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Modular colorings of join of two special graphs

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Abstract

For $k \geq 2$, a modular k-coloring of a graph G without isolated vertices is a coloring of the vertices of G with the elements in $\mathbb{Z}_k$ having the property that for every two adjacent vertices of G, the sums of the colors of their neighbors are different in $\mathbb{Z}_k$. The minimum k for which G has a modular k-coloring is the modular chromatic number of G. In this paper, we determine the modular chromatic number of join of two special graphs.

Keywords: modular coloring; modular chromatic number; join of two graphs
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1. Introduction

For graph-theoretical terminology and notation, we in general follow [1]. For a vertex v of a graph G, let $N_G(v)$, the *neighborhood of* v, denote the set of vertices adjacent to v in G. For a graph G without isolated vertices, let $c : V(G) \to \mathbb{Z}_k$, $k \geq 2$, be a vertex coloring of G where adjacent vertices may be colored the same. The *color sum* $\sigma(v) = \sum_{u \in N_G(v)} c(u)$ of a vertex v of G is the sum of the colors of the vertices in $N_G(v)$. The coloring c is called a *modular k-coloring* of G if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$ for all pairs x, y of adjacent vertices in G. The *modular chromatic number* $mc(G)$ of G is the minimum k for which G has a modular k-coloring. This concept was introduced by Okamoto, Salehi and Zhang [2].

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Okamoto, Salehi and Zhang proved, in [2], that: every nontrivial connected graph G has a modular k-coloring for some integer $k \geq 2$ and $mc(G) \geq \chi(G)$, where $\chi(G)$ denotes the chromatic number of G; for the cycle C_n of length n, $mc(C_n)$ is 2 if $n \equiv 0 \ mod\ 4$ and it is 3 otherwise; every nontrivial tree has modular chromatic number 2 or 3; for the complete multipartite graph G, $mc(G) = \chi(G)$; for the cartesian product $G = K_r \Box K_2$, $mc(G)$ is r if $r \equiv 2 \ mod\ 4$ and it is $r+1$ otherwise; for the wheel $W_n = C_n \vee K_1$, $n \geq 3$, $mc(W_n) = \chi(W_n)$, where $\vee$ denotes the join of two graphs; for $n \geq 3$, $mc(C_n \vee K_2^c) = \chi(C_n \vee K_2^c)$, where G^c denotes the complement of G; and for $n \geq 2$, $mc(P_n \vee K_2) = \chi(P_n \vee K_2)$, where P_n denotes the path of length $n-1$; and in [3] that: for $m, n \geq 2$, $mc(P_m \Box P_n) = 2$.

For graphs G_1 and G_2, their union $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. For vertex-disjoint graphs G_1 and G_2, their *join* $G_1 \vee G_2$ is the supergraph of $G_1 \cup G_2$ in which each vertex of G_1 is adjacent to every vertex of G_2 and both G_1 and G_2 are induced subgraphs.

In this paper, we compute the modular chromatic number of join of two special graphs.

2. Join of two bipartite graphs

2.1. Sufficient condition for $mc = 4$

Let $i, j, k \in \mathbb{Z}_4$ with $i \neq j$. Set $M_4^{i,j;k} = \{G : G$ is a bipartite graph such that G has a modular 4-coloring c with the property that for every $v \in V(G)$, $\sigma(v)(mod\ 4) \in \{i, j\}$ and $\sum_{v \in V(G)} c(v) \equiv k \ mod\ 4 \}$.

Lemma 2.1. *Let G and H be two vertex-disjoint nonempty bipartite graphs. If any one of the following holds, then $mc(G \vee H) = 4$.*

(1) $G \in M_4^{0,1;0}$ and $H \in M_4^{2,3;0}$;
(2) $G \in M_4^{0,1;1}$ and $H \in M_4^{2,3;1}$;
(3) $G \in M_4^{0,1;2}$ and $H \in M_4^{2,3;2}$;
(4) $G \in M_4^{0,1;3}$ and $H \in M_4^{2,3;3}$;
(5) $G \in M_4^{0,2;0}$ and $H \in M_4^{1,3;0}$;
(6) $G \in M_4^{0,2;1}$ and $H \in M_4^{1,3;1}$;
(7) $G \in M_4^{0,2;2}$ and $H \in M_4^{1,3;2}$;
(8) $G \in M_4^{0,2;3}$ and $H \in M_4^{1,3;3}$;
(9) $G \in M_4^{0,3;0}$ and $H \in M_4^{1,2;0}$;
(10) $G \in M_4^{0,3;1}$ and $H \in M_4^{1,2;1}$;
(11) $G \in M_4^{0,3;2}$ and $H \in M_4^{1,2;2}$;
(12) $G \in M_4^{0,3;3}$ and $H \in M_4^{1,2;3}$;
(13) $G \in M_4^{0,1;0}$ and $H \in M_4^{0,1;2}$;
(14) $G \in M_4^{0,1;1}$ and $H \in M_4^{0,1;3}$;
(15) $G \in M_4^{0,3;0}$ and $H \in M_4^{0,3;2}$;
(16) $G \in M_4^{0,3;1}$ and $H \in M_4^{0,3;3}$;
(17) $G \in M_4^{1,2;0}$ and $H \in M_4^{1,2;2}$;
(18) $G \in M_4^{1,2;1}$ and $H \in M_4^{1,2;3}$;
(19) $G \in M_4^{2,3;0}$ and $H \in M_4^{2,3;2}$;

