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Self-dual embeddings of $K_{4m,4n}$ in different orientable and nonorientable pseudosurfaces with the same Euler characteristic[1]

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Abstract

A proper embedding of a graph G in a pseudosurface P is an embedding in which the regions of the complement of G in P are homeomorphic to discs and a vertex of G appears at each pinch-point in P; we say that a proper embedding of G in P is self dual if there exists an isomorphism from G to its dual graph. We give an explicit construction of a self-dual embedding of the complete bipartite graph $K_{4m,4n}$ in an orientable pseudosurface for all $m,\ n \geq 1$; we show that this embedding maximizes the number of umbrellas of each vertex and has the property that for any vertex v of $K_{4m,4n}$, there are two faces of the constructed embedding that intersect all umbrellas of v. Leveraging these properties and applying a lemma of Bruhn and Diestel, we apply a surgery introduced here or a different known surgery of Edmonds to each of our constructed embeddings for which at least one of m or n is at least 2. The result of these surgeries is that there exist distinct orientable and nonorientable pseudosurfaces with the same Euler characteristic that feature a self-dual embedding of $K_{4m,4n}$.

Keywords: graph embedding, self-dual embedding, surgeries, complete bipartite graph, pseudosurface
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[1]This article is dedicated to the memory of Dan Archdeacon.
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1. Introduction

To us, a graph is a finite and connected multigraph, and a surface is a compact and connected 2-manifold without boundary; we let G denote a graph and S denote a surface. A *cellular embedding* of G in S is an embedding in which the complement of G in S is a disjoint union of regions (called *faces*), each of which is homeomorphic to a disc. We will let $G \to S$ denote a cellular embedding of G in S. Following [11], given $G \to S$, we define the *dual graph* G^* and the *dual embedding* $(G \to S)^*$ as follows: the vertices of G^* are the "centers" of the faces of G in S, and each edge e^* corresponds bijectively to an edge e of G and connects the vertice(s) of G^* corresponding to the face(s) on either side of e. We say that two embeddings of G in surfaces S and T, denoted $i\colon G \to S$ and $j\colon G \to T$, are equivalent if there is an orientation-preserving homeomorphism $f\colon S \to T$ such that $f \circ i = j$. Furthermore, $G \to S$ is *self dual* if the cellular structure of vertices, edges, and faces (commonly called 0-cells, 1-cells, and 2-cells, respectively) given by G in S is isomorphic to the cellular structure given by G^* in S; this implies that G and G^* are isomorphic. By [11, §1.4.8], this also implies that $((G \to S)^*)^*$ and $G \to S$ are equivalent embeddings. An example of a self-dual graph embedding in a surface is given in Figure 1.

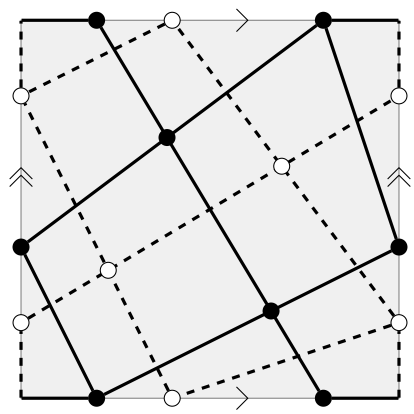

Figure 1. The self-dual embedding of the complete graph K_5 in the torus. The dual graph is drawn with dashed edges joining hollow vertices.

The purpose of this article is to begin an investigation of self-dual embeddings and embeddability of graphs in pseudosurfaces. To place this discussion in some context, we describe some of the research concerning self-dual embeddings of graphs in surfaces, which is quite rich. Recently, Abrams and Slilaty have merged the study of self-dual graph embeddings in surfaces with the study of symmetries of cellular decompositions of surfaces [1]. Moreover, the self-dual graph embeddings in various surfaces have been cataloged from different viewpoints: in the sphere by Abrams and Slilaty [1], and by B. Servatius and H. Sevatius [16, 17]; in the projective plane by Abrams and Slilaty [1], and by Archdeacon and Negami [7]; and in all other surfaces of Euler characteristic at least -1 [1]. Archdeacon and Hartsfield in [6] gave results concerning the orientable and nonorientable self-dual embeddability of complete bipartite graphs in surfaces; the current work can be viewed as the first step in extending their results to pseudosurfaces. Of similar interest is another article by Archdeacon [3] in which he shows that one can use ordinary voltage graph theory to create self-dual graph embeddings in surfaces, and other non self-dual embeddings in surfaces for which the dual is predictable and has very specific properties.

Following [2] a closed, connected *pseudosurface* is a connected topological space obtained from a disjoint union of surfaces via a finite number of point identifications, called *pinches*. A surface is therefore a special case of a pseudosurface. The points that are identified with other points are called *pinchpoints*. A small-enough neighborhood of a pinchpoint is homeomorphic to the union of discs identified at a point; each identified disc is called an *umbrella* of the pinchpoint. A *proper embedding* of a graph G in a pseudosurface P is an embedding in which each of the regions of the complement of G in P is homeomorphic to a disc and a vertex of G appears at each pinchpoint of P. We shall let $G \to P$ denote a proper embedding of G in P. We let $F(G \to P)$ denote the set of faces of $G \to P$. Given $G \to P$, the definitions of the dual graph and the dual embedding of $G \to P$ are immediate natural extensions of the definition of the dual graph and dual embedding of $G \to S$, respectively: $(G \to P)^*$ captures the incidence of faces and edges of G in P. However, as evidenced by Figure 2, $((G \to P)^*)^*$ is not necessarily well defined since $(G \to P)^*$ is not a proper embedding if P has any pinchpoints. We therefore give a weaker notion of graph self-duality for pseudosurfaces. For a pseudosurface P with at least one pinchpoint, we say that $G \to P$ is *self dual* if G^* is isomorphic to G, which still requires that the incidence of faces and edges in $G \to P$ is isomorphic to the incidence of vertices and edges in G.

We should note that there has been some attention given to the study of embeddings and embeddability of graphs in pseudosurfaces. Archdeacon's survey article [4, §5.7] and the introduction of [2] (which itself is about the embeddability of graphs in pseudosurfaces) contain some relevant references. Among other relevant works is [5], in which Archdeacon and Bonnington give the list of 21 graphs that form all obstructions of embeddability of cubic graphs in the pinched sphere. Part of the study of graph embeddings in pseudosurfaces has been about proving theorems in design theory. Among these efforts are works by Garman [10] and White [18]. Finally, the current authors and E. Rarity in [14] found the smallest simple graphs with a self-dual embedding in a pseudosurface with at least one pinchpoint.

