 in [2] in the following two sections. This is a corollary to Theorem 6 [24] whose proof uses the linear-algebraic notion of translation of subsets $S \subset \mathbb{Z}^n$. The modification in question (of the corollary) is given as Proposition 5.1 below and it is tailored in order to complete the proofs of the results in Section 4. The additive-group epimorphism technique starts by having:

- **(a)** a lattice L in $(\mathbb{Z}^n, +)$ generated by elements $u_1, \dots, u_n \in \mathbb{Z}^n$ such that $L = \{\alpha_1 u_1 + \dots + \alpha_n u_n;\ \alpha_i \in \mathbb{Z},\ i = 1, \dots, n\}$;
- **(b)** a set $T \subseteq \mathbb{Z}^n$ containing one element from each coset of $\mathbb{Z}^n$ mod L such that $\{T + u\,; u \in L\}$ is a partition of $\mathbb{Z}^n$ into subsets of size $|\mathbb{Z}^n/L|$, with the induced subgraphs $[T + u]$ of $T + u$ in Λ_n pairwise isomorphic, where $u \in L$.

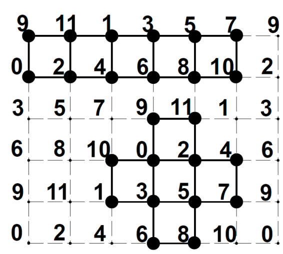

Figure 4. Accompanying example: two possible selections for $[T]$.

Given a lattice L, we can split $\mathbb{Z}^n$ into subsets with their induced subgraphs having different shapes depending on the choice of T. For example, $L = \{\alpha_1(3, 2) + \alpha_2(0, 4); \alpha_i \in \mathbb{Z}, i = 1, 2\}$ in Λ_2 has $(\mathbb{Z}^n, +)/L = \mathbb{Z}_{12}$. The graph $[T]$ might be either the cartesian product $P_6\square\, P_2$ or the closed neighborhood of a 2-cube $Q_2 = P_2\square\, P_2$, as shown in Figure 1.

Let $D = (V, E)$ be an induced subgraph of Λ_n. We are looking for a partition (tiling) of Λ_n into copies of D. We need to find a lattice L for the required selection of the set T with $[T] = D$. The following construction leads to the sought tiling of Λ_n.

If there is an abelian group $(G, +)$ of order $|V|$ and elements $g_1, \dots, g_n$ of G such that the restriction of the epimorphism $\Phi : \mathbb{Z}^n \to G$ defined by $\Phi((a_1, \dots, a_n)) = a_1\Phi(e_1) + \dots + a_n\Phi(e_n) = a_1 g_1 + \dots + a_n g_n$ to V is a bijection then there is a partition of Λ_n into copies of D.

In other words, we need to find an abelian group G of order $|V|$ and assign elements $g_1, \dots, g_n$ of G to the vertices $e_1, \dots, e_n$ of Λ_n such that $\Phi((a_1, \dots, a_n)) = a_1\Phi(e_1) + \dots + a_n\Phi(e_n) =$

$a_1g_1 + \ldots + a_ng_n$ is a bijection on V. Since the kernel of a group epimorphism $\Phi : \mathbb{Z}^n \to G$ is a subgroup of $\mathbb{Z}^n$, then the elements w of $\mathbb{Z}^n$ for which $\Phi(w) = 0$ form a lattice L in $(\mathbb{Z}^n, +)$. In addition, $(\mathbb{Z}^n, +)/L = G$; also, V has exactly one element in each coset of $(\mathbb{Z}^n, +)/L$. Thus we can set $T = V$.