(20) $G \in M_4^{2,3;1}$ *and* $H \in M_4^{2,3;3}$;
(21) $G \in M_4^{0,2;0}$ *and* $H \in M_4^{0,2;1}$;
(22) $G \in M_4^{0,2;1}$ *and* $H \in M_4^{0,2;2}$;
(23) $G \in M_4^{0,2;2}$ *and* $H \in M_4^{0,2;3}$;
(24) $G \in M_4^{0,2;3}$ *and* $H \in M_4^{0,2;0}$;
(25) $G \in M_4^{1,3;0}$ *and* $H \in M_4^{1,3;1}$;
(26) $G \in M_4^{1,3;1}$ *and* $H \in M_4^{1,3;2}$;
(27) $G \in M_4^{1,3;2}$ *and* $H \in M_4^{1,3;3}$;
(28) $G \in M_4^{1,3;3}$ *and* $H \in M_4^{1,3;0}$;
(29) $G \in M_4^{0,1;0}$ *and* $H \in M_4^{0,3;1}$;
(30) $G \in M_4^{0,1;1}$ *and* $H \in M_4^{0,3;2}$;
(31) $G \in M_4^{0,1;2}$ *and* $H \in M_4^{0,3;3}$;
(32) $G \in M_4^{0,1;3}$ *and* $H \in M_4^{0,3;0}$;
(33) $G \in M_4^{0,1;0}$ *and* $H \in M_4^{1,2;3}$;
(34) $G \in M_4^{0,1;1}$ *and* $H \in M_4^{1,2;0}$;
(35) $G \in M_4^{0,1;2}$ *and* $H \in M_4^{1,2;1}$;
(36) $G \in M_4^{0,1;3}$ *and* $H \in M_4^{1,2;2}$;
(37) $G \in M_4^{0,2;0}$ *and* $H \in M_4^{1,3;2}$;
(38) $G \in M_4^{0,2;1}$ *and* $H \in M_4^{1,3;3}$;
(39) $G \in M_4^{0,2;2}$ *and* $H \in M_4^{1,3;0}$;
(40) $G \in M_4^{0,2;3}$ *and* $H \in M_4^{1,3;1}$;
(41) $G \in M_4^{0,3;0}$ *and* $H \in M_4^{2,3;1}$;
(42) $G \in M_4^{0,3;1}$ *and* $H \in M_4^{2,3;2}$;
(43) $G \in M_4^{0,3;2}$ *and* $H \in M_4^{2,3;3}$;
(44) $G \in M_4^{0,3;3}$ *and* $H \in M_4^{2,3;0}$;
(45) $G \in M_4^{1,2;0}$ *and* $H \in M_4^{2,3;3}$;
(46) $G \in M_4^{1,2;1}$ *and* $H \in M_4^{2,3;0}$;
(47) $G \in M_4^{1,2;2}$ *and* $H \in M_4^{2,3;1}$;
(48) $G \in M_4^{1,2;3}$ *and* $H \in M_4^{2,3;2}$.

Proof. Clearly, $mc(G \vee H) \geq \chi(G \vee H) = \chi(G) + \chi(H) = 4$. We prove (48) and the proofs of (1) to (47) are similar. $G \in M_4^{1,2;3}$ implies that G has a modular 4-coloring c' such that for every $v \in V(G)$, $\sigma'(v)(mod\ 4) \in \{1,2\}$ and $\sum_{v \in V(G)} c'(v) \equiv 3\ mod\ 4$ and $H \in M_4^{2,3;2}$ implies that H has a modular 4-coloring c'' such that for every $v \in V(H)$, $\sigma''(v)(mod\ 4) \in \{2,3\}$ and $\sum_{v \in V(H)} c''(v) \equiv 2\ mod\ 4$. Define $c : V(G \vee H) \to \mathbb{Z}_4$ by $c(v) = c'(v)$ for $v \in V(G)$ and $c(v) = c''(v)$ for $v \in V(H)$. Then, for every $v \in V(G)$, $\sigma(v)(mod\ 4) = 3 \Leftrightarrow \sigma'(v)(mod\ 4) = 1$, and $\sigma(v)(mod\ 4) = 0 \Leftrightarrow \sigma'(v)(mod\ 4) = 2$; and for every $v \in V(H)$, $\sigma(v)(mod\ 4) = 1 \Leftrightarrow \sigma''(v)(mod\ 4) = 2$, and $\sigma(v)(mod\ 4) = 2 \Leftrightarrow \sigma''(v)(mod\ 4) = 3$. Hence, c is a modular 4-coloring of $G \vee H$. Consequently, $mc(G \vee H) \leq 4$. □

Using Lemma 2.1, we compute $mc(G \vee H)$ for some special graphs G and H.

2.2. Join of two paths

Theorem 2.1. *For $m \geq 2$ and $n \geq 2$, $mc(P_m \vee P_n) = 4$.*

Proof. Case 1. $n \not\equiv 1 \, mod \, 4$.

First, we claim that $P_m \in M_4^{0,2;0} \cup M_4^{0,2;2}$. To see this, for $m \equiv 0 \, mod \, 4$, label the vertices of P_m by $0,0,2,0,\ 0,0,2,0,\ \ldots,\ 0,0,2,0$ in order; for $m \equiv 1 \, mod \, 4$, label the vertices of P_m by $0,0,2,0,\ 0,0,2,0,\ \ldots,\ 0,0,2,0,\ 0$ in order; for $m \equiv 2 \, mod \, 4$, label the vertices of P_m by $2,0,0,0,\ 2,0,0,0,\ \ldots,\ 2,0,0,0,\ 2,0$ in order; and for $m \equiv 3 \, mod \, 4$, label the vertices of P_m by $0,0,2,0,\ 0,0,2,0,\ldots,\ 0,0,2,0,\ 0,0,2$ in order.

Next, we claim that $P_n \in M_4^{1,3;0}$. To see this, for $n \equiv 0 \, mod \, 4$, label the vertices of P_n by $0,1,3,0,\ 0,1,3,0,\ \ldots,\ 0,1,3,0$ in order; for $n \equiv 2 \, mod \, 4$, label the vertices of P_n by $1,3,0,0,\ 1,3,0,0,\ \ldots,\ 1,3,0,0,\ 1,3$ in order; and for $n \equiv 3 \, mod \, 4$, label the vertices of P_n by $0,1,3,0,\ 0,1,3,0,\ \ldots,\ 0,1,3,0,\ 0,1,3$ in order.

Finally, apply Lemma 2.1 (5) and (39).

Now, by symmetry, assume that both m and n are $\equiv 1 \, mod \, 4$. Again, by symmetry, it is enough if we consider the following cases.

Case 2. $m \equiv 1 \, mod \, 16$ and $n \equiv 1 \, mod \, 16$.

First, we claim that $P_m \in M_4^{0,3;0}$. To see this, label the vertices of P_m by $0,0,3,0,\ 0,0,3,0,\ \ldots,\ 0,0,3,0,\ 0$ in order. Next, we claim that $P_n \in M_4^{0,1;3}$. To see this, label the vertices of P_n by $3,0,2,0,\ 3,0,2,0,\ \ldots,\ 3,0,2,0,\ 3$ in order. Finally, apply Lemma 2.1 (32).

Case 3. $m \equiv 1 \, mod \, 16$ and $n \equiv 9 \, mod \, 16$.

First, we claim that $P_m \in M_4^{0,3;1}$. To see this, label the vertices of P_m by $1,0,2,0,\ 1,0,2,0,\ \ldots,\ 1,0,2,0,\ 1$ in order. Next, we claim that $P_n \in M_4^{0,3;3}$. To see this, label the vertices of P_n by $1,0,2,0,\ 1,0,2,0,\ \ldots,\ 1,0,2,0,\ 1$ in order. Finally, apply Lemma 2.1 (16).

