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On total edge product cordial labeling of fullerenes

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Abstract

For a simple graph $G = (V, E)$ this paper deals with the existence of an edge labeling $\varphi : E(G) \to \{0, 1, \dots, k-1\}$, $2 \le k \le |E(G)|$, which induces a vertex labeling $\varphi^* : V(G) \to \{0, 1, \dots, k-1\}$ in such a way that for each vertex v, assigns the label $\varphi(e_1) \cdot \varphi(e_2) \cdot \ldots \cdot \varphi(e_n) \pmod{k}$, where $e_1, e_2, \dots, e_n$ are the edges incident to the vertex v. The labeling φ is called a k-total edge product cordial labeling of G if $|(e_\varphi(i) + v_{\varphi^*}(i)) - (e_\varphi(j) + v_{\varphi^*}(j))| \le 1$ for every i, j, $0 \le i < j \le k-1$, where $e_\varphi(i)$ and $v_{\varphi^*}(i)$ is the number of edges and vertices with $\varphi(e) = i$ and $\varphi^*(v) = i$, respectively. The paper examines the existence of such labelings for toroidal fullerenes and for Klein-bottle fullerenes.

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1. Introduction

Let $G = (V, E)$ be a finite graph without loops and multiple edges, where $V(G)$ and $E(G)$ are its vertex set and edge set, respectively. A general reference for graph-theoretic notions is [19].

A labeling of a graph is any mapping that sends some set of graph elements to a set of numbers or colors. If we label only vertices (respectively edges), we call such a labeling a *vertex* (respectively *edge*) *labeling*.

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A vertex labeling $\phi : V(G) \to \{0,1\}$ induces an edge labeling $\phi^* : E(G) \to \{0,1\}$ defined by $\phi^*(uv) = |\phi(u) - \phi(v)|$. For a vertex labeling ϕ and $i \in \{0,1\}$, a vertex v is an i-vertex if $\phi(v) = i$ and an edge e is an i-edge if $\phi^*(e) = i$. Denote the numbers of 0-vertices, 1-vertices, 0-edges, and 1-edges of G under ϕ and ϕ^* by $v_\phi(0)$, $v_\phi(1)$, $e_{\phi^*}(0)$, and $e_{\phi^*}(1)$, respectively. A vertex labeling ϕ is called *cordial* if $|v_\phi(0) - v_\phi(1)| \leq 1$ and $|e_{\phi^*}(0) - e_{\phi^*}(1)| \leq 1$.

The notion of the cordial labeling was first introduced by Cahit [2] as a weaker version of graceful labeling. He proved in [3] that every tree is cordial, the complete bipartite graph $K_{m,n}$ is cordial for all m and n, and the complete graph K_n is cordial if and only if $n \leq 3$. Cordial labelings of various families of graphs were studied in [7, 10, 13]. For related results see [8, 14] and for generalizations see [5, 9]. Cairnie and Edwards [4] determined the computational complexity of cordial labelings. They proved a conjecture of Kirchherr [11] that deciding whether a graph admitting a cordial labeling is NP-complete.

A binary vertex labeling $\phi : V(G) \to \{0,1\}$ with induced edge labeling $\phi^* : E(G) \to \{0,1\}$ defined by $\phi^*(uv) = \phi(u)\phi(v)$ is called a *product cordial labeling* if $|v_\phi(0) - v_\phi(1)| \leq 1$ and $|e_{\phi^*}(0) - e_{\phi^*}(1)| \leq 1$. The concept of the product cordial labeling was introduced by Sundaram et al. [15]. Some labelings with variations in cordial theme, namely an edge product cordial labeling and a total edge product cordial labeling have been introduced by Vaidya and Barasara in [17, 18].

Let k be an integer, $2 \leq k \leq |E(G)|$. An edge labeling $\varphi : E(G) \to \{0,1,\dots,k-1\}$ with induced vertex labeling $\varphi^* : V(G) \to \{0,1,\dots,k-1\}$ defined by $\varphi^*(v) = \varphi(e_1)\cdot\varphi(e_2)\cdot\ldots\cdot\varphi(e_n)$ $(\mathrm{mod}\ k)$, where $e_1, e_2, \dots, e_n$ are the edges incident to the vertex v, is called a *k-total edge product cordial labeling* of G if $|(e_\varphi(i) + v_{\varphi^*}(i)) - (e_\varphi(j) + v_{\varphi^*}(j))| \leq 1$ for every i, j, $0 \leq i < j \leq k-1$.

The concept of the k-total edge product cordial labeling was introduced by Azaizeh et al. in [1]. A graph G with a k-total edge product cordial labeling is called a *k-total edge product cordial graph*.

In the paper, we investigate the existence of 3-total edge product cordial labeling for toroidal fullerenes and for Klein-bottle fullerenes.