TABLE I

			$\mathbb{Z}^3$					$\xrightarrow{\Phi}$			G				
⋯	⋯	122	222	⋯	⋯	⋯	⋯	⋯	⋯	322	422	⋯	⋯	⋯	⋯
⋯	⋯	112	212	⋯	⋯	⋯	⋯	⋯	⋯	212	312	⋯	⋯	⋯	⋯
⋯	⋯	131	231	⋯	⋯	⋯	⋯	⋯	⋯	401	501	⋯	⋯	⋯	⋯
⋯	021	**121**	**221**	321	⋯	⋯	⋯	⋯	221	**321**	**421**	521	⋯	⋯	⋯
⋯	011	**111**	**211**	311	⋯	⋯	⋯	⋯	111	**211**	**311**	411	⋯	⋯	⋯
⋯	001	101	201	301	⋯	⋯	⋯	⋯	001	101	201	301	⋯	⋯	⋯
⋯	⋯	120	220	⋯	420	520	⋯	⋯	⋯	320	420	⋯	020	120	⋯
⋯	010	110	210	310	410	510	⋯	⋯	110	210	310	410	510	010	⋯
$\bar{1}$00	**000**	100	200	**300**	400	⋯	⋯	500	**000**	100	200	**300**	400	⋯	⋯
⋯	0$\bar{1}$0	⋯	⋯	3$\bar{1}$0	⋯	⋯	⋯	⋯	520	⋯	⋯	220	⋯	⋯	⋯
⋯	⋯	⋯	⋯	⋯	43$\bar{1}$	53$\bar{1}$	⋯	⋯	⋯	⋯	⋯	⋯	102	202	⋯
⋯	⋯	⋯	⋯	32$\bar{1}$	**42$\bar{1}$**	**52$\bar{1}$**	62$\bar{1}$	⋯	⋯	⋯	⋯	522	**022**	**122**	222
⋯	⋯	⋯	⋯	31$\bar{1}$	**41$\bar{1}$**	**51$\bar{1}$**	61$\bar{1}$	⋯	⋯	⋯	⋯	412	**512**	**012**	112
⋯	00$\bar{1}$	⋯	⋯	30$\bar{1}$	40$\bar{1}$	50$\bar{1}$	⋯	⋯	002	⋯	⋯	302	402	502	⋯
⋯	⋯	⋯	⋯	⋯	42$\bar{2}$	52$\bar{2}$	⋯	⋯	⋯	⋯	⋯	⋯	021	121	⋯
⋯	⋯	⋯	⋯	⋯	41$\bar{2}$	51$\bar{2}$	⋯	⋯	⋯	⋯	⋯	⋯	511	011	⋯

As mentioned, Corollary 2 of [2] can be modified as follows.

Proposition 5.1. *Let $1 \leq t_i \in \mathbb{Z}$ and $1 \leq m_i \in \mathbb{Z}$. Let H_i be a finite subgraph of Λ_n, for $i = 0, 1$. Let H be the disjoint union in Λ_n of m_0 copies of H_0 and m_1 copies of H_1. Let an induced supergraph H^* of H in Λ_n be such that a vertex v is in H^* if and only if there is just a copy H' of H_i that is a component of H with the least $\rho(v, H') \leq t_i$, for a fixed $i = 0, 1$ dependent only on v. Let $D = (V, E)$ be a copy of H^* in Λ_n that contains vertices $O, e_1, \ldots, e_n$. Then there is a lattice-like (t_0, t_1)-*RPDS$[H_0, H_1; m_0, m_1]$ *if there exists an abelian group G of order $|V|$ and a group epimorphism $\Phi : \mathbb{Z}^n \to G$ such that the restriction of Φ to V is a bijection.*

6. PROOF OF THEOREM 4.1

If $x < y$ in $\mathbb{Z}$, then let $[x, y] = \{z \in \mathbb{Z}; x \leq z \leq y\}$. Consider the subset $X \subset \mathbb{Z}^3$ of vertices of Λ_3 whose coordinates are divisible by 3. Clearly, X is a lattice of $\mathbb{Z}^3$. Each element of X is in a subset $\tau_{x_1,x_2,x_3} = [3x_1, 2 + 3x_1] \times [3x_2, 2 + 3x_2] \times [3x_3, 2 + 3x_3]$ of $\mathbb{Z}^3$, with $x_1, x_2, x_3 \in \mathbb{Z}$. Such a subset is a constellation of the lattice X and from now on will be called a 3-*grenade*.

TABLE II

	···	···	···	···	0110	···	···	···	···
	···	001⊢	···	⊢010	0010	1010	···	0011	···
	···	···	···	···	0⊢10	···	···	···	···
	···	010⊢	···	⊢100	0100	1100	···	0101	···
$\mathbb{Z}^4$	⊢00⊢	000⊢	100⊢	⊢000	**0000**	1000	⊢001	0001	1001
	···	0⊢0⊢	···	⊢⊢00	0⊢00	1⊢00	···	0⊢01	···
	···	···	···	···	01⊢0	···	···	···	···
	···	00⊢⊢	···	⊢0⊢0	00⊢0	10⊢0	···	00⊢1	···
	···	···	···	···	0⊢⊢0	···	···	···	···
$\downarrow^{\Phi}$									
	···	···	···	···	2110	···	···	···	···
	···	1012	···	0010	1010	2010	···	1011	···
	···	···	···	···	0210	···	···	···	···
	···	1102	···	0100	1100	2100	···	1101	···
G	5002	0002	1002	5000	**0000**	1000	5001	0001	1001
	···	5202	···	4200	5200	0200	···	5201	···
	···	···	···	···	0120	···	···	···	···
	···	5022	···	4020	5020	0020	···	5021	···
	···	···	···	···	4220	···	···	···	···

