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Spectra of graphs and the spectral criterion for property (T)

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Abstract

For a finite connected graph X, we consider the graph RX obtained from X by associating a new vertex to every edge of X and joining by edges the extremities of each edge of X to the corresponding new vertex. We express the spectrum of the Laplace operator on RX as a function of the corresponding spectrum on X. As a corollary, we show that X is a complete graph if and only if $\lambda_1(RX) > \frac{1}{2}$. We give a re-interpretation of the correspondence $X \mapsto RX$ in terms of the right-angled Coxeter group defined by X.

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1. Introduction

Let $X = (V, E)$ be a finite, connected graph. Denote by $\sim$ the adjacency relation on V; that is, $x \sim y$ if and only if $\{x, y\} \in E$. Endow the space $\mathbb{R}V$ of real-valued functions on V with the scalar product $\langle f|g\rangle = \sum_{x\in V} f(x)g(x)\deg(x)$, where $\deg(x)$ is the number of neighbors of x.

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The *combinatorial Laplace operator* of X is the operator Δ_X on $\mathbb{R}V$, defined by

$$(\Delta_X f)(x) = f(x) - \frac{1}{\deg(x)} \sum_{y\sim x} f(y)$$

($f \in \mathbb{R}V,\ x \in V$). It is classical that Δ_X is self-adjoint with respect to $\langle .|. \rangle$ (that is, $\langle \Delta_X f|g\rangle = \langle f|\Delta_X g\rangle$ for every $f, g \in \mathbb{R}V$), and has spectrum contained in $[0, 2]$; the associated quadratic form is given by:

$$\langle \Delta_X f|f\rangle = \frac{1}{2} \sum_{x,y:\, x\sim y} (f(x) - f(y))^2$$

($f \in \mathbb{R}V$); see [4] for all this. Then 0 is a multiplicity 1 eigenvalue of Δ_X, and we denote by $\lambda_1(X)$ the smallest non-zero eigenvalue of X.

We denote by RX the graph with vertex set $V \sqcup E$ (the disjoint union of V and E) and adjacency relation given by:

- if $x, y \in V :\ x \sim y \Leftrightarrow \{x, y\} \in E$;
- if $x \in V,\ e \in E : x \sim e \Leftrightarrow x \in e$;
- if $e, e' \in E$, then e, e' are not adjacent in RX.

Graphically, this means that every edge $e = \{x, y\}$ in X gets replaced in RX by a triangle $\{x, y, e\}$ (with $\deg e = 2$)[1]. This operation on graphs was considered by Cvetkovic [5], who computed, in case X is regular, the spectrum of the adjacency operator of RX as a function of the corresponding spectrum for X (see Theorem 3 in [5]).

The purpose of this note is twofold. First, we explain the relevance of the transformation $X \mapsto RX$ in terms of Cayley graphs for the right-angled Coxeter group associated with X. Second, we compute the spectrum $Sp\,\Delta_{RX}$ of Δ_{RX} in terms of the spectrum $Sp\,\Delta_X$ of Δ_X, without regularity assumption on X. Observe that, for $f \in \mathbb{R}(V \sqcup E)$:

$$(\Delta_{RX} f)(y) = \begin{cases} f(y) - \frac{1}{2\deg(y)}\left[\sum_{x\in V, x\sim y} f(x) + \sum_{e\in E, y\in E} f(e)\right] & \text{if} \quad y \in V \\ f(y) - \frac{1}{2}\sum_{x\in y} f(x) & \text{if} \quad y \in E \end{cases} \tag{1}$$

The following result will be proved in Section 3:

Proposition 1.1. *Let X be a finite connected graph with n vertices and m edges. A real number $\lambda \in [0, 2]$ is an eigenvalue of Δ_{RX} if and only some of the following cases occurs:*

- $\lambda = 1$ *(this case occurs only if $m > n$);*
- $\lambda = \frac{3}{2}$*;*

[1] The graph RX should NOT be confused with the total graph TX, whose set of vertices is also $V \sqcup E$ but the 3rd condition above gets replaced by: there is an edge between $e, e' \in E$ if and only if e and e' are incident in X. So RX is a spanning subgraph of TX.

- 2λ *is an eigenvalue of* Δ_X.

Taking into account the fact that, for the complete graph K_n on n vertices, we have $Sp(\Delta_{K_n}) = \{0, \frac{n}{n-1}\}$, and that $\lambda_1 > 1$ characterizes complete graphs (see Lemma 1.7 in [4]), we get as an immediate corollary:

Corollary 1.1. *Let X be a finite connected graph. The following are equivalent:*

i) $\lambda_1(RX) > \frac{1}{2}$*;*
ii) X *is a complete graph.*

However, it is possible to give a direct, group-theoretic proof of the implication $(i) \Rightarrow (ii)$ in Corollary 1.1: this will be done in Section 2.

2. Cayley graphs and property (T)

Recall that a finitely generated group Γ has property (T) if every affine isometric action of Γ on a Hilbert space, has a fixed point. We refer to [2] for examples, characterizations and applications of property (T).

Let Γ be a finitely generated group and let S be a finite generating subset such that $S = S^{-1}$ and $1 \notin S$. Let $\mathcal{G}(\Gamma, S)$ be the Cayley graph of Γ with respect to S; that is, the vertex set is Γ, and two vertices $x, y \in \Gamma$ are adjacent if $x^{-1}y \in S$. Let X_S be the graph induced by $\mathcal{G}(\Gamma, S)$ on S; that is, the vertex set of X_S is S, and two elements $s, t \in S$ are adjacent if $s^{-1}t \in S$. The *spectral criterion for property (T)* (see [1], [6], [7]; see also [2], Theorem 5.5.2) is the statement that, if X_S is connected and $\lambda_1(X_S) > \frac{1}{2}$, then Γ has property (T).