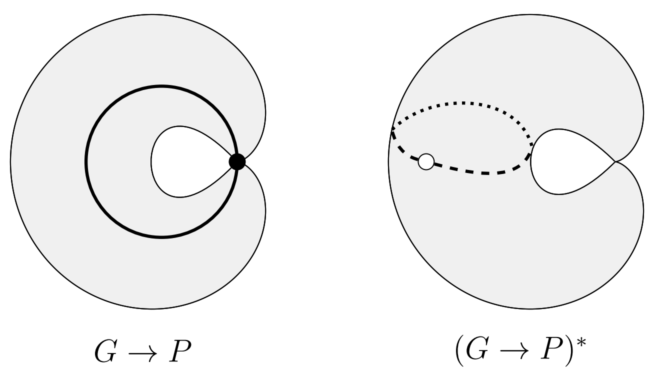

Figure 2. An example of a self-dual embedding of a graph in the pinched sphere, and the corresponding dual embedding.

In Section 2, we construct self-dual embeddings of $K_{4m,4n}$ in orientable pseudosurfaces for all $m, n \geq 1$. In Section 3, we introduce our surgery and state that of Edmonds, including the necessary background information. Finally, in Section 4, we apply the surgeries of Section 3 to our embeddings from Section 2 to produce self-dual embeddings of $K_{4m,4n}$ in different orientable and nonorientable pseudosurfaces with the same Euler characteristic.

2. Orientable self-dual embeddings of $K_{4m,4n}$ in pseudosurfaces

2.1. Background necessary for our embeddings of $K_{4m,4n}$

Recall that G is a finite and connected multigraph. We orient the edges of G and let $\vec{E}(G)$ denote the set of directed edges of G; each directed edge we will call a *dart* on an edge of G. For each edge e of G, we will call one of the two corresponding darts the *positive dart* and the other the *negative dart*; the positive dart on an edge is assumed to correspond to the given orientation on that edge. For $e \in E(G)$, we will let e also denote the positive dart on that edge and e^- the negative dart. If G is a simple graph, then for $v_1, v_2 \in V(G)$, we let $\{v_1, v_2\}$ denote an undirected edge joining v_1 and v_2, and we let v_1v_2 denote the dart from v_1 to v_2. If $e \in \vec{E}(G) = v_1v_2$, then we say that the *head of* e and *tail of* e, denoted $h(e)$ and $t(e)$, are v_2 and v_1, respectively; $e^- = v_2v_1$, and $(e^-)^- = e$. A *walk* in a graph G is a sequence of darts $e_1e_2 \ldots e_n$, which in general may include negative darts, such that $h(e_i) = t(e_{i+1})$. If $h(e_n) = t(e_1)$, then we say that W is a *closed walk*. We will use a sequence of vertices to denote a walk in a simple graph since a simple graph has at most one edge joining two vertices; if a walk $W = e_1e_2 \ldots e_n$ satisfies $h(e_i) = v_i$ and $t(e_i) = v_{i-1}$, then we may let W be denoted by $v_0v_1 \ldots v_n$. If $W_1 = e_1 \ldots e_n$ and $W_2 = e'_1 \ldots e'_m$ are walks having the property that $h(e_n) = t(e'_1)$, then we let $W_1W_2 = e_1 \ldots e_ne'_1 \ldots e'_m$.

A surface or pseudosurface is said to be *orientable* if it can be triangulated by a proper graph embedding whose faces are compatibly oriented.

One way to encode a proper graph embedding in an orientable surface is with a description of a cyclic permutation of the edges incident to each vertex, with a convention that all cyclic permutations capture the edges in a counterclockwise or clockwise order; the same order must apply to all vertices in order to combinatorially capture the embedding. This is called a *rotation scheme* or *rotation system*. In order to describe a proper embedding in a pseudosurface with pinchpoints this way, one must be sure to describe the cyclic ordering of edges incident to each vertex v in each umbrella of v. In the case of proper embeddings in orientable pseudosurfaces with pinchpoints, one must add additional information to the encoding; for our purposes, this additional information will be captured in the collection of facial boundary walks. In the case of simple graphs, since there are no parallel edges joining two vertices, we may characterize the cyclic permutations of edges incident to a vertex by listing its neighboring vertices instead. We give here an example of a rotation of a vertex v in a simple graph with the neighboring vertices of v representing the edges appearing in each of three umbrellas represented by distinct parenthetical encapsulations.

$$v: \quad (v_5v_0v_{11}v_6)(v_1v_2v_9v_{10})(v_3v_4v_7v_8)$$

For a complete treatment of rotation schemes for graphs embedded in surfaces, including nonorientable surfaces, the reader is referred to [11]. Since we will not be representing nonorientable embeddings as rotation schemes, we do not introduce the corresponding notation.

Let $A = \{a_0, ..., a_{r-1}\}$ and $B = \{b_0, ..., b_{s-1}\}$ be the bipartition sets of $K_{r,s}$, where $r \leq s$. We say that the dart $a_k b_\ell$ has *slope* $\ell - k$ (reduced modulo s).

2.2. The construction

Definition 1. *Consider $G \to P$. We say that a facial boundary walk F* covers *a vertex v if F visits every umbrella of v, and we say that $G \to P$ is* doubly covered *if every vertex of G is covered by two different facial boundary walks.*

Consider $G \to P$. If an umbrella of a vertex v contains only one or two neighbors of v, then the dual graph of that embedding will contain a loop or two parallel edges, respectively. Moreover, if an umbrella of v contains precisely three neighbors of v, then the dual graph contains a triangle. Thus, since $K_{4m,4n}$ is a simple triangle-free graph, every umbrella of a vertex in a self-dual embedding of $K_{4m,4n}$ must include at least four ends of edges incident to four distinct neighbors of that vertex, and it follows that the maximum number of umbrellas at any vertex of degree d is $\lfloor d/4 \rfloor$.

Definition 2. *Consider $K_{4m,4n}$ and $V(K_{4m,4n}) = A \cup B$, where $A = \{a_0, ..., a_{4m-1}\}$ and $B = \{b_0, ..., b_{4n-1}\}$ are the bipartition sets of $K_{4m,4n}$. We call an embedding $K_{4m,4n} \to P$* full *if every vertex in A has n umbrellas and every vertex in B has m umbrellas.*

It is immediate from Definition 2 and the preceding paragraph that if a proper embedding of $K_{4m,4n}$ is full, then we may not add any more umbrellas to any more vertices and still expect to have a proper embedding of $K_{4m,4n}$ featuring a simple dual graph without any cycles of length 3.