A lattice-like 1-RPDS$[Q_2, Q_0; 2, 2]$ as claimed in Theorem 4.1 is composed by X and a subset Y defined as follows. We select in each 3-grenade τ_{x_1,x_2,x_3} the 2-cube (or square) $\sigma_{x_1,x_2,x_3} = [1 + 3x_1, 2 + 3x_1] \times [1 + 3x_2, 2 + 3x_2] \times \{1 + \delta + 3x_3\}$, where δ is the rest of dividing $x_1 + x_2$ by 2. Then Y is given by the union of all squares σ_{x_1,x_2,x_3} with $x_1, x_2, x_3 \in \mathbb{Z}$. Theorem 4.1 is a direct corollary of the following lemma.

Lemma 6.1. *$X \cup Y$ is a lattice-like $(1, 1)$-RPDS$[Q_2, Q_0; 2, 2]$ of Λ_3. Its induced components are centers of the copies of respective spheres $W^{\rho}_{3,2,1}$ and $W^{\rho}_{3,0,1}$ in a specific tiling of Λ_3.*

Proof. We will construct the claimed RPDS$[Q_2, Q_0; 2, 2]$ by applying Proposition 5.1. Let H be given by the union of $\sigma_{0,0,0}$, $\sigma_{1,0,-1}$, $\{3e_1\}$ and $\{O\}$. Let $D = (V, E)$ be as in the statement of Proposition 5.1 in out present situation. The graph H^* is represented in Figure 2 with: **(i)** edges between vertices in each component of D in thick black trace, **(ii)** the remaining edges in H^* in thick dark-gray trace, **(iii)** the convex hulls of shown parts of 3-grenades in $\mathbb{R}^3$ in light gray and **(iv)** dominating copies of Q_2 in black.

TABLE III

	···	···	···	1231	2231	···	···	···
	···	···	···	1131	2131	···	···	···
	···	···	···	1321	2321	···	···	···
	1220	2220	0221	**1221**	**2221**	3221	1222	2222
$\mathbb{Z}^4$	1120	2120	0121	**1121**	**2121**	3121	1122	2122
	···	···	···	1021	2021	···	···	···
	···	···	···	1311	2311	···	···	···
	1210	2210	0211	**1211**	**2211**	3211	1212	2212
	1110	2110	0111	**1111**	**2111**	3111	1112	2112
	···	···	···	1011	2011	···	···	···
	···	···	···	1201	2201	···	···	···
	···	···	···	1101	2101	···	···	···
$\downarrow^{\Phi}$								
	···	···	···	0201	1201	···	···	···
	···	···	···	5101	0101	···	···	···
	···	···	···	0021	1021	···	···	···
	5220	0220	4221	**5221**	**0221**	1221	5222	0222
G	4120	5120	3121	**4121**	**5121**	0121	4122	5122
	···	···	···	3021	4021	···	···	···
	···	···	···	5011	0011	···	···	···
	4210	5210	3211	**4211**	**5211**	0211	4212	5212
	3110	4110	2111	**3111**	**4111**	5111	3112	4112
	···	···	···	2011	3011	···	···	···
	···	···	···	3201	4201	···	···	···
	···	···	···	2101	3101	···	···	···