Case 4. $m \equiv 5 \, mod \, 16$ and $n \equiv 5 \, mod \, 16$.

First, we claim that $P_m \in M_4^{0,1;1}$. To see this, label the vertices of P_m by $0,0,1,0,\ 0,0,1,0,\ \ldots,\ 0,0,1,0,\ 0$ in order. Next, we claim that $P_n \in M_4^{0,3;2}$. To see this, label the vertices of P_n by $3,0,0,0,\ 3,0,0,0,\ \ldots,\ 3,0,0,0,\ 3$ in order. Finally, apply Lemma 2.1 (30).

Case 5. $m \equiv 5 \, mod \, 16$ and $n \equiv 13 \, mod \, 16$.

First, we claim that $P_m \in M_4^{0,1;1}$. To see this, label the vertices of P_m by $0,0,1,0,\ 0,0,1,0,\ \ldots,\ 0,0,1,0,\ 0$ in order. Next, we claim that $P_n \in M_4^{0,3;2}$. To see this, label the vertices of P_n by $1,0,2,0,\ 1,0,2,0,\ \ldots,\ 1,0,2,0,\ 1$ in order. Finally, apply Lemma 2.1 (30).

Case 6. $m \equiv 9 \, mod \, 16$ and $n \equiv 9 \, mod \, 16$.

First, we claim that $P_m \in M_4^{0,3;2}$. To see this, label the vertices of P_m by $0,0,3,0,\ 0,0,3,0,\ \ldots,\ 0,0,3,0,\ 0$ in order. Next, we claim that $P_n \in M_4^{0,1;1}$. To see this, label the vertices of P_n by $3,0,2,0,\ 3,0,2,0,\ \ldots,\ 3,0,2,0,\ 3$ in order. Finally, apply Lemma 2.1 (30).

Case 7. $m \equiv 13 \, mod \, 16$ and $n \equiv 13 \, mod \, 16$.

First, we claim that $P_m \in M_4^{0,1;0}$. To see this, label the vertices of P_m by $1,0,0,0,\ 1,0,0,0,\ \ldots,\ 1,0,0,0,\ 1$ in order. Next, we claim that $P_n \in M_4^{0,3;1}$. To see this, label the vertices of P_n by $0,0,3,0,\ 0,0,3,0,\ \ldots,\ 0,0,3,0,\ 0$ in order. Finally, apply Lemma 2.1 (29).

Cases 8.1. $m \equiv 1 \, mod \, 16$ and $n \equiv 5 \, mod \, 16$;

8.2. $m \equiv 13 \, mod \, 16$ and $n \equiv 1 \, mod \, 16$;

8.3. $m \equiv 5 \, mod \, 16$ and $n \equiv 9 \, mod \, 16$;

8.4. $m \equiv 9 \, mod \, 16$ and $n \equiv 13 \, mod \, 16$.

First, label the vertices of P_m by $3, 0, 2, 0,\ 3, 0, 2, 0,\ \dots,\ 3, 0, 2, 0,\ 3$ in order. This shows that $P_m \in M_4^{0,1;3}$ if $m \equiv 1 \, mod \, 16$, $P_m \in M_4^{0,1;0}$ if $m \equiv 5 \, mod \, 16$, $P_m \in M_4^{0,1;1}$ if $m \equiv 9 \, mod \, 16$, and $P_m \in M_4^{0,1;2}$ if $m \equiv 13 \, mod \, 16$.

Next, label the vertices of P_n by $0, 0, 1, 0,\ 0, 0, 1, 0,\ \dots,\ 0, 0, 1, 0,\ 0$ in order. This implies that $P_n \in M_4^{0,1;0}$ if $n \equiv 1 \, mod \, 16$, $P_n \in M_4^{0,1;1}$ if $n \equiv 5 \, mod \, 16$, $P_n \in M_4^{0,1;2}$ if $n \equiv 9 \, mod \, 16$, and $P_n \in M_4^{0,1;3}$ if $n \equiv 13 \, mod \, 16$.

If $m \equiv 13 \, mod \, 16$ and $n \equiv 1 \, mod \, 16$ or if $m \equiv 5 \, mod \, 16$ and $n \equiv 9 \, mod \, 16$, then apply Lemma 2.1 (13). If $m \equiv 1 \, mod \, 16$ and $n \equiv 5 \, mod \, 16$, or if $m \equiv 9 \, mod \, 16$ and $n \equiv 13 \, mod \, 16$, then apply Lemma 2.1 (14). □

2.3. Join of a path and an even cycle

Theorem 2.2. *For $m \geq 2$ and $n \geq 2$, $mc(P_m \vee C_{2n}) = 4$.*

Proof. First, label the vertices of C_{2n}, $n \equiv 0 \, mod \, 2$, by $0, 1, 3, 0, 0, 1, 3, 0, \dots, 0, 1, 3, 0$ in cyclic order. This shows that $C_{2n} \in M_4^{1,3;0}$ if $n \equiv 0 \, mod \, 2$.

Next, label the vertices of C_{2n}, $n \equiv 1 \, mod \, 2$, by $1, 0, 1, 0, 1, 0, 1, 0, \dots, 1, 0, 1, 0, 1, 0$ in cyclic order. This shows that $C_{2n} \in M_4^{0,2;1}$ if $n \equiv 1 \, mod \, 4$ and $C_{2n} \in M_4^{0,2;3}$ if $n \equiv 3 \, mod \, 4$.

Finally, for $m \equiv 0 \, mod \, 4$, label the vertices of P_m by $0, 0, 2, 0,\ 0, 0, 2, 0,\ \dots,\ 0, 0, 2, 0$ in order; for $m \equiv 1 \, mod \, 4$, label the vertices of P_m by $0, 0, 2, 0,\ 0, 0, 2, 0, \dots, 0, 0, 2, 0,\ 0$ in order; for $m \equiv 2 \, mod \, 4$, label the vertices of P_m by $2, 0, 0, 0,\ 2, 0, 0, 0, \dots, 2, 0, 0, 0,\ 2, 0$ in order; and for $m \equiv 3 \, mod \, 4$, label the vertices of P_m by $0, 0, 2, 0,\ 0, 0, 2, 0,\ \dots,\ 0, 0, 2, 0,\ 0, 0, 2$ in order. This shows that $P_m \in M_4^{0,2;0} \cup M_4^{0,2;2}$.