Theorem 2.1. *There exists a doubly-covered, full, self-dual embedding of $K_{4m,4n}$ in an orientable pseudosurface for all $m, n \geq 1$.*

Proof. We construct the boundary walk of each face in the embedding. Let $A = \{a_0, a_1, \ldots, a_{4m-1}\}$ and $B = \{b_0, b_1, \ldots, b_{4n-1}\}$ be the bipartition sets of $K_{4m,4n}$. For each i satisfying $0 \leq i \leq m-1$ define the closed walks $X_{i,0}, X_{i,1}, X_{i,2}, X_{i,3}$ as follows.

$$\begin{aligned}
X_{i,0}: &\quad (a_{4i} b_0 a_{4i+1} b_1 a_{4i} b_2 a_{4i+1} b_3 \cdots a_{4i} b_{2n-2} a_{4i+1} b_{2n-1}) \\
X_{i,1}: &\quad (a_{4i} b_{2n} a_{4i+1} b_{2n+1} a_{4i} b_{2n+2} a_{4i+1} b_{2n+3} \cdots a_{4i} b_{4n-2} a_{4i+1} b_{4n-1}) \\
X_{i,2}: &\quad (a_{4i+2} b_0 a_{4i+3} b_1 a_{4i+2} b_2 a_{4i+3} b_3 \cdots a_{4i+2} b_{2n-2} a_{4i+3} b_{2n-1}) \\
X_{i,3}: &\quad (a_{4i+2} b_{2n} a_{4i+3} b_{2n+1} a_{4i+2} b_{2n+2} a_{4i+3} b_{2n+3} \cdots a_{4i+2} b_{4n-2} a_{4i+3} b_{4n-1})
\end{aligned}$$

For each j satisfying $0 \leq j \leq n-1$ define the closed walks $Y_{j,0}, Y_{j,1}, Y_{j,2}, Y_{j,3}$ as follows.

$$\begin{aligned}
Y_{j,0}: &\quad (b_{2j} a_0 b_{4n-2j-1} a_3 b_{2j} a_4 b_{4n-2j-1} a_7 \cdots b_{2j} a_{4m-4} b_{4n-2j-1} a_{4m-1}) \\
Y_{j,1}: &\quad (b_{2j} a_{4m-2} b_{4n-2j-1} a_{4m-3} b_{2j} a_{4m-6} b_{4n-2j-1} a_{4m-7} \cdots b_{2j} a_2 b_{4n-2j-1} a_1) \\
Y_{j,2}: &\quad (b_{2j+1} a_{4m-1} b_{4n-2j-2} a_{4m-4} b_{2j+1} a_{4m-5} b_{4n-2j-2} a_{4m-8} \cdots \\
&\quad b_{2j+1} a_3 b_{4n-2j-2} a_0) \\
Y_{j,3}: &\quad (b_{2j+1} a_1 b_{4n-2j-2} a_2 b_{2j+1} a_5 b_{4n-2j-2} a_6 \cdots b_{2j+1} a_{4m-3} b_{4n-2j-2} a_{4m-2})
\end{aligned}$$

We show that the collection

$$\mathcal{F} = \{X_{i,0}, X_{i,1}, X_{i,2}, X_{i,3} \,|\, 0 \le i \le m-1\} \cup \{Y_{j,0}, Y_{j,1}, Y_{j,2}, Y_{j,3} \,|\, 0 \le j \le n-1\}$$

forms a family of facial boundary walks that yields a self-dual embedding of $K_{4m,4n}$ in an orientable pseudosurface with the desired properties.

Note that each pair of X-walks is edge-disjoint and each pair of Y-walks is edge-disjoint. The intersection of any X-walk with any Y-walk is given by the undirected edge below, where i and j satisfy $0 \le i \le m-1$ and $0 \le j \le n-1$, respectively.

$$\begin{array}{ll}
X_{i,0} \cap Y_{j,0} = \{a_{4i}, b_{2j}\} & X_{i,0} \cap Y_{j,1} = \{a_{4i+1}, b_{2j}\} \\
X_{i,0} \cap Y_{j,2} = \{a_{4i}, b_{2j+1}\} & X_{i,0} \cap Y_{j,3} = \{a_{4i+1}, b_{2j+1}\} \\
X_{i,1} \cap Y_{j,0} = \{a_{4i}, b_{4n-2j-1}\} & X_{i,1} \cap Y_{j,1} = \{a_{4i+1}, b_{4n-2j-1}\} \\
X_{i,1} \cap Y_{j,2} = \{a_{4i}, b_{4n-2j-2}\} & X_{i,1} \cap Y_{j,3} = \{a_{4i+1}, b_{4n-2j-2}\} \\
X_{i,2} \cap Y_{j,0} = \{a_{4i+3}, b_{2j}\} & X_{i,2} \cap Y_{j,1} = \{a_{4i+2}, b_{2j}\} \\
X_{i,2} \cap Y_{j,2} = \{a_{4i+3}, b_{2j+1}\} & X_{i,2} \cap Y_{j,3} = \{a_{4i+2}, b_{2j+1}\} \\
X_{i,3} \cap Y_{j,0} = \{a_{4i+3}, b_{4n-2j-1}\} & X_{i,3} \cap Y_{j,1} = \{a_{4i+2}, b_{4n-2j-1}\} \\
X_{i,3} \cap Y_{j,2} = \{a_{4i+3}, b_{4n-2j-2}\} & X_{i,3} \cap Y_{j,3} = \{a_{4i+2}, b_{4n-2j-2}\}
\end{array}$$

From this it is clear that each undirected edge of the form $\{a_k, b_\ell\}$ is contained in precisely one X-walk and precisely one Y-walk, so when 2-cells are glued following the facial boundary walks, what results is a 2-complex, homeomorphic to a pseudosurface, whose 1-skeleton is $K_{4m,4n}$. Take each walk to be oriented from left to right as written above; then every dart $a_k b_\ell$ of even slope is oriented in the forward direction in an X-walk and in the reverse direction in a Y-walk. Similarly, every dart $a_k b_\ell$ of odd slope is oriented in the reverse direction in an X-walk and in the forward direction in a Y-walk. Thus, since the orientations are compatible, the embedding is in an orientable pseudosurface. In the dual graph of this embedding, the vertices corresponding to the faces bounded by X-walks form the independent set of size $4m$ in $K_{4m,4n}$ and the vertices corresponding to the faces bounded by Y-walks form the independent set of size $4n$. The intersections shown above verify that every vertex corresponding to an X-walk is adjacent to every vertex corresponding to a Y-walk. Thus, the embedding is self dual.