We place $D = (V, E)$ in such a way that V comprises: **(a)** the vertices $e_1+e_2+e_3$, $2e_1+e_2+e_3$, $e_1 + 2e_2 + e_3$ and $2e_1 + 2e_2 + e_3$ of $\sigma_{0,0,0}$; **(b)** $4e_1 + e_2 - e_3$, $5e_1 + e_2 - e_3$, $4e_1 + 2e_2 - e_3$ and $5e_1 + 2e_2 - e_3$ of $\sigma_{1,0,-1}$; **(c)** $3e_1$; **(d)** O. This yields a total of 10 vertices, to which we must add their 44 neighbors, namely, respectively: **(a′)** $e_1 + e_2$, $2e_1 + e_2$, $e_1 + 2e_2$,$2e_1 + 2e_2$, $e_1 + e_2 + 2e_3$, $2e_1 + e_2 + 2e_3$, $e_1 + 2e_2 + 2e_3$, $2e_1 + 2e_2 + 2e_3$, $e_2 + e_3$, $3e_1 + e_2 + e_3$, $2e_2 + e_3$, $3e_1 + 2e_2 + e_3$, $e_1 + e_3$, $2e_1 + e_3$, $e_1 + 3e_2 + e_3$ and $2e_1 + 3e_2 + e_3$; **(b′)** $4e_1 + e_2$, $5e_1 + e_2$, $4e_1 + 2e_2$,$5e_1 + 2e_2$, $4e_1+e_2+2e_3$, $5e_1+e_2+2e_3$, $4e_1+2e_2+2e_3$, $5e_1+2e_2+2e_3$, $3e_1+e_2+e_3$, $6e_1+e_2+e_3$, $5e_2+e_3$, $6_1 + 2e_2 + e_3$, $4e_1 + e_3$, $5e_1 + e_3$, $4e_1 + 3e_2 + e_3$ and $5e_1 + 3e_2 + e_3$; **(c′)** $2e_1$, $4e_1$, $3e_1 + e_2$ and $3e_1 - e_2$, $3e_1 + e_3$ and $3e_1 - e_3$; **(d′)** $-e_1$, e_1, $-e_2$, e_2, e_3 and $-e_3$. Thus, $|V| = 54$ and D contains the vertices O, e_1, e_2, e_3 as required by Proposition 5.1. We choose $G = \mathbb{Z}_6 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$. The element g_i of G that is assigned to the vertex e_i, for $i = 1, 2, 3$, is given by expressing it without parentheses or commas, as follows: $g_1 = \Phi(e_1) = 100$, $g_2 = \Phi(e_2) = 110$, and $g_3 = \Phi(e_3) = 001$.

We need to show that the restriction of the mapping $\Phi((a_1,\dots,a_n)) = \Phi(e_1)^{a_1}\circ\ldots\circ\Phi(e_n)^{a_n} = a_1g_1+\ldots+a_ng_n$ to V is a bijection. This can be verified by means of Table I, where elements of $V\subset\mathbb{Z}^3$ are disposed on the left-hand side (in slices for constant $x_3 = 2, 1, 0, -1, -2$) and their images via Φ in G accordingly on the right-hand side; parentheses and commas avoided both for the elements of V and for those of G, with $O := 000$, $e_1 = 100$, $e_2 = 010$, $e_3 = 001$, $\dots$ to save space, and where we indicated -1 and -2 respectively by $\bar{1}$ and $\bar{2}$; the positions of elements of $\mathbb{Z}^3\setminus V$ and of their images via Φ are indicated by means of ellipsis, for a better reference, and those vertices in items (a)-(d) above are in bold trace. □

7. PROOF OF THEOREM 4.2

In order to extend the construction of Section 6, consider in Λ_n $(n>3)$ the subset $X\subset\mathbb{Z}^n$ of vertices whose coordinates are divisible by 3. Clearly, X is a lattice of $\mathbb{Z}^n$. Each element of X is in a subset $[3x_1, 2+3x_1]\times[3x_2, 2+3x_2]\times\cdots\times[3x_n, 2+3x_n]$ of $\mathbb{Z}^n$, with $x_1, x_2, \dots, x_n\in\mathbb{Z}$. Such a subset is a constellation of the lattice X and from now on will be called an *n-grenade*.

A lattice-like $(1, n-2)$-RPDS$[Q_{n-1}, Q_0; 2, 2]$ as claimed in Theorem 4.2 is composed by X and a subset Y as follows. We select in each n-grenade $\tau_{x_1,x_2,\dots,x_n}$ the $(n-1)$-cube $\sigma_{x_1,x_2,\dots,x_n} = [1+x_1, 2+x_1]\times[1+x_2, 2+x_2]\times\cdots\times[1+x_{n-1}, 2+x_{n-1}]\times\{1+\delta+x_n\}$, where δ is the rest of dividing $x_1+x_2+\cdots+x_{n-1}$ by 2. Then Y given by the union of all $(n-1)$-cubes $\sigma_{x_1,x_2,\dots,x_n}$ with $x_1, x_2, \dots, x_n\in\mathbb{Z}$. Theorem 4.2 is a direct corollary of the following lemma.