The discovery of the fullerene molecules and related forms of carbon such as nanotubes has generated an explosion of activity in chemistry, physics, and materials science. Classical fullerene is an all-carbon molecule in which the atoms are arranged on a pseudospherical framework made up entirely of pentagons and hexagons. Its molecular graph is a finite trivalent graph embedded on the surface of a sphere with only hexagonal and (exactly 12) pentagonal faces. Deza et al. [6] considered fullerene's extension to other closed surfaces and showed that only four surfaces are possible, namely sphere, torus, Klein bottle and projective plane. Unlike spherical fullerenes, toroidal and Klein bottle's fullerenes have been regarded as tessellations of entire hexagons on their surfaces since they must contain no pentagons, see [6, 12].

Let L be a regular hexagonal lattice and let P_m^n be an $m \times n$ quadrilateral section (with $m \geq 2$ hexagons on the top and bottom sides and $n \geq 2$ hexagons on the lateral sides, n is even) cut from the regular hexagonal lattice L, (see Figure 1).

If we identify two lateral sides of P_m^n then we form a cylinder. If we identify the top and bottom sides of the cylinder such that we identify the vertices u_i^0 and u_i^n, and the vertices v_i^0 and v_i^n, for $i = 1, 2, \dots, m$, we are able to obtain the *toroidal fullerene* (toroidal polyhex) $\mathbb{H}_m^n$ with mn hexagons. We can see that the toroidal fullerene is a cubic bipartite graph embedded on the torus such that each face is a hexagon. If we identify the top and bottom sides of the cylinder in such

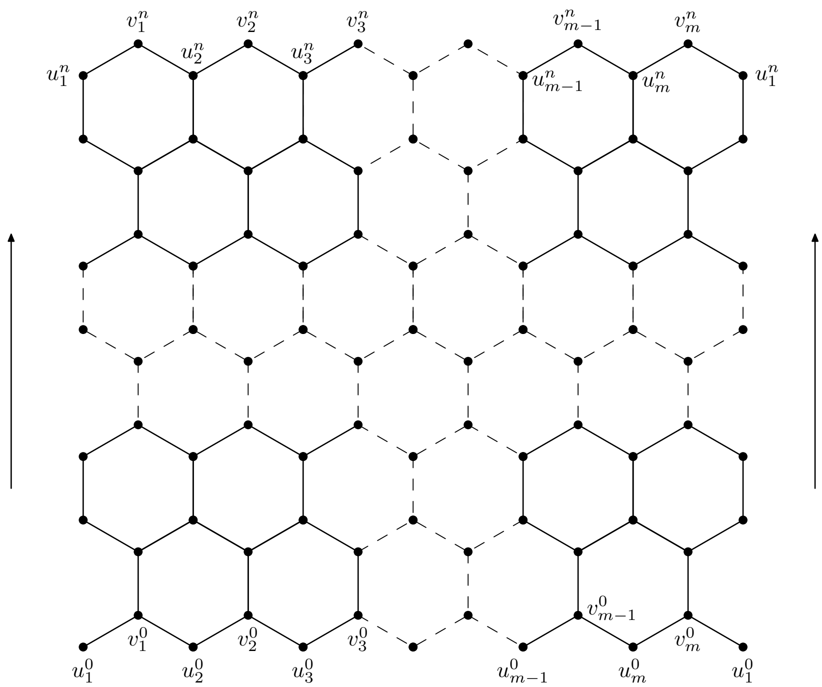

Figure 1. Quadrilateral section P_m^n cut from the regular hexagonal lattice L.

a way that we identify the vertices u_1^0 and u_1^n, the vertices u_i^0 and u_{m+2-i}^n, for $i = 2, 3, \ldots, m$, and the vertices v_i^0 and v_{m+1-i}^n, for $i = 1, 2, \ldots, m$, we obtain the *Klein-bottle fullerene* (Klein-bottle polyhex) $\mathbb{KB}_m^n$ with mn hexagons. In this case $\mathbb{KB}_m^n$ is a cubic bipartite graph of order $2mn$ and size $3mn$ embedded on the Klein-bottle and contains only hexagons.

2. Product cordial labeling of toroidal polyhex $\mathbb{H}_m^n$

Under an *open edge* we mean an edge with only one end vertex. For a graph containing one or more open edges we use the notation a *segment*. By the symbol $\oplus_v$ we mean an operation of gluing two segments/graphs in the vertical direction. Analogously, the symbol $\oplus_h$ is used for an operation of gluing two segments/graphs in the horizontal direction. Under the operation a *gluing* of a segment and a graph/segment we mean that we attach the open edge (edges) of a segment to a vertex of graph/segment. Note, that gluing of a segment and a graph results either to a graph or to a segment while when gluing two segments we always obtain a segment.