Next, we consider the rotation scheme for this embedding of $K_{4m,4n}$. Note that if an oriented facial walk contains the sequence $(\cdots u\, v\, w \cdots)$, then the rotation of v contains a permutation with the transition uw. The rotation at each vertex decomposes into a disjoint union of permutations of length four, and each permutation corresponds to an umbrella in the embedding. The complete rotation scheme is given below, where p and q satisfy $0 \le p \le 2m-1$ and $0 \le q \le n-1$, respectively; the first transition listed in each permutation corresponds to a sequence from an X-walk.

$$\begin{array}{rl}
a_{2p}: & \{(b_{2n-1} b_0 b_{4n-1} b_{2n})\} \cup \{(b_{2j-1} b_{2j} b_{4n-2j-1} b_{4n-2j}) \,|\, 1 \le j \le n-1\} \\
a_{2p+1}: & \{(b_{2j} b_{2j+1} b_{4n-2j-2} b_{4n-2j-1}) \,|\, 0 \le j \le n-1\} \\
b_{2q}: & \{(a_{4i} a_{4i+1} a_{4i-2} a_{4i-1}) \,|\, 0 \le i \le m-1\} \\
b_{2n+2q}: & \{(a_{4i} a_{4i+1} a_{4i+2} a_{4i+3}) \,|\, 0 \le i \le m-1\} \\
b_{2q+1}: & \{(a_{4i+1} a_{4i} a_{4i-1} a_{4i-2}) \,|\, 0 \le i \le m-1\} \\
b_{2n+2q+1}: & \{(a_{4i+1} a_{4i} a_{4i+3} a_{4i+2}) \,|\, 0 \le i \le m-1\}
\end{array}$$

For each vertex v, every umbrella of v contains 4 edges, each incident to a distinct neighbor of v. Thus, each vertex in A has m umbrellas and each vertex in B has n umbrellas, so the embedding is full.

Finally, note that a_{4i} and a_{4i+1} are both covered by $X_{i,0}$ and $X_{i,1}$, a_{4i+2} and a_{4i+3} are both covered by $X_{i,2}$ and $X_{i,3}$, b_{2j} and $b_{4n-2j-1}$ are both covered by $Y_{j,0}$ and $Y_{j,1}$, and b_{2j+1} and $b_{4n-2j-2}$ are both covered by $Y_{j,2}$ and $Y_{j,3}$, so the embedding is doubly covered. This completes the proof. $\square$

Example 1. *The following closed walks form the facial boundary walks of the orientable self-dual embedding of $K_{8,12}$ described in Theorem 2.1.*

$$\begin{array}{ll}
X_{0,0}: & (a_0b_0a_1b_1a_0b_2a_1b_3a_0b_4a_1b_5) \\
X_{0,1}: & (a_0b_6a_1b_7a_0b_8a_1b_9a_0b_{10}a_1b_{11}) \\
X_{0,2}: & (a_2b_0a_3b_1a_2b_2a_3b_3a_2b_4a_3b_5) \\
X_{0,3}: & (a_2b_6a_3b_7a_2b_8a_3b_9a_2b_{10}a_3b_{11}) \\
X_{1,0}: & (a_4b_0a_5b_1a_4b_2a_5b_3a_4b_4a_5b_5) \\
X_{1,1}: & (a_4b_6a_5b_7a_4b_8a_5b_9a_4b_{10}a_5b_{11}) \\
X_{1,2}: & (a_6b_0a_7b_1a_6b_2a_7b_3a_6b_4a_7b_5) \\
X_{1,3}: & (a_6b_6a_7b_7a_6b_8a_7b_9a_6b_{10}a_7b_{11})
\end{array}
\qquad
\begin{array}{ll}
Y_{0,0}: & (b_0a_0b_{11}a_3b_0a_4b_{11}a_7) \\
Y_{0,1}: & (b_0a_6b_{11}a_5b_0a_2b_{11}a_1) \\
Y_{0,2}: & (b_1a_7b_{10}a_4b_1a_3b_{10}a_0) \\
Y_{0,3}: & (b_1a_1b_{10}a_2b_1a_5b_{10}a_6) \\
Y_{1,0}: & (b_2a_0b_9a_3b_2a_4b_9a_7) \\
Y_{1,1}: & (b_2a_6b_9a_5b_2a_2b_9a_1) \\
Y_{1,2}: & (b_3a_7b_8a_4b_3a_3b_8a_0) \\
Y_{1,3}: & (b_3a_1b_8a_2b_3a_5b_8a_6) \\
Y_{2,0}: & (b_4a_0b_7a_3b_4a_4b_7a_7) \\
Y_{2,1}: & (b_4a_6b_7a_5b_4a_2b_7a_1) \\
Y_{2,2}: & (b_5a_7b_6a_4b_5a_3b_6a_0) \\
Y_{2,3}: & (b_5a_1b_6a_2b_5a_5b_6a_6)
\end{array}$$

The rotation scheme for this embedding of $K_{8,12}$ is given below. Figure 3 shows a neighborhood of a_0, and Figure 4 shows a neighborhood of b_1. In Figures 3 and 4, we use the distinction of hollow and solid vertices to highlight the bipartition of the vertices of $K_{8,12}$.

$$\begin{array}{rl}
v \in \{a_0, a_2, a_4, a_6\}: & (b_5b_0b_{11}b_6)(b_1b_2b_9b_{10})(b_3b_4b_7b_8) \\
v \in \{a_1, a_3, a_5, a_7\}: & (b_0b_1b_{10}b_{11})(b_2b_3b_8b_9)(b_4b_5b_6b_7) \\
v \in \{b_0, b_2, b_4\}: & (a_0a_1a_6a_7)(a_4a_5a_2a_3) \\
v \in \{b_6, b_8, b_{10}\}: & (a_0a_1a_2a_3)(a_4a_5a_6a_7) \\
v \in \{b_1, b_3, b_5\}: & (a_1a_0a_7a_6)(a_5a_4a_3a_2) \\
v \in \{b_7, b_9, b_{11}\}: & (a_1a_0a_3a_2)(a_5a_4a_7a_6)
\end{array}$$

3. The surgeries

3.1. Background necessary for Edmonds' surgery and ours

The content of this subsection is adapted from [8, Section 2]. Recall that $\vec{E}(G)$ denotes the set of darts (directed edges) of G, and let $\mathcal{E}(G)$ denote the real vector space of all functions $\phi\colon \vec{E}(G) \to \mathbb{R}$ satisfying $\phi(e^-) = -\phi(e)$. A *cycle* is a connected 2-regular graph. When $W = e_1e_2\ldots e_n$ is a closed walk (which may include negative directed edges) traversing every edge it traverses at most once, we call the function mapping the positive directed edges of W to