Lemma 7.1. *$X\cup Y$ is a lattice-like $(1, n-2)$-*RPDS$[Q_{n-1}, Q_0; 2, 2]$ *of Λ_n. Its induced components are centers of the copies of respective spheres $W^\rho_{n,n-1,1}$ and $W^\rho_{n,0,n-2}$ in a specific tiling of Λ_n.*

Proof. We will construct the claimed $(1, n-2)$-RPDS$[Q_{n-1}, Q_0; 2, 2]$ by applying Proposition 5.1. Let H be given by the union of $\sigma_{0,0,\dots,0,0}$, $\sigma_{1,0,\dots,0,-1}$, $\{3e_1\}$ and $\{O\}$. We place the graph $D = (V, E)$ isomorphic to H^* so that V comprises two copies of Q_{n-1} with vertices of the form: **(a)** $\beta_1e_1+\beta_2e_2+\cdots+\beta_{n-1}e_{n-1}+e_n$ in $\sigma_{0,0,\dots,0,0}$, where $\beta_i\in\{1,2\}$ for $1\le i<n$, and **(b)** $(3+\beta_1)e_1+\beta_2e_2+\cdots+\beta_{n-1}e_{n-1}-e_n$ in $\sigma_{1,0,\dots,0,-1}$, and the isolated vertices **(c)** $3e_1$ and **(d)** O. This yields a total of 2^n+2 vertices, to which we must add their $2\times3^n-2^n-2$ neighbors, namely the vertices of the forms, respectively: **(a′)** $\beta_1e_1+\beta_2e_2+\cdots+\beta_{n-1}e_{n-1}+e_n\pm e_n$, $\beta_2e_2+\cdots+\beta_{n-1}e_{n-1}+e_n$, $3e_1+\beta_2e_2+\cdots+\beta_{n-1}e_{n-1}+e_n$, $\beta_1e_1+\beta_3e_3+\cdots+\beta_{n-1}e_{n-1}+e_n$, $\beta_1e_1+3e_2+\beta_3e_3+\cdots+\beta_{n-1}e_{n-1}+e_n$, …, $\beta_1e_1+\cdots+\beta_{n-2}e_{n-2}+e_n$, $\beta_1e_1+\cdots+\beta_{n-2}e_{n-2}+3e_{n-1}+e_n$; **(b′)** $(3+\beta_1)e_1+\beta_2e_2+\cdots+\beta_{n-1}e_{n-1}-e_n\pm e_n$, $3e_1+\beta_2e_2+\cdots+\beta_{n-1}e_{n-1}-e_n$, $6e_1+\beta_2e_2+\cdots+\beta_{n-1}e_{n-1}-e_n$, $(3+\beta_1)e_1+\beta_3e_3+\cdots+\beta_{n-1}e_{n-1}-e_n$, $(3+\beta_1)e_1+3e_2+\beta_3e_3+\cdots+\beta_{n-1}e_{n-1}-e_n$, …, $(3+\beta_1)e_1+\cdots+\beta_{n-2}e_{n-2}-e_n$, $(3+\beta_1)e_1+\cdots+\beta_{n-2}e_{n-2}+3e_{n-1}-e_n$; **(c′)** $3e_1$ added to any of $\pm e_1$, …, $\pm e_n$ and the sums of up to $n-2$ of $\pm e_1$, …, $\pm e_n$, namely $3e_1\pm e_1$, $3e_1\pm e_2$, …, $3e_1\pm e_3\pm e_4\pm\cdots\pm e_n$; **(d′)** $\pm e_1$, …, $\pm e_n$ and the sums of up to $n-2$ of $\pm e_1$, …, $\pm e_n$, namely $\pm e_1\pm e_2$, …, $\pm e_3\pm e_4\pm\cdots\pm e_n$. Counting these items yields the following numbers.

Subtotal of vertices in (a') and (b'): $2[2^n+2(n-1)2^{n-2}] = 2^{n+1}+n2^n-2^n$. Subtotal of vertices in (c') and (d'): $2\left[\sum_{i=1}^{n-2}2^i\binom{n}{i}\right] = 2[(1+2)^n-2^n-n2^{n-1}-1] = 2[3^n-2^n-n2^{n-1}-1] = 2\times3^n-2^{n+1}-n2^n-2$. Total of vertices: $2^{n+1}+n2^n-2^n+2x3^n-2^{n+1}-n2^n-2 = 2\times3^n-2^n-2$.