If $n \equiv 0 \, mod \, 2$, then apply Lemma 2.1 (5) and (39). If $n \equiv 1 \, mod \, 4$, then apply Lemma 2.1 (21) and (22). If $n \equiv 3 \, mod \, 4$, then apply Lemma 2.1 (23) and (24). □

2.4. Join of a path and a complete bipartite graph

Theorem 2.3. *For integers $n \geq 2$, $r \geq 1$, and $s \geq 1$, $mc(P_n \vee K_{r,s}) = 4$.*

Proof. Let $P_n := u_1 u_2 \dots u_n$, $X = \{x_1, x_2, \dots, x_r\}$, $Y = \{y_1, y_2, \dots, y_s\}$, and the bipartition of $K_{r,s}$ be (X, Y). We consider four cases.

Case 1. $n \equiv i \, mod \, 16$, $i \in \{1, 2, 3, 4, 5\}$.

First, we claim that $K_{r,s} \in M_4^{2,3;1}$. To see this, label the vertex x_1 of $K_{r,s}$ by 2, the vertex y_1 of $K_{r,s}$ by 3 and all other vertices of $K_{r,s}$ by 0. Next, we claim that $P_n \in M_4^{0,1;1}$. To see this, for $n \equiv 1 \, mod \, 16$, label the vertices of P_n by $1, 0, 0, 0,\ 1, 0, 0, 0,\ \dots,\ 1, 0, 0, 0,\ 1$ in order; for $n \equiv 2 \, mod \, 16$, label the vertices of P_n by $1, 0, 0, 0,\ 1, 0, 0, 0,\ \dots,\ 1, 0, 0, 0,\ 1, 0$ in order; for $n \equiv 3 \, mod \, 16$, label the vertices of P_n by $1, 0, 0, 0,\ 1, 0, 0, 0,\ \dots,\ 1, 0, 0, 0,\ 1, 0, 0$ in order; for $n \equiv 4 \, mod \, 16$, label the vertices of P_n by $0, 0, 1, 0,\ 0, 0, 1, 0,\ \dots,\ 0, 0, 1, 0$ in order; for $n \equiv 5 \, mod \, 16$, label the vertices of P_n by $0, 0, 1, 0,\ 0, 0, 1, 0,\ \dots,\ 0, 0, 1, 0,\ 0$ in order. Finally, apply Lemma 2.1 (2).

Case 2. $n \equiv i \, mod \, 16$, $i \in \{6, 7, 8\}$.

First, we claim that $K_{r,s} \in M_4^{0,3;3}$. To see this, label the vertex y_1 of $K_{r,s}$ by 3 and all other vertices of $K_{r,s}$ by 0. Next, we claim that $P_n \in M_4^{0,1;2}$. To see this, for $n \equiv 6 \, mod \, 16$, label the vertices of P_n by $0, 1, 0, 0,\ 0, 1, 0, 0,\ \dots,\ 0, 1, 0, 0,\ 0, 1$ in order; for $n \equiv 7 \, mod \, 16$, label the vertices of P_n by $0,1,0,0,\ 0,1,0,0,\ \ldots,\ 0,1,0,0,\ 0,1,0$ in order; for $n \equiv 8\,mod\,16$, label the vertices of P_n by $0,1,0,0,\ 0,1,0,0,\ \ldots,\ 0,1,0,0$ in order. Finally, apply Lemma 2.1 (31).

Case 3. $n \equiv i\,mod\,16, i \in \{9,10,11,12,13\}$.

First, we claim that $K_{r,s} \in M_4^{0,1;1}$. To see this, label the vertex y_1 of $K_{r,s}$ by 1 and all other vertices of $K_{r,s}$ by 0. Next, we claim that $P_n \in M_4^{0,1;3}$. To see this, for $n \equiv 9\,mod\,16$, label the vertices of P_n by $1,0,0,0,\ 1,0,0,0,\ \ldots,\ 1,0,0,0,\ 1$ in order; for $n \equiv 10\,mod\,16$, label the vertices of P_n by $1,0,0,0,\ 1,0,0,0,\ \ldots,\ 1,0,0,0,\ 1,0$ in order; for $n \equiv 11\,mod\,16$, label the vertices of P_n by $1,0,0,0,\ 1,0,0,0,\ \ldots,\ 1,0,0,0,\ 1,0,0$ in order; for $n \equiv 12\,mod\,16$, label the vertices of P_n by $0,0,1,0,\ 0,0,1,0,\ \ldots,\ 0,0,1,0$ in order; for $n \equiv 13\,mod\,16$, label the vertices of P_n by $0,0,1,0,\ 0,0,1,0,\ \ldots,\ 0,0,1,0,\ 0$ in order. Finally, apply Lemma 2.1 (14).

Case 4. $n \equiv i\,mod\,16, i \in \{0,14,15\}$.

First, we claim that $K_{r,s} \in M_4^{1,2;3}$. To see this, label the vertex x_1 of $K_{r,s}$ by 1, the vertex y_1 of $K_{r,s}$ by 2 and all other vertices of $K_{r,s}$ by 0. Next, we claim that $P_n \in M_4^{0,1;0}$. To see this, for $n \equiv 14\,mod\,16$, label the vertices of P_n by $0,1,0,0,\ 0,1,0,0,\ \ldots,\ 0,1,0,0,\ 0,1$ in order; for $n \equiv 15\,mod\,16$, label the vertices of P_n by $0,1,0,0,\ 0,1,0,0,\ \ldots,\ 0,1,0,0,\ 0,1,0$ in order; for $n \equiv 0\,mod\,16$, label the vertices of P_n by $0,1,0,0,\ 0,1,0,0,\ \ldots,\ 0,1,0,0$ in order. Finally, apply Lemma 2.1 (33). □

2.5. Join of an even cycle and a complete bipartite graph

Let $X = \{x_1, x_2, \ldots, x_r\}$, $Y = \{y_1, y_2, \ldots, y_s\}$, and the bipartition of $K_{r,s}$ be (X, Y).

Theorem 2.4. *For integers $n \geq 1$, $r \geq 1$, and $s \geq 1$, $mc(C_{4n} \vee K_{r,s}) = 4$.*

Proof. Let $C_{4n} := u_1u_2\ldots u_{4n}u_1$. Label the vertices of C_{4n} by $1,0,0,0,\ 1,0,0,0,\ldots,\ 1,0,0,0$ in cyclic order. We consider four cases.

Case 1. $n \equiv 1\,mod\,4$.

$C_{4n} \in M_4^{0,1;1}$ and by the proof of Theorem 2.3, $K_{r,s} \in M_4^{2,3;1}$. Apply Lemma 2.1 (2).