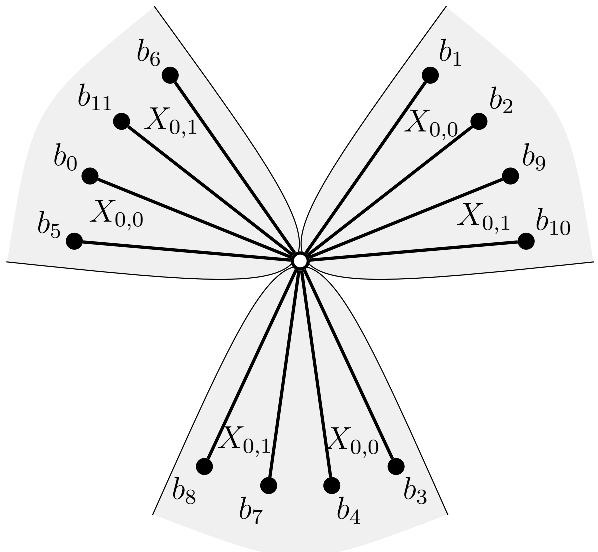

Figure 3. A neighborhood of a_0 (the hollow vertex) in the embedding of $K_{8,12}$ from Example 1 (corresponding neighborhoods of a_2, a_4, and a_6 are similar). The orientation of each face is counterclockwise, so that the permutations of neighbors in the rotation of a_0 appear in clockwise order in each umbrella. The faces that cover a_0 are labeled.

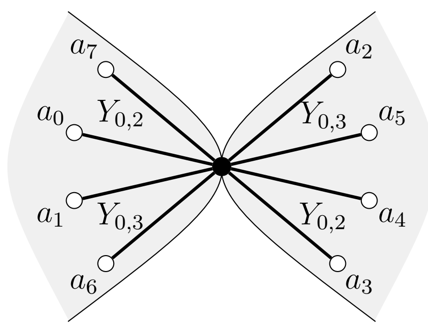

Figure 4. A neighborhood of b_1 (the solid vertex) in the embedding of $K_{8,12}$ given in Example 1 (corresponding neighborhoods of b_3 and b_5 are similar). The orientation of each face is counterclockwise, so that the permutations of neighbors in the rotation at b_1 appear in clockwise order in each umbrella. The faces that cover b_1 are labeled.

1, the negative directed edges of W to -1, and all other directed edges of G to 0 an *oriented circuit*. The subspace of $\mathcal{E}(G)$ generated by the set of oriented circuits of G is called the *oriented cycle space*. Recall that a surface S is *orientable* if it can be triangulated by a graph embedding whose faces can be compatibly oriented. As pointed out in [8], equivalent conditions are that every triangulation has this property, and that the surface does not contain a Möbius strip. We say that a family of walks $\mathcal{W}$ *double covers* the edges of G if each edge is traversed twice in total by the walks of $\mathcal{W}$.

For a walk W, we let $\vec{c}(W)$ denote the function that assigns to each directed edge of $\vec{E}(G)$

the number of times that W traverses its corresponding positive directed edge minus the number of times that W traverses its corresponding negative directed edge, and assigns 0 to any edge not appearing in W. The *oriented dimension* of a family of walks $\mathcal{W}$, denoted $\vec{\dim}(\mathcal{W})$, is the dimension of the subspace of $\mathcal{E}(G)$ spanned by the functions $\vec{c}(W)$ with $W \in \mathcal{W}$.

Lemma 3.1 is an adaptation of [8, Lemma 9]. Since orientability of a pseudosurface is determined by the orientations of the directed edges of the facial boundary walks and not the incidence of these directed edges and the vertices of the embedded graph, the proof of Lemma 3.1 is a straightforward adaptation of the proof of [8, Lemma 9].

Lemma 3.1. *If $\mathcal{W}$ is the family of facial-boundary walks of $G \to P$, then P is nonorientable if and only if $\vec{dim}(\mathcal{W}) < |\mathcal{W}|$.*

Following [1] we say that a pseudosurface P is *face connected* if any proper embedding of a connected graph in P induces a 2-complex K such that for any faces f_a, f_b of K, there is a sequence of faces $f_a = f_1 f_2 \dots f_n = f_b$ such that for each $i \in \{1, 2 \dots n-1\}$, f_i and f_{i+1} share a common boundary edge. Note that if $G \to P$ is an embedding in a non-face-connected pseudosurface, then the dual graph is not connected. For the remainder of this article, P shall denote a face-connected pseudosurface.

We let $\chi(P)$ denote the Euler characteristic of P, which, as an invariant of P, depends on neither G nor any proper embedding of G in P. Given $G \to P$,

$$\chi(P) = |V(G)| - |E(G)| + |F(G \to P)|.$$

3.2. Edmonds' surgery

For a walk W, let $\overline{W}$ be the walk consisting of the darts opposite the darts of W, and appearing in reverse order relative to their counterparts in W. For example: if $W = e_1^- e_2 e_3^-$, then $\overline{W} = e_3 e_2^- e_1$. We give a somewhat revised proof of Theorem 3.1 for the sake of completeness since the original statement of [9, Theorem 1] did not include all the conclusions that one could derive from its proof.

Theorem 3.1. *[9, Theorem 1] Consider $G \to P$ having a pinchpoint vertex v with the property that two umbrellas of v are intersected by the same face f of $G \to P$. There exists a proper embedding of G in a different face-connected pseudosurface P' such that: the dual graphs of $G \to P$ and $G \to P'$ are identical, there is one fewer umbrella of v, and $\chi(P) = \chi(P')$.*

Proof. For each face of $G \to P$, choose one of the two possible facial boundary walks, and let $\mathcal{W}$ be the set of these chosen facial boundary walks. Choose two umbrellas U_1, U_2 of v intersected by f, and choose one intersection of f with each chosen umbrella. Let $U^*(v)$ denote the intersection of an arbitrarily small open neighborhood of v with $U_1 \cup U_2$, whose intersection with G consists only of v and the ends of edges incident to v. We let f_i denote the chosen intersection of f with $U_i \cap U^*(v)$.