Thus, $|V| = 2 \times 3^n$ and D contains O and e_i, for $i = 1, \ldots, n$, as required by Proposition 5.1. We choose $G = \mathbb{Z}_6 \oplus (\mathbb{Z}_3)^{n-1}$. The element g_i of G that is assigned to the vertex e_i, for $i = 1, 2, \ldots, n$, is given by expressing it without parentheses or commas, as follows: $\Phi(e_1) = 10\cdots0$, $\Phi(e_2) = 110\cdots0$, $\Phi(e_3) = 1010\cdots0$, …, $\Phi(e_{n-1}) = 10\cdots010$, and $\Phi(e_n) = 0\cdots01$. We need to show that the restriction of the mapping $\Phi((a_1, \ldots, a_n)) = \Phi(e_1)^{a_1} \circ \ldots \circ \Phi(e_n)^{a_n} = a_1g_1 + \ldots + a_ng_n$ to V is a bijection. To help in visualizing the construction, we present tables for the case $n = 4$. Table II shows the assignment Φ restricted to the sphere $W\rho_{4,0,2}$ around $O = 0000$ (items (d) and (d′)), with the corresponding values in G presented in the lower half of the table.

From Table II, a similar table is obtained for the sphere $W^{\rho}_{4,0,2}$ around $3e_1 = 3000$ (cases (c) and (c′)) by adding 3 to the first entry of the 4-tuples in the upper half of the table, and adding 3 mod 6 to the first entry of the 4-tuples in the lower half. Table III shows the assignment Φ restricted to the sphere $W^{\rho}_{4,3,1}$ spanned by the vertices in items (a) and (a′), with the corresponding values in G presented in the lower half of the table.

From Table III, a similar table is obtained for the sphere $W^{\rho}_{4,3,1}$ spanned by the vertices in items (b) and (b′) by adding 3 to the first entry of the 4-tuples in the upper half of the table and modifying accordingly the last entry, i.e. $1 \to (-1)$; $0 \to (-2)$; $2 \to 0$, and by adding 3 mod 6 to the first entry of the 4-tuples in the lower half and applying the permutation that exchanges the last entries 1 and 2, with null last entries kept fixed. By combining the four tables obtained, it can be seen that Φ is indeed as required. These four tables, of which just two are displayed, will be denoted A, B, C, D, the same letters (capitalized now) corresponding to the lower-case ones used. We separate the first entry α_1 of an element $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of $G = \mathbb{Z}_6 \oplus (\mathbb{Z}_3)^{n-1}$ from the remaining entries, considering for each of the resulting $(n-1)$-tuples $\alpha' = (\alpha_2, \ldots, \alpha_n)$ in $G' = (\mathbb{Z}_3)^{n-1}$ a corresponding terminal $(n-1)$-tuple $\beta_2, \ldots, \beta_n$ in $\mathbb{Z}^{n-1}$ of a pre-image n-tuple $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ in $\mathbb{Z}^n$ via Φ, in order to establish, for each terminal $(n-1)$-tuple α', a correspondence from the first entries α_1 to the first entries β_1, that can be grouped depending on the corresponding tables A, B, C, D.

In Table IV, for $n = 3$ and 4, we show that this grouping depends on the tables A, B, C, D above, for the four rainbow spheres involved in V, in four corresponding columns. In each of these four columns we can see three sub-columns showing from left to right: the different possible values of $\alpha_1 = \alpha_1^{\xi}$, ($\xi = A < B < C < D$, without separating commas) that pre-fixed to the row-heading α' yields a corresponding α; the corresponding values of $\beta_1 = \beta_1^{\xi}$ (again without separating commas); and a uniquely corresponding $\beta' = \beta'_B$.

In general, for any $n \geq 3$ we find it is necessary to consider six cases of α' as appearing in Table IV, namely:

(1) $\alpha' \in \{1,2\}^{n-2} \times \{1\}$; here $\beta_1^B \in \{0,1,2,3\}$, $\beta'_B = \alpha'$ and $\alpha_1^B = \beta_1^B + \alpha_2 + \cdots + \alpha_{n-1} \in \mathbb{Z}_6$; also $\beta_1^D \in \{4,5\}$, $\beta'_D = \alpha' - (0, \ldots, 0, 3)$ and $\alpha_1^D = \beta_1^D + \alpha_2 + \cdots + \alpha_{n-1} \in \mathbb{Z}_6$; moreover, nothing is contributed for $\xi = A, C$.