Case 2. $n \equiv 2\,mod\,4$.

$C_{4n} \in M_4^{0,1;2}$ and by the proof of Theorem 2.3, $K_{r,s} \in M_4^{0,3;3}$. Apply Lemma 2.1 (31).

Case 3. $n \equiv 3\,mod\,4$.

$C_{4n} \in M_4^{0,1;3}$ and by the proof of Theorem 2.3, $K_{r,s} \in M_4^{0,1;1}$. Apply Lemma 2.1 (14).

Case 4. $n \equiv 0\,mod\,4$.

$C_{4n} \in M_4^{0,1;0}$ and by the proof of Theorem 2.3, $K_{r,s} \in M_4^{1,2;3}$. Apply Lemma 2.1 (33). □

Theorem 2.5. *For integers $n \geq 1$, $r \geq 1$, and $s \geq 1$, $mc(C_{4n+2} \vee K_{r,s}) = 4$.*

Proof. Label the vertex x_1 of $K_{r,s}$ by 2 and all other vertices of $K_{r,s}$ by 0. This shows that $K_{r,s} \in M_4^{0,2;2}$. Let $C_{4n+2} := u_1u_2\ldots u_{4n+2}u_1$. Label the vertices of C_{4n+2} by $1,0,1,0,\ 1,0,1,0,\ \ldots,\ 1,0,1,0,\ 1,0$ in cyclic order.

If $n \equiv 1\,mod\,2$, then $C_{4n+2} \in M_4^{0,2;3}$. Now apply Lemma 2.1 (23).

If $n \equiv 0\,mod\,2$, then $C_{4n+2} \in M_4^{0,2;1}$. Now apply Lemma 2.1 (22). □

2.6. Join of two regular bipartite graphs

Theorem 2.6. *Let G be a k-regular bipartite graph with $k \equiv 1 \, mod \, 2$ and let H be an ℓ-regular bipartite graph with $\ell \equiv 1 \, mod \, 2$. We have $mc(G \vee H) = 4$.*

Proof. Let (X, Y) be the bipartition of G with $|X| = |Y| = m$, and let (U, V) be the bipartition of H with $|U| = |V| = n$. Define $c : V(G \vee H) \to \mathbb{Z}_4$ by $c(x) = 0$ for $x \in X$, $c(y) = 2$ for $y \in Y$, $c(u) = 1$ for $u \in U$, $c(v) = 3$ for $v \in V$. Then, $\sigma(x) = (n + 3n + 2k)(mod\,4) = 2$ for $x \in X$ and $\sigma(y) = (n + 3n)(mod\,4) = 0$ for $y \in Y$. We consider 2 cases.

Case 1. Either $m \equiv 0 \, mod \, 2$ and $\ell \equiv 1 \, mod \, 4$ or $m \equiv 1 \, mod \, 2$ and $\ell \equiv 3 \, mod \, 4$.

In this case, $\sigma(u) = (2m + 3\ell)(mod\,4) = 3$ for $u \in U$ and $\sigma(v) = (2m + \ell)(mod\,4) = 1$ for $v \in V$.

Case 2. Either $m \equiv 0 \, mod \, 2$ and $\ell \equiv 3 \, mod \, 4$ or $m \equiv 1 \, mod \, 2$ and $\ell \equiv 1 \, mod \, 4$.

In this case, $\sigma(u) = (2m + 3\ell)(mod\,4) = 1$ for $u \in U$ and $\sigma(v) = (2m + \ell)(mod\,4) = 3$ for $v \in V$. □

Theorem 2.7. *Let G be a k-regular bipartite graph with $k \equiv 2 \, mod \, 4$ and let H be an ℓ-regular bipartite graph with $\ell \equiv 1 \, mod \, 2$. We have $mc(G \vee H) = 4$.*

Proof. Let (X, Y) be the bipartition of G with $|X| = |Y| = m$, and let (U, V) be the bipartition of H with $|U| = |V| = n$. We consider two cases.

Case 1. $m \equiv 1 \, mod \, 2$.

Define $c : V(G \vee H) \to \mathbb{Z}_4$ by $c(x) = 0$ for $x \in X$, $c(y) = 1$ for $y \in Y$, $c(u) = 0$ for $u \in U$, $c(v) = 2$ for $v \in V$. Then, $\sigma(x) = (2n+k)(mod\,4) = (2n+2)(mod\,4)$ for $x \in X$, $\sigma(y) = 2n \, mod \, 4$ for $y \in Y$, $\sigma(u) = (m+2\ell)(mod\,4) = (m+2)(mod\,4)$ for $u \in U$ and $\sigma(v) = m \, mod \, 4$ for $v \in V$. Note that $\{2n, 2n+2\}(mod\,4) = \{0, 2\}$ and as $m \equiv 1 \, mod \, 2$, $\{m, m+2\}(mod\,4) = \{1, 3\}$.

Case 2. $m \equiv 0 \, mod \, 2$.

Define $c : V(G \vee H) \to \mathbb{Z}_4$ by $c(x) = 0$ for $x \in X$, $c(y) = 1$ for $y \in Y$, $c(u) = 1$ for $u \in U$, $c(v) = 3$ for $v \in V$. Then, $\sigma(x) = (4n + k)(mod\,4) = 2$ for $x \in X$, $\sigma(y) = 4n \, mod \, 4 = 0$ for $y \in Y$, $\sigma(u) = (m + 3\ell)(mod\,4)$ for $u \in U$ and $\sigma(v) = (m + \ell)(mod\,4)$ for $v \in V$.

Note that (i) if either $m \equiv 0 \, mod \, 4$ and $\ell \equiv 1 \, mod \, 4$ or $m \equiv 2 \, mod \, 4$ and $\ell \equiv 3 \, mod \, 4$, then $\sigma(u) = 3$ for $u \in U$ and $\sigma(v) = 1$ for $v \in V$, (ii) if either $m \equiv 2 \, mod \, 4$ and $\ell \equiv 1 \, mod \, 4$ or $m \equiv 0 \, mod \, 4$ and $\ell \equiv 3 \, mod \, 4$, then $\sigma(u) = 1$ for $u \in U$ and $\sigma(v) = 3$ for $v \in V$. □

We propose:

Problem 1. *Let G be a k-regular bipartite graph with $k \equiv 0 \, mod \, 4$ and let H be an ℓ-regular bipartite graph with $\ell \equiv 1 \, mod \, 2$. Find $mc(G \vee H)$.*

Problem 2. *Let G be a k-regular bipartite graph with $k \equiv 0 \, mod \, 2$ and let H be an ℓ-regular bipartite graph with $\ell \equiv 0 \, mod \, 2$. Find $mc(G \vee H)$.*

2.7. Join of two even cycles

Theorem 2.8. *For* $m \geq 1$ *and* $n \geq 1$, $mc(C_{4m} \vee C_{4n}) = 4$.