Let W denote the chosen facial boundary walk of f and let $W = \omega_1\omega_2$ for closed walks ω_1 and ω_2, as in Figure 5; ω_1 begins by traversing edge end 4 and ends after traversing edge end 1, and ω_2 begins by traversing edge end 2 and ends after traversing edge end 3. We surgically

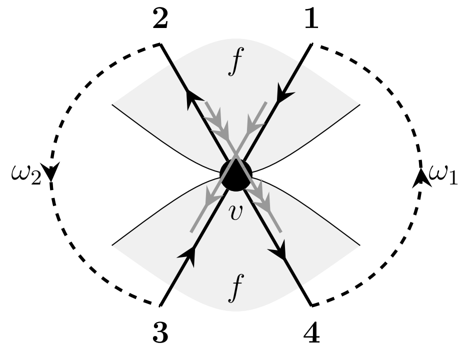

Figure 5. The applicaton of Edmonds' surgery turning W into W' by reversing the subwalk ω_2. The surgical modifications are drawn in gray.

modify W to produce another facial boundary walk; let $W' = \omega_1\overline{\omega}_2$ (as in Figure 5), and let $\mathcal{W}' = (\{W'\} \cup \mathcal{W}) \setminus \{W\}$. Let $\mathcal{W}'$ be the set of facial boundary walks of $G \to P'$.

To see that the dual graphs of $G \to P$ and $G \to P'$ are identical, note that the darts of W and the darts of W' are darts on identical edges of G. Therefore, the incidence of faces and edges of $G \to P$ is the same as that of $G \to P'$. It follows that the dual graphs of $G \to P$ and $G \to P'$ are identical and that P is also face connected.

To show that there is one fewer umbrella of v, we define an auxiliary graph that captures the incidence of certain regions of $G \to P$ with G. Let $R_1 = U_1 \setminus f_1$ and $R_2 = U_2 \setminus f_2$. We define a graph H whose vertex set is $\{f_1, f_2, R_1, R_2\}$ and with four edges e_1, e_2, e_3, e_4. Consider Figure 6; we define the edge e_i to be incident to f_j and R_k if the edge end i is common to the boundaries of f_j and R_k. The graph H is comprised of two disjoint 2-cycles, one corresponding to each umbrella of v in $U^*(v)$.

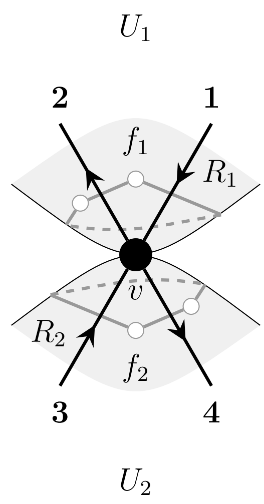

Figure 6. The subset $U^*(v)$ of P. The graph H is drawn in gray. Each edge e_i is drawn transversely crossing the edge end i.

Because of the reversal of ω_2 to produce the facial walk W', W' traverses the following ordered pairs of edge ends as W' enters and exits $U^*(v)$: 1 and then 3, and 2 and then 4. Let f' be the face of $G \to P'$ bounded by W'. We define f_1' and f_2' to be the regions of $f' \cap U^*(v)$ bounded by edge ends 1 and 3 and 2 and 4, respectively. Note that the process of producing W' from W does not affect the boundaries of R_1 and R_2. We define another analogous graph H' for $G \to P'$ whose vertex set is $\{R_1, R_2, f_1', f_2'\}$. It is easy to see that this graph is a single cycle, and so we conclude that there is one less umbrella of v in $G \to P'$.

Lastly, since $|\mathcal{W}| = |\mathcal{W}'|$, and G is unaffected by the surgery that created $\mathcal{W}'$, we may conclude that $\chi(P) = \chi(P')$.

□

Corollary 3.1. *The pseudosurface P' in Theorem 3.1 is nonorientable.*

Proof. This is an application of [8, Lemma 7]. The reader is advised that the surgery appearing in the proof of Theorem 3.1 is the same surgery that appears in the proof of [8, Lemma 7], which Bruhn and Diestel show results in a nonorientable pseudosurface. □

3.3. Our surgery

Our surgery, which is given in the proof of Theorem 3.2, is applicable to fewer embeddings than the surgery of Edmonds described in Theorem 3.1 because it requires a face of $G \to P$ to intersect three umbrellas of a pinchpoint vertex v, whereas Edmonds' surgery requires two. However, our surgery does have the advantage that the orientability of the resulting pseudosurface is the same as the orientability of P. By Corollary 3.1, Edmonds' surgery necessarily produces a nonorientable embedding, no matter the orientability of the embedding to which Edmonds' surgery is applied.

Theorem 3.2. *Consider $G \to P$ having a pinchpoint vertex v with the property that three umbrellas of v are intersected by the same face f. There exists a proper embedding of G in a different face-connected pseudosurface P' such that: the dual graphs of $G \to P$ and $G \to P'$ are identical, P' has the same orientability as P, there are two fewer umbrellas of v in $G \to P'$, and $\chi(P) = \chi(P')$.*

Proof. For each face of $G \to P$, choose one of the two possible facial boundary walks, and let $\mathcal{W}$ be the set of chosen facial boundary walks. Choose three umbrellas U_1, U_2, and U_3 of v intersected by f, and choose one intersection of f with each chosen umbrella. Let $U^*(v)$ denote the intersection of an arbitrarily small neighborhood with $U_1 \cup U_2 \cup U_3$, whose intersection with G consists of v and ends of edges incident to v. We let f_i denote the chosen intersection of f with $U_i \cap U^*(v)$.

Let W denote the chosen facial boundary walk of f and let $W = \omega_1\omega_2\omega_3$ for closed walks ω_1, ω_2, and ω_3, as in Figure 7: ω_1 is a closed walk beginning by traversing edge end 6 as indicated, and ending at v after traversing the edge end 1 as indicated; ω_2 is a closed walk beginning by traversing the edge end 2 as indicated, and ending at v after traversing the edge end 3 as indicated; ω_3 is a closed walk beginning by traversing the edge end 4 as indicated, and ending at v after traversing the edge end 5 as indicated.

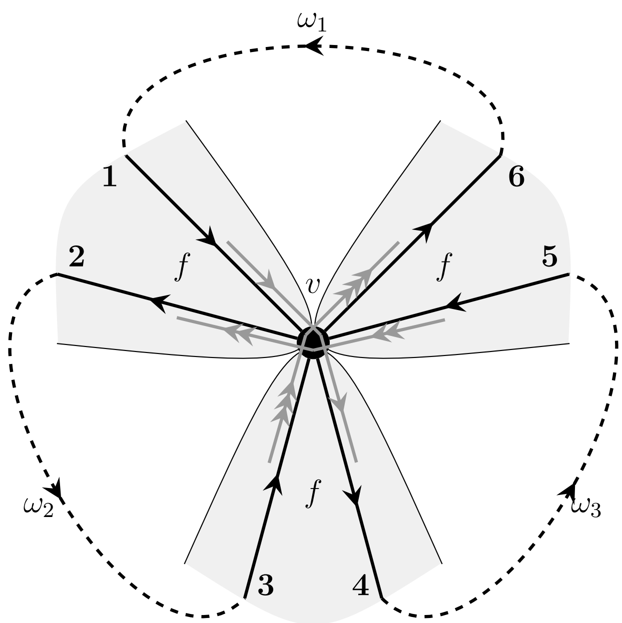

Figure 7. The facial boundary walk W' passing through three umbrellas of v, with surgical modifications in gray.