TABLE IV

α'	α_1^A	β_1^A	β'_A	α_1^B	β_1^B	β'_B	α_1^C	β_1^C	β'_C	α_1^D	β_1^D	β'_D
−−	−−	−−	−−	−−	−−	−−	−−	−−	−−	−−	−−	−−
11				1234	0123	11				50	45	12-
21				2345	0123	21				01	45	22-
111				2345	0123	111				01	45	112-
211				3450	0123	211				12	45	212-
121				3450	0123	121				12	45	121
221				4501	0123	221				23	45	222-
12				23	12	12				4501	3450	1⊢
22				34	12	22				5012	3450	1⊢
112				34	12	112				5012	3450	11⊢
212				45	12	212				0123	3450	21⊢
122				45	12	122				0123	3450	12⊢
222				50	12	222				1234	3450	22⊢
10	1	0	10	23	12	10	4	3	10	50	45	10
20	5	0	⊢0	34	12	20	2	3	⊢0	01	45	20
110	2	0	110	34	12	110	5	3	110	01	45	110
210	0	0	⊢10	45	12	210	3	3	⊢10	12	45	210
120	0	0	1⊢0	45	12	120	3	3	1⊢0	12	45	120
220	4	0	⊢⊢0	50	12	220	1	3	⊢⊢0	23	45	220
01	0	0	01	12	12	01	3	3	01			
				45	12	31						
101	1	0	101	23	12	101	4	3	101			
				50	12	131						
201	5	0	⊢01	34	12	201	2	3	⊢01			
				01	12	231						
011	1	0	011	23	12	011	4	3	011			
				50	12	311						
021	5	0	0⊢1	34	12	021	2	3	0⊢1			
				01	12	321						
02	0	0	0⊢				3	3	0⊢	12	45	3⊢
										45	45	0⊢
102	1	0	10⊢				4	3	10⊢	23	45	13⊢
										50	45	01⊢
202	5	0	⊢0⊢				2	3	⊢0⊢	34	45	23⊢
										01	45	20⊢
012	1	0	01⊢				4	3	01⊢	23	45	31⊢
										50	45	01⊢
022	5	0	0⊢⊢				2	3	0⊢⊢	34	45	32⊢
										01	45	02⊢
00	501	⊢01	00				234	234	00			
000	501	⊢01	000				234	234	000			
100	012	⊢01	100				345	234	100			
200	450	⊢01	⊢00				123	234	⊢00			
010	012	⊢01	010				345	234	010			
020	450	⊢01	010				123	234	0⊢0			
001	501	⊢01	001				234	234	001			
002	501	⊢01	00⊢				234	234	00⊢			

(2) $\alpha' \in \{1,2\}^{n-2} \times \{2\}$; here $\beta_1^D \in \{3,4,5,0\}$, $\beta'_D = \alpha' - (0,\dots,0,3)$ and $\alpha_1^D = \beta_1^D + \alpha_2 + \cdots + \alpha_{n-1} \in \mathbb{Z}_6$; also $\beta_1^B \in \{1,2\}$, $\beta'_B = \alpha'$ and $\alpha_1^B = \beta_1^B + \alpha_2 + \cdots + \alpha_{n-1} \in \mathbb{Z}_6$; moreover, nothing is contributed for $\xi = A, C$.

(3) $\alpha' \in \{1,2\}^{n-2} \times \{0\}$; here $\beta_1^A = 0$, $\beta_1^B \in \{1,2\}$, $\beta_1^C = 3$ and $\beta_1^D \in \{4,5\}$; $\beta'_\xi = \alpha'$, for

$\xi \in \{B, D\}$; for $\xi \in \{A, C\}$, $\beta' = \alpha''$, where α'' differs from α' just in that each entry γ valued 2 in α' is modified to $\gamma' = -1$ in α''; for any other entry γ of α', we set $\gamma' = \gamma$; then α_1^A is the sum mod 6 of the values γ' corresponding to the entries γ of α', and $\alpha_1^C = 3 + \alpha_1^A$ mod 6; if δ is the sum of the entries of α', then to each feasible β_1^ξ corresponds $\alpha_1^\xi = \beta_1^\xi + \delta$, where $\xi = B, D$.