Proof. First, label the vertices of C_{4m} by $0, 0, 2, 0,\ 0, 0, 2, 0,\ \ldots,\ 0, 0, 2, 0$ in cyclic order. This shows that $C_{4m} \in M_4^{0,2;0} \cup M_4^{0,2;2}$. Next, label the vertices of C_{4n} by $1, 3, 0, 0,\ 1, 3, 0, 0,\ \ldots,$ $1, 3, 0, 0$ in cyclic order. This implies that $C_{4n} \in M_4^{1,3;0}$. Finally, apply Lemma 2.1 (5) and (39). □

Theorem 2.9. *For* $m \geq 1$ *and* $n \geq 1$, $mc(C_{4m} \vee C_{4n+2}) = 4$.

Proof. By the proof of Theorem 2.3, $C_{4m} \in M_4^{0,2;0} \cup M_4^{0,2;2}$. Labelling the vertices of C_{4n+2} by $1, 0, 1, 0,\ 1, 0, 1, 0,\ \ldots,\ 1, 0, 1, 0,\ 1, 0$ in cyclic order shows that $C_{4n+2} \in M_4^{0,2;1} \cup M_4^{0,2;3}$. Application of Lemma 2.3 (21), (22), (23) and (24) completes the proof. □

Lemma 2.2. *Let* $\ell \geq 1$. *If* $C_{4\ell+2} \in M_4^{i,j;k}$, *then* $\{i, j\} = \{0, 2\}$ *and* $k \in \{1, 3\}$.

Proof. Let $C_{4\ell+2} = z_1 z_2 \ldots z_{4\ell+2} z_1$. $C_{4\ell+2} \in M_4^{i,j;k}$ implies $C_{4\ell+2}$ has a modular 4-coloring c such that $\sigma(z_{2p-1}) = i$ and $\sigma(z_{2p}) = j$ for $p \in \{1, 2, \ldots, 2\ell+1\}$, and $\sum_{q=1}^{4\ell+2} c(z_q) \equiv k \ mod\ 4$. We consider four cases.

Case 1. $i = 0$ and $j = 1$.

The σ-value 1 for the vertices $z_2, z_4, z_6, \ldots, z_{4\ell}$ in order implies $(c(z_1), c(z_3), c(z_5), c(z_7), \ldots, c(z_{4\ell+1}))$ is one of the following: $(0, 1, 0, 1, \ldots, 0)$, $(1, 0, 1, 0, \ldots, 1)$, $(2, 3, 2, 3, \ldots, 2)$, $(3, 2, 3, 2, \ldots, 3)$. But then $\sigma(z_{4\ell+2}) \neq 1$, a contradiction.

Case 2. $i \in \{0, 2\}$ and $j = 3$.

The σ-value 3 for the vertices $z_2, z_4, z_6, \ldots, z_{4\ell}$ in order implies $(c(z_1), c(z_3), c(z_5), c(z_7), \ldots, c(z_{4\ell+1}))$ is one of the following: $(0, 3, 0, 3, \ldots, 0)$, $(3, 0, 3, 0, \ldots, 3)$, $(1, 2, 1, 2, \ldots, 1)$, $(2, 1, 2, 1, \ldots, 2)$. But then $\sigma(z_{4\ell+2}) \neq 3$, a contradiction.

Case 3. $i = 1$ and $j \in \{2, 3\}$.

The σ-value 1 for the vertices $z_3, z_5, z_7, \ldots, z_{4\ell+1}$ in order implies $(c(z_2), c(z_4), c(z_6), c(z_8), \ldots, c(z_{4\ell+2}))$ is one of the following: $(0, 1, 0, 1, \ldots, 0)$, $(1, 0, 1, 0, \ldots, 1)$, $(2, 3, 2, 3, \ldots, 2)$, $(3, 2, 3, 2, \ldots, 3)$. But then $\sigma(z_1) \neq 1$, a contradiction.

Case 4. $i = 0$ and $j = 2$.

The σ-value 0 for the vertices $z_3, z_5, z_7, \ldots, z_{4\ell+1}$ in order implies $(c(z_2), c(z_4), c(z_6), c(z_8), \ldots, c(z_{4\ell+2}))$ is one of the following: $(0, 0, 0, 0, \ldots, 0)$, $(2, 2, 2, 2, \ldots, 2)$, $(1, 3, 1, 3, \ldots, 1)$, $(3, 1, 3, 1, \ldots, 3)$. $\sigma(z_1) = 0$ implies that $(c(z_2), c(z_4), c(z_6), c(z_8), \ldots, c(z_{4\ell+2})) = (0, 0, 0, 0, \ldots, 0)$ or $(2, 2, 2, 2, \ldots, 2)$.

The σ-value 2 for the vertices $z_2, z_4, z_6, \ldots, z_{4\ell}$ in order implies $(c(z_1), c(z_3), c(z_5), c(z_7), \ldots, c(z_{4\ell+1}))$ is one of the following: $(0, 2, 0, 2, \ldots, 0)$, $(2, 0, 2, 0, \ldots, 2)$, $(1, 1, 1, 1, \ldots, 1)$, $(3, 3, 3, 3, \ldots, 3)$. $\sigma(z_{4\ell+2}) = 2$ implies that $(c(z_1), c(z_3), c(z_5), c(z_7), \ldots, c(z_{4\ell+1})) = (1, 1, 1, 1, \ldots, 1)$ or $(3, 3, 3, 3, \ldots, 3)$.

Hence the c-values for the vertices $z_1, z_2, \ldots z_{4\ell+2}$ in cyclic order are: $0, 1, 0, 1, \ldots, 0, 1;\ 0, 3, 0, 3, \ldots, 0, 3;\ 2, 1, 2, 1, \ldots, 2, 1;\ 2, 3, 2, 3, \ldots, 2, 3$. □

Theorem 2.10. *For* $m \geq 1$ *and* $n \geq 1$, $mc(C_{4m+2} \vee C_{4n+2}) = 5$.