We alter the order of ω_1, ω_2 and ω_3 to produce a new facial boundary walk W'. We let $W' = \omega_1\omega_3\omega_2$, and let the set of facial boundary walks of $G \to P$ be $\mathcal{W}' = (\{W'\} \cup \mathcal{W}) \setminus \{W\}$. The reordering of ω_1, ω_2, and ω_3 to produce W' is described in Figure 7.

To see that the dual graphs of $G \to P$ and $G \to P'$ are identical, note that the darts of W and the darts of W' are darts corresponding to the same edges of G. Therefore, the incidence of faces and edges of $G \to P$ is the same as that of $G \to P'$. It follows that the dual graphs of $G \to P$ and $G \to P'$ are identical, and that P' is also face connected.

To see that P' has the same orientability as P, note that $c(W') = c(W)$, and so $\overrightarrow{\dim}(\mathcal{W}') = \overrightarrow{\dim}(\mathcal{W})$. Lemma 3.1 allows us to conclude that P' is orientable if and only if P is orientable.

To see that the three chosen umbrellas of the vertex v have been merged into one, we will define an auxiliary graph that captures the incidence of certain regions of $G \to P$ with G. Let $R_1 = U_1 \setminus f_1$, $R_2 = U_2 \setminus f_2$, and $R_3 = U_3 \setminus f_3$. We define a graph H whose vertex set is $\{f_1, f_2, f_3, R_1, R_2, R_3\}$, and with six edges $\{e_1, e_2, e_3, e_4, e_5, e_6\}$. Consider Figure 8; we define the edge e_i to be incident to f_j and R_k if the edge end i is common to the boundaries of f_j and R_k. Note that H consists of three disjoint 2-cycles.

Because of the reordering of $\omega_1,\ \omega_2,\ \omega_3$ to produce $W' = \omega_1\omega_3\omega_2$, the new facial boundary walk W' traverses the following ordered pairs of edge ends as as W' enters and exits $U^*(v)$: 1 and then 4, 5 and then 2, 3 and then 6, and in this order. Let f' be the face of $G \to P'$ bounded by the facial boundary walk W'. We define f'_1, f'_2, f'_3 to be the regions of $f' \cap U^*(v)$ bounded by edge ends 1 and 4, 5 and 2, 3 and 6, respectively.

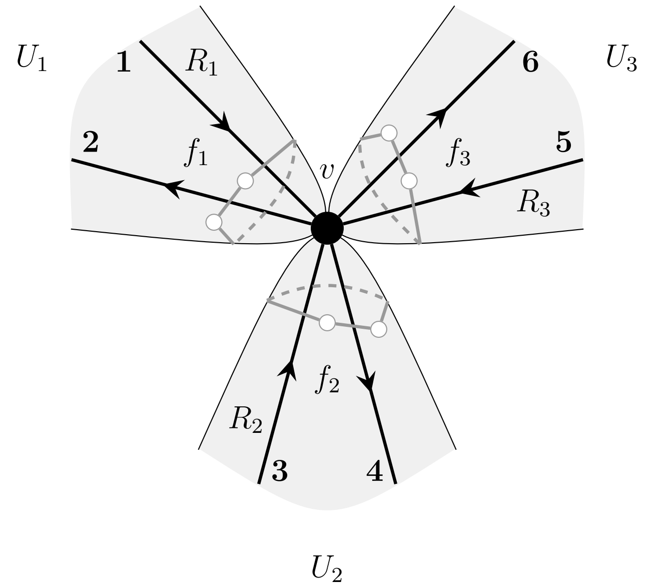

Figure 8. The subset $U^*(v)$ of P. The graph H is drawn in gray. Each edge e_i is drawn transversely crossing the edge end i.

Note that the process of producing W' does not affect the boundaries of R_1, R_2 and R_3. We define another, analogous graph H' for $G \to P'$ whose vertex set is $\{R_1, R_2, R_3, f_1', f_2', f_3'\}$. It is easy to verify that this graph is a single cycle. We conclude that there are two fewer umbrellas of v in $G \to P'$.

Lastly, since $|\mathcal{W}| = |\mathcal{W}'|$, and G is unaffected by the surgery that created $\mathcal{W}'$, we may conclude that $\chi(P) = \chi(P')$. □

4. Applications of the surgeries to our embeddings

It is not difficult to show (see the proof of [15, Theorem 1.2]) that for a closed face-connected pseudosurface P with h handles, c crosscaps, and p pinches the Euler characteristic of P is

$$\chi(P) = 2 - 2h - c - p. \tag{1}$$

Since, in a self-dual proper embedding $G \to P$ it must be true that $|V(G)| = |F(G \to P)|$, it follows that

$$\chi(P) = 2|V(G)| - |E(G)| = 2 - 2h - c - p. \tag{2}$$

For $K_{4m,4n}$ to have a self-dual embedding $K_{4m,4n} \to P$ in some pseudosurface P, it follows from Equations 1 and 2 that

$$\chi(P) = 2 - 2h - c - p = 8m + 8n - 16mn. \tag{3}$$

We now apply the surgeries described in Theorems 3.1 and 3.2 to embeddings of $K_{4m,4n}$ given in Section 2.2. Depending on m and n, there are many different face-connected pseudosurfaces whose Euler characteristic is $8m + 8n - 16mn$, and our surgeries and construction can be used to find self-dual embeddings in many, but not all of them. Recall our choice of $A = \{a_0, \ldots, a_{4m-1}\}$ and $B = \{b_0, \ldots, b_{4n-1}\}$ denoting the bipartition sets of vertices of $K_{4m,4n}$. We first treat the case for which the surgically-produced pseudosurface is orientable.

Theorem 4.1. *For each integer i such that $0 \le i \le 4m-1$, let x_i be an integer having the property that $n - 2x_i \ge 1$. For each integer j such that $0 \le j \le 4n-1$, let y_j be an integer having the property hat $m - 2y_j \ge 1$. There exists a self-dual proper embedding of $K_{4m,4n}$ in an orientable face-connected pseudosurface P with h handles and p pinches such that:*

1. $8m + 8n - 16mn = 2 - 2h - p$,
2. *the vertex a_i has $n - 2x_i$ umbrellas and the vertex b_j has $m - 2y_j$ umbrellas, and*
3. $p = \sum_i (n - 2x_i - 1) + \sum_j (m - 2y_j - 1)$.