(4) α' obtained from $\alpha'' \in \{1,2\}^{n-2} \times \{1\}$ by changing one entry $\neq \alpha_n$ to 0; here $\beta_1^A = 0$, $\beta_1^C = 3$ and $\beta_1^B \in \{1, 2\}$; β'_ξ is obtained from α' as in item 3 above, for $\xi = A, C$; for each of $\beta_1^B = 1, 2$, there are two instances for β'_B, namely $\beta'_{B'} = \alpha'$ and $\beta'_{B''}$ obtained from α' by replacing each null entry by 3; then, to each feasible β_1^B corresponds $\alpha_1^{B'} = \beta_1^B + \alpha_2 + \cdots + \alpha_{n-1} \in \mathbb{Z}_6$ and $\alpha_1^{B''} = \alpha_1^{B'} + 3$ (mod 6), respectively for $\beta'_{B'}$ and for $\beta'_{B''}$; moreover, nothing is contributed for $\xi = D$.

(5) α' obtained from $\alpha''' \in \{1,2\}^{n-2} \times \{2\}$ by changing one entry $\neq \alpha_n$ to 0; here $\beta_1^A = 0$, $\beta_1^C = 3$ and $\beta_1^D \in \{4, 5\}$; β'_ξ is obtained from α' as in item 3 above, for $\xi = A, C$; for each of $\beta_1^D = 4, 5$, there are two instances for β'_D, namely $\beta'_{D'} = \alpha'$ and $\beta'_{D''}$, obtained from α' by replacing each null entry by 3; then, to each feasible β_1^D corresponds $\alpha_1^{D'} = \beta_1^D + \alpha_2 + \cdots + \alpha_{n-1} \in \mathbb{Z}_6$ and $\alpha_1^{D''} = \alpha_1^{D'} + 3$ (mod 6), respectively for $\beta'_{D'}$ and for $\beta'_{D''}$; moreover, nothing is contributed for $\xi = B$.

(6) α' with at least two null entries; here $\beta_1^A \in \{-1, 0, 1\}$ and $\beta_1^C \in \{2, 3, 4\}$; β'_ξ is obtained from α' as in item 3 above, for $\xi = A, C$; to each β_1^ξ as above corresponds $\alpha_1^\xi = \beta'_\xi + \alpha_2 + \cdots + \alpha_{n-1} \in \mathbb{Z}_6$; moreover, nothing is contributed for $\xi = B, D$.

By combining these six cases, it is seen that the restriction of the additive-group epimorphism $\Phi : \mathbb{Z}^6 \to G = \mathbb{Z}_6 \oplus (\mathbb{Z}_3)^{n-1}$ is effectively a bijection, for every $n \geq 3$. Indeed, the cardinalities of those $\alpha' \in (\mathbb{Z}_3)^{n-1}$ are respectively: **(1)** 2^{n-2}; **(2)** 2^{n-2}; **(3)** 2^{n-2}; **(4)** $(n-2)2^{n-3}$; **(5)** $(n-2)2^{n-3}$; and **(6)** $2(3^n) - 3(2^{n-2}) - 2(n-2)2^{n-3}$. These cardinalities add up to $|V| = 2(3^n) = |G|$, as required. □

8. CONCLUSION AND OPEN PROBLEMS

After reviewing previous work on perfect dominating sets and perfect distance dominating sets, we continued here with the novel notion of rainbow distance in graph lattices. This was done in order to introduce rainbow perfect dominating sets or RPDSs in those graphs as well as in their quotient toroidal graphs. These are cartesian products of cycles, with possible applications to parallel computers.

Let $0 < n \in \mathbb{Z}$. Two constructions of lattice-like RPDSs were presented in this work having their induced components C with:

(i) vertex sets $V(C)$ whose convex hulls are n-parallelotopes (resp., both $(n-1)$- and 0-cubes) and,

(ii) each $V(C)$ contained in a corresponding rainbow sphere centered at C with radius n (resp., radii 1 and $n-2$).

These rainbow spheres form a partition of $\mathbb{Z}^n$, in each one of the two constructions. Such a partition can be projected into partitions of the quotient toroidal graphs.

We find it not clear that similar lattice RPDS results as in (i) above hold with r-parallelotopes, for $0 < r < n$. So this is a source of open problems on the existence or nonexistence of such RPDSs.

It has to be seen whether the construction of Theorem 4.1 is unique or not. (A result in this vein was obtained in [14] to the effect that there is but one PDS in Λ_3 inducing just square components). The same may be inquired for the constructions obtained in Theorem 4.2.

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