Proof. Let $C_{4m+2} = x_1x_2\ldots x_{4m+2}x_1$ and $C_{4n+2} = y_1y_2\ldots y_{4n+2}y_1$. Define $c : V(C_{4m+2} \vee C_{4n+2}) \to \mathbb{Z}_5$ by $c(x_{2p}) = 1$ and $c(x_{2p-1}) = 4$ for $p \in \{1, 2, \ldots, 2m+1\}$ and $c(y_{2q}) = 2$ and $c(y_{2q-1}) = 3$ for $q \in \{1, 2, \ldots, 2n+1\}$. Then, $\sigma(x_{2p}) = 3$ and $\sigma(x_{2p-1}) = 2$ for $p \in \{1, 2, \ldots, 2m+1\}$ and $\sigma(y_{2q}) = 1$ and $\sigma(y_{2q-1}) = 4$ for $q \in \{1, 2, \ldots, 2n+1\}$. Hence c is a modular 5-coloring and therefore $mc(C_{4m+2} \vee C_{4n+2}) \leq 5$.

Suppose there exists a modular 4-coloring c for $C_{4m+2} \vee C_{4n+2}$. This induces a modular 4-coloring $c' = c|_{\{x_1,x_2,\ldots,x_{4m+2}\}}$ such that $\sigma'(x_{2p-1}) = i'$ and $\sigma'(x_{2p}) = j'$ for $p \in \{1, 2, \ldots, 2m+1\}$, and $\sum_{q=1}^{4m+2} c'(x_q) \equiv k' \,(mod\, 4)$ for some i', j', k'; and a modular 4-coloring $c'' = c|_{\{y_1,y_2,\ldots,y_{4n+2}\}}$ such that $\sigma''(y_{2p-1}) = i''$ and $\sigma''(y_{2p}) = j''$ for $p \in \{1, 2, \ldots, 2n+1\}$, and $\sum_{q=1}^{4n+2} c''(y_q) \equiv k'' \,(mod\, 4)$ for some i'', j'', k''.

By the proof of Lemma 2.2, both the sequences $\{c'(x_p)\}_{p=1}^{4m+2}$ and $\{c''(y_p)\}_{p=1}^{4n+2}$ are in $\{(0,1,0,1,\ldots,0,1), (0,3,0,3,\ldots,0,3), (2,1,2,1,\ldots,2,1), (2,3,2,3,\ldots,2,3)\}$.

If $\{c'(x_p)\}_{p=1}^{4m+2} = \{c''(y_p)\}_{p=1}^{4n+2}$ is one of: $(0,1,0,1,\ldots,0,1)$, $(0,3,0,3,\ldots,0,3)$, $(2,1,2,1,\ldots,2,1)$, $(2,3,2,3,\ldots,2,3)$, then $\{\sigma(x_p)\}_{p=1}^{4m+2} = \{\sigma(y_p)\}_{p=1}^{4n+2} = (1,3,1,3,\ldots,1,3)$, a contradiction.

If $\{\{c'(x_p)\}_{p=1}^{4m+2}, \{c''(y_p)\}_{p=1}^{4n+2}\} \in \{\{(0,1,0,1,\ldots,0,1), (0,3,0,3,\ldots,0,3)\}$, $\{(0,1,0,1,\ldots,0,1), (2,1,2,1,\ldots,2,1)\}$, $\{(0,1,0,1,\ldots,0,1), (2,3,2,3,\ldots,2,3)\}$, $\{(0,3,0,3,\ldots,0,3), (2,1,2,1,\ldots,2,1)\}$, $\{(0,3,0,3,\ldots,0,3), (2,3,2,3,\ldots,2,3)\}$, $\{(2,1,2,1,\ldots,2,1), (2,3,2,3,\ldots,2,3)\}$, then $\{\sigma(x_p)\}_{p=1}^{4m+2} = \{\sigma(y_p)\}_{p=1}^{4n+2} = (1,3,1,3,\ldots,1,3)$, again a contradiction.

From these two contradictions, we have $mc(C_{4m+2} \vee C_{4n+2}) \geq 5$. □

2.8. Join of a regular bipartite graph and an empty graph

Okamoto, Salehi and Zhang [2] have shown that for any integer $n \geq 3$, $mc(C_n \vee K_1) = \chi(C_n \vee K_1)$. Using the proof technique of this result, we prove Theorem 2.11.

Theorem 2.11. *Let G be an r-regular bipartite graph with $r \equiv 1$ or 2 (mod 3). Then, for any positive integer s, $mc(G \vee sK_1) = 3$.*

Proof. Let (X, Y) be the bipartition of G. Construct $G \vee sK_1$ from G by joining new vertices $w_1, w_2, \ldots, w_s$ to every vertex of G. Define $c : V(G \vee sK_1) \to \mathbb{Z}_3$ by $c(w_i) = 0$ for $i \in \{1, 2, \ldots, s\}$, $c(x) = 1$ if $x \in X$ and $c(y) = 2$ if $y \in Y$. Then, for $i \in \{1, 2, \ldots, s\}$, $\sigma(w_i) = |X| + 2|Y| = |X| + 2|X| \equiv 0 \,mod\, 3$; for $x \in X$, $\sigma(x) = 2r$ is 2 if $r \equiv 1 \,mod\, 3$ and it is 1 if $r \equiv 2 \,mod\, 3$; for $y \in Y$, $\sigma(y) = r$ is 1 if $r \equiv 1 \,mod\, 3$ and it is 2 if $r \equiv 2 \,mod\, 3$. Hence, c is a modular 3-coloring of G, and therefore $mc(G \vee sK_1) \leq 3$. But, $mc(G \vee sK_1) \geq \chi(G \vee sK_1) = 3$. Thus, $mc(G \vee sK_1) = 3$. □

Remark 2.1. Let G be an r-regular bipartite graph with $r \equiv 0 \,mod\, 3$ and let (X, Y) be the bipartition of G. Suppose there exist partitions $\{X_1, X_2\}$ of X and $\{Y_1, Y_2\}$ of Y and integers t' and t'' both not congruent to $0 \,mod\, 3$ such that t' is not congruent to $t''(mod\ 3)$, the subgraph induced by $X_1 \cup Y_1$ is t'-regular and the subgraph induced by $X_2 \cup Y_2$ is t''-regular, then $mc(G \vee sK_1) = 3$.