Proof. Let $K_{4m,4n} \to P$ denote the orientable self-dual embedding given in Theorem 2.1. Recall that by Theorem 2.1, $K_{4m,4n} \to P$, as a doubly covered embedding, has the property that each vertex is covered by a facial boundary walk. The result now follows by applying Theorem 3.2 x_i times to the vertex a_i and y_j times to the vertex b_j. Each application of the surgery reduces the number of pinches by two while producing a pseudosurface with the same Euler characteristic as the pseudosurface to which the surgery was applied. It follows from Equation 2 that the number of handles must increase by one each time the surgery is applied. □

We now turn to finding all nonorientable pseudosurfaces for which we may produce a self-dual embedding of $K_{4m,4n}$ using the construction given in Theorem 2.1 and the surgery given in Theorem 3.1. For this, we will need a few more definitions and lemmas.

For surfaces S_1 and S_2, we let the *connected sum* of S_1 and S_2 be the surface formed by removing a disc from S_1, another disc from S_2, and identifying the two remaining spaces along their boundaries.

To fully appreciate the proof of Lemma 4.1, one must be familiar with the presentation of a surfaces as a polygon with its edges pasted together. The reader will find a full treatment of this topic in [12, Chapter 1] or [13, Chapter 12].

Lemma 4.1. *A nonorientable surface S with $h > 1$ handles and $c > 0$ crosscaps is homeomorphic to a nonorientable surface with $h - 1$ handles and $c + 2$ crosscaps.*

Proof. By [13, Theorem 77.5], S is homeomorphic to a connected sum of $d = 2h + c$ projective planes. By invoking [13, Lemma 77.4] $h - 1$ times (exchanging two projective projective planes for one handle each time) one may form a presentation of S as the connected sum of $h - 1$ tori and $c + 2$ projective planes; equivalently a surface with $h - 1$ handles and $c + 2$ crosscaps. □

Lemma 4.2. *Let P be a nonorientable face-connected pseudosurface with h handles, c crosscaps, and n pinchpoints p_1, p_2, ..., p_n such that p_i has u_i umbrellas. There is a homeomorphism $\phi\colon P \to P'$ such that P' is a face-connected pseudosurface P' with $c + 2$ crosscaps, $h - 1$ handles, and n pinchpoints p'_1, p'_2, ... p'_n such that p'_i has u_i umbrellas.*

Proof. We know P is obtained from a surface S with h handles and c crosscaps by identifying u_i unique points to form each pinchpoint p_i. For each p_i, we let $X_i = \{x_k : 1 \leq k \leq u_i\}$ be the set of points thus identified. It follows that no two X_i have any points in common. Let $Q\colon S \to P$ be the quotient map that for each i identifies the points of X_i. Let S' be a surface with $c+2$ crosscaps and $h-1$ handles, and let $f\colon S \to S'$ be the homeomorphism whose existence is implied in 4.1. Let $Q'\colon S' \to P'$ be the quotient map that identifies the points of $f(X_i)$ to p'_i for all i. Let $\phi\colon P \to P'$ be defined by $\phi = Q' \circ f \circ Q^{-1}$. It is easy to see that ϕ is a homeomorphism mapping the pinchpoint p_i to the pinchpoint p'_i. □

Corollary 4.1. *If G is properly embedded in P, then the homeomorphism $\phi\colon P \to P'$ from Lemma 4.2 induces a proper embedding of G in P' such that the dual graphs of $G \to P$ and $G \to P'$ are isomorphic.*

Proof. Since the complement of G in P is homeomorphic to a disjoint union of regions, each homeomorphic to a disc, it follows that since ϕ is a homeomorphism, the intersections of the boundaries of any faces are preserved. Since the vertices and edges of G form the boundaries of the faces of $G \to P$, it follows that the incidence of faces and edges is preserved by ϕ. It follows that the dual graphs of $G \to P$ and $G \to P'$ are isomorphic. □

Theorem 4.2. *For each integer i such that $0 \leq i \leq 4m-1$, let x_i be an integer having the property that $n - x_i \geq 1$. For each integer j such that $0 \leq j \leq 4n-1$, let y_j be an integer satisfying $m - y_j \geq 1$. Suppose further that at least one of the x_i or y_j is nonzero. There is a self-dual embedding of $K_{4m,4n}$ in the nonorientable face connected pseudosurface P' having h handles, c crosscaps, and p pinches such that:*

1. $8m + 8n - 16mn = 2 - 2h - c - p$,
2. *the vertex a_i has $n - x_i$ umbrellas and the vertex b_j has $m - y_j$ umbrellas, and*
3. $p = \sum_i (n - x_i - 1) + \sum_j (m - y_j - 1)$.

Proof. Consider the orientable self-dual embedding of $K_{4m,4n} \to P$ given in Theorem 2.1. Assume without loss of generality that x_{i^*} is nonzero. Applying the surgery given in Theorem 3.1 x_i times at the vertex a_i and y_j times at the vertex b_j, we see in light of Corollary 3.1, since the surgery was applied at least once at a_{i^*}, that we have produced a nonorientable self-dual embedding in a pseudosurface with the pinchpoint vertices having the desired number of umbrellas. Each application of the surgery reduces the number pinches by 1 while producing a pseudosurface with the same Euler characteristic as the one to which the surgery was applied. Invoking Lemma 4.2 and Corollary 4.1 enough times, we can find a self-dual embedding of $K_{4m,4n}$ in a nonorientable pseudosurface with the desired number of handles and crosscaps; we exchange two crosscaps for a handle (or vice versa) each time. □

In light of Lemma 4.2 we may conclude that there is only one nonorientable face-connected pseudosurface of a given Euler characteristic with a specific number of umbrellas at each pinchpoint. Theorem 4.2 helps us classify exactly which of these pseudosurfaces admits a self-dual embedding of $K_{4m,4n}$: every pseudosurface P that satisfies $\chi(P) = 8m + 8n - 16mn$ and has an admissible number of pinchpoints and umbrellas admits a self-dual embedding of $K_{4m,4n}$ except for the unique pseudosurface of Euler characteristic $8m + 8n - 16mn$ that has exactly $4m$ pinchpoints with n umbrellas and exactly $4n$ pinchpoints with m umbrellas. In other words, Theorem 4.2 does not guarantee the existence of a full self-dual embedding of $K_{4m,4n}$ in a nonorientable pseudosurface.

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