Proof of $(i) \Rightarrow (ii)$ in Corollary 1.1: Let $X = (V, E)$ be a finite connected graph and let W_X be the right-angled Coxeter group associated with X; this is the group defined by the presentation:

$$W_X = \langle s \in V | s^2 = 1 \ (s \in V); \ st = ts \ (\{s, t\} \in E) \rangle.$$

We will need two standard facts about Coxeter groups:

(a) An infinite Coxeter group does not have property (T) (see [3]);
(b) If $\{s, t\} \notin E$, then st has infinite order in W_X.

We define a new generating set of W_X by:

$$S =: X \cup \{st = ts : \ \{s, t\} \in E\}.$$

Observe that, if $st = ts$, then for any two distinct $x, y \in \{s, t, st\}$, the quotient $x^{-1}y$ is still in $\{s, t, st\}$. In other words, the graph X_S induced by $\mathcal{G}(W_X, S)$ on S, is isomorphic to RX.

So, if we assume $\lambda_1(RX) > \frac{1}{2}$, then W_X has property (T) by the spectral criterion. By fact (a) above, W_X is a finite group, which of course implies that st has finite order for every $s, t \in V$. By fact (b), we must have $s \sim t$ for every $s, t \in V$; that is, X is a complete graph. $\square$

3. The Laplace operator on RX

Recall that the Laplace operator on X, as a matrix indexed by $V \times V$, is:

$$(\Delta_X)_{xy} = \begin{cases} 1 & \text{if} \quad x = y \\ -\frac{1}{\deg(x)} & \text{if} \quad x \sim y \\ 0 & \text{if} \quad x \neq y, x \nsim y \end{cases}$$

Set $|V| = n$ and $|E| = m$. Turning to RX with vertex set $E \sqcup V$, recall that $\deg_{RX}(e) = 2$ for $e \in E$ and $\deg_{RX}(x) = 2\deg(x)$ for $x \in V$. So, from (1), the Laplace operator Δ_{RX} on RX is a $(m+n) \times (m+n)$ matrix:

$$\Delta_{RX} = \begin{pmatrix} 1_m & B \\ A & 1_n - \frac{1}{2}M_X \end{pmatrix}$$

where $(M_X f)(x) = \frac{1}{\deg(x)} \sum_{y \sim x} f(y)$ is the Markov operator on X (with $f \in \mathbb{R}V$) and, for $x \in V, e \in E$:

$$A_{xe} = \begin{cases} -\frac{1}{2\deg(x)} & \text{if} \quad x \in e \\ 0 & \text{if} \quad x \notin e \end{cases}$$

$$B_{ex} = \begin{cases} -\frac{1}{2} & \text{if} \quad x \in e \\ 0 & \text{if} \quad x \notin e \end{cases}$$

Observe that

$$(AB)_{xy} = \begin{cases} 0 & \text{if} \quad x \neq y, x \nsim y \\ \frac{1}{4\deg(x)} & \text{if} \quad x \sim y \\ \frac{1}{4} & \text{if} \quad x = y \end{cases}$$

So that

$$AB = \frac{1}{4}(1_n + M_X) = \frac{1}{4}(2 \cdot 1_n - \Delta_X). \tag{2}$$

The characteristic polynomial of Δ_{RX} is

$$P_{RX}(\lambda) = \det(\Delta_{RX} - \lambda \cdot 1_{m+n}) = \det \begin{pmatrix} (1-\lambda)1_m & B \\ A & (\frac{1}{2} - \lambda)1_n + \frac{1}{2}\Delta_X \end{pmatrix}$$

For $\lambda \neq 1$, multiply on the left by the unimodular matrix $\begin{pmatrix} 1_m & 0 \\ -(1-\lambda)^{-1}A & 1_n \end{pmatrix}$ to get

$$P_{RX}(\lambda) = \det \begin{pmatrix} (1-\lambda)1_m & B \\ 0 & (\frac{1}{2} - \lambda)1_n + \frac{\Delta_X}{2} - (1-\lambda)^{-1}AB \end{pmatrix}$$

$$= (1-\lambda)^m \det[(\frac{1}{2} - \lambda)1_n + \frac{\Delta_X}{2} - (1-\lambda)^{-1}AB]$$

$$= (1-\lambda)^{m-n} \det[(1-\lambda)(\frac{1}{2} - \lambda)1_n + \frac{(1-\lambda)\Delta_X}{2} - AB]$$

By Equation (2):

$$P_{RX}(\lambda) = (1-\lambda)^{m-n}\det[(1-\lambda)(\frac{1}{2}-\lambda)1_n + \frac{(1-\lambda)\Delta_X}{2} - \frac{1}{4}(2\cdot 1_n - \Delta_X)]$$

$$= (1-\lambda)^{m-n}\det[(\lambda-\frac{3}{2})(\lambda-\frac{\Delta_X}{2})]$$

$$= 2^{-n}(1-\lambda)^{m-n}(\lambda-\frac{3}{2})^n\det(2\lambda-\Delta_X)$$

So

$$P_{RX}(\lambda) = 2^{-n}(1-\lambda)^{m-n}(\frac{3}{2}-\lambda)^n P_X(2\lambda) \tag{3}$$

Proposition 1.1 immediately follows from equation (3).

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