To see this, define $c : V(G \vee sK_1) \to \mathbb{Z}_3$ by $c(w_i) = 0$ for $i \in \{1, 2, \ldots, s\}$, $c(x) = 1$ if $x \in X_1$, $c(x) = 2$ if $x \in X_2$, $c(y) = 2$ if $y \in Y_1$, and $c(y) = 1$ if $y \in Y_2$. Then, for $i \in \{1, 2, \ldots, s\}$, $\sigma(w_i)$ $= |X_1| + 2|X_2| + 2|Y_1| + |Y_2| = 3|X| = 0 \, mod \, 3$; for $x \in X_1$, $\sigma(x) = 2t' + (r - t') = r + t'$; for $x \in X_2$, $\sigma(x) = t'' + 2(r - t'') = 2r - t''$; for $y \in Y_1$, $\sigma(y) = t' + 2(r - t') = 2r - t'$; and for $y \in Y_2$, $\sigma(y) = 2t'' + (r - t'') = r + t''$. By hypothesis, in $\mathbb{Z}_3$, $0 \notin \{r + t', r + t'', 2r - t', 2r - t''\}$ and $\{r + t', 2r - t''\} \cap \{r + t'', 2r - t'\} = \phi$.

3. Join of a path and a complete graph

Theorem 3.1. *For integers $n \geq 3$ and $p \geq 3$, $mc(P_n \vee K_p) = p + 2$.*

Proof. Let $P_n := u_1 u_2 \ldots u_n$ and $V(K_p) = \{v_1, v_2, \ldots, v_p\}$. Define $c : V(P_n \vee K_p) \to \mathbb{Z}_{p+2}$ as follows: Label the vertices of P_n, in order, by $0, 1, 0, 0,\ 0, 1, 0, 0,\ \ldots,\ 0, 1, 0, 0$ for $n \equiv 0 \, mod \, 4$; by $1, 0, 0, 0,\ 1, 0, 0, 0,\ \ldots,\ 1, 0, 0, 0,\ 1$ for $n \equiv 1 \, mod \, 4$; by $1, 0, 0, 0,\ 1, 0, 0, 0, \ldots,\ 1, 0, 0, 0,\ 1, 0$ for $n \equiv 2 \, mod \, 4$; and by $1, 0, 0, 0,\ 1, 0, 0, 0,\ \ldots,\ 1, 0, 0, 0,\ 1, 0, 0$ for $n \equiv 3 \, mod \, 4$. Label the p vertices of K_p by the p numbers in

$$\{0, 1, 2, \ldots, p + 1\} \backslash \{(\left\lceil \frac{n}{4} \right\rceil - 1)(mod\ (p + 2)), \left\lceil \frac{n}{4} \right\rceil (mod\ (p + 2))\}.$$

The σ- values of the vertices of P_n in $P_n \vee K_p$ are, alternately, $(\frac{(p+1)(p+2)}{2} - 2 \left\lceil \frac{n}{4} \right\rceil + 1)(mod\ (p+2))$ and $(\frac{(p+1)(p+2)}{2} - 2 \left\lceil \frac{n}{4} \right\rceil + 2)(mod\ (p + 2))$. The σ- values of the p vertices of K_p in $P_n \vee K_p$ are the p numbers in $\{(\frac{(p+1)(p+2)}{2} - \left\lceil \frac{n}{4} \right\rceil + 1 - i)(mod\ (p + 2)) : i \in \{0, 1, 2, \ldots, p + 1\} \backslash \{(\left\lceil \frac{n}{4} \right\rceil - 1)(mod\ (p + 2)), \left\lceil \frac{n}{4} \right\rceil (mod\ (p + 2))\}\}$. Note that the σ- value of any vertex of P_n in $P_n \vee K_p$ is different from that of any vertex of K_p in $P_n \vee K_p$. This completes the proof. □

4. Join of an even cycle and a complete graph

Theorem 4.1. *For integers $n \geq 2$ and $p \geq 3$, $mc(C_{2n} \vee K_p) = p + 2$.*

Proof. Let $C_{2n} := u_1 u_2 \ldots u_{2n} u_1$ and $V(K_p) = \{v_1, v_2, \ldots, v_p\}$. Define $c : V(C_{2n} \vee K_p) \to \mathbb{Z}_{p+2}$ as follows: We consider two cases:
Case 1. $n \equiv 0 \, mod \, 2$.

Label the vertices of C_{2n}, in cyclic order, by $1, 0, 0, 0,\ 1, 0, 0, 0,\ \ldots,\ 1, 0, 0, 0$. Label the p vertices of K_p by the p numbers in

$$\{0, 1, 2, \ldots, p + 1\} \backslash \{(\frac{n}{2} - 1)(mod\ (p + 2)), \frac{n}{2}(mod\ (p + 2))\}.$$

The σ- values of the vertices of C_{2n} in $C_{2n} \vee K_p$ are, alternately, $(\frac{(p+1)(p+2)}{2} - n + 1)(mod\ (p+2))$ and $(\frac{(p+1)(p+2)}{2} - n + 2)(mod\ (p + 2))$. The σ- values of the p vertices of K_p in $C_{2n} \vee K_p$ are the p numbers in $\{(\frac{(p+1)(p+2)}{2} - \frac{n}{2} + 1 - i)(mod\ (p + 2)) : i \in \{0, 1, 2, \ldots, p + 1\} \backslash \{(\frac{n}{2} - 1)(mod\ (p + 2)), \frac{n}{2}(mod\ (p + 2))\}\}$. Observe that the σ- value of any vertex of C_{2n} in $C_{2n} \vee K_p$ is different from that of any vertex of K_p in $C_{2n} \vee K_p$.
Case 2. $n \equiv 1 \, mod \, 2$.

Label the vertices of C_{2n}, in cyclic order, by $1, 0, 1, 0,\ 1, 0, 1, 0,\ \ldots,\ 1, 0, 1, 0$. Label the p vertices of K_p by the p numbers in $\{0, 1, 2, \ldots, p+1\}\backslash\{(n-2)(mod\ (p+2)), n\ mod\ (p+2)\}$. The σ- values of the vertices of C_{2n} in $C_{2n} \vee K_p$ are, alternately, $(\frac{(p+1)(p+2)}{2} - 2n+2)(mod\ (p+2))$ and $(\frac{(p+1)(p+2)}{2} - 2n+4)(mod\ (p+2))$. The σ- values of the p vertices of K_p in $C_{2n} \vee K_p$ are the p numbers in $\{(\frac{(p+1)(p+2)}{2} - n + 2 - i)(mod\ (p+2)) : i \in \{0, 1, 2, \ldots, p+1\}\backslash\{(n-2)(mod\ (p+2)), n\ mod\ (p+2)\}$. Clearly, the σ- value of any vertex of C_{2n} in $C_{2n} \vee K_p$ is different from that of any vertex of K_p in $C_{2n} \vee K_p$.

This completes the proof. □

5. Conclusion

We have seen that $mc(G \vee H) = \chi(G) + \chi(H)$ for every join graph $G \vee H$, except for the join graph $C_{4m+2} \vee C_{4n+2}$, that we have encountered in this paper.

References

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