The next theorem shows that the toroidal polyhex $\mathbb{H}_3^n$, n even, admits a 3-total edge product cordial labeling.

Theorem 2.1. *For n even, $n \geq 4$, the toroidal polyhex $\mathbb{H}_3^n$ is 3-total edge product cordial.*

Proof. For obtaining the toroidal polyhex $\mathbb{H}_3^n$ we will use the labeled segments A_3^2, B_3^2 and C_3^2 illustrated in Figure 2. We can see that each labeled segment has the same number of 0, 1 and 2, namely in the segment A_3^2 every number of zeros, ones and twos is used 14 times, in the segment B_3^2 every number is used 10 times and in the segment C_3^2 every number is used 6 times as a label.

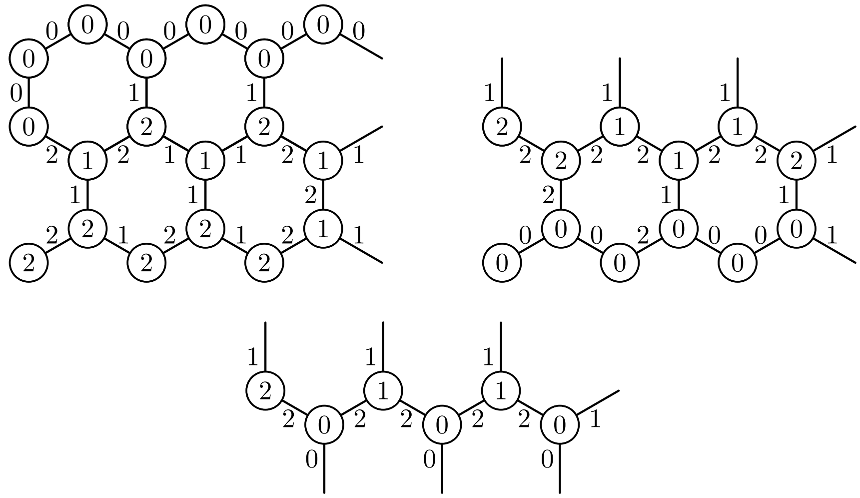

Figure 2. The labeled segments A_3^2, B_3^2 and C_3^2.

First we glue $(n/2-2)$ segments B_3^2 together in the vertical direction. Since the open edges in the segment B_3^2 are labeled with number 1 it follows that by gluing these segments we do not change the vertex labels in the segment $B_3^2 \oplus_v B_3^2 \oplus_v \cdots \oplus_v B_3^2 = (n/2-2)B_3^2$. Then we glue in the vertical direction the segment A_3^2 to the segment $(n/2-2)B_3^2$ to obtain $A_3^2 \oplus_v (n/2-2)B_3^2$. Finally we glue the segment $A_3^2 \oplus_v (n/2-2)B_3^2$ in the vertical direction to the segment C_3^2. All open edges used in the gluing operations are labeled with the number 1 therefore these operations do not have any impact to the vertex labels in the resulting segment H_3^n, where

$$H_3^n = \begin{bmatrix} A_3^2 \\ \oplus_v \\ (\frac{n}{2}-2)B_3^2 \\ \oplus_v \\ C_3^2 \end{bmatrix}.$$

Now we identify two lateral sides of the segment H_3^n to form a cylinder or nanotube. Then we identify the top and bottom sides of the cylinder such that we join the open edges in the bottom side of the cylinder labeled by 0 to the corresponding vertices in the top side of the cylinder (labeled by 0) and we obtain the toroidal polyhex $\mathbb{H}_3^n$ with $3n$ hexagons. Table 1 shows multiplicity of numbers 0, 1 and 2 used in the segments A_3^2, B_3^2, C_3^2 and H_3^n.

It is only routine checking that the toroidal polyhex $\mathbb{H}_3^n$ contains every number of 0, 1 and 2 exactly $5n$ times. □

Next we extend the construction of the segment H_3^n for arbitrary m to obtain the segment H_m^n and then to obtain the graph of the toroidal polyhex $\mathbb{H}_m^n$.

Theorem 2.2. *For n even, $n \geq 4$ and $m \geq 3$ the toroidal polyhex $\mathbb{H}_m^n$ is 3-total edge product cordial.*

segment	$e_\varphi(0)+v_{\varphi^*}(0)$	$e_\varphi(1)+v_{\varphi^*}(1)$	$e_\varphi(2)+v_{\varphi^*}(2)$
A_3^2	14	14	14
B_3^2	10	10	10
C_3^2	6	6	6
$(\frac{n}{2}-2)B_3^2$	$5n-20$	$5n-20$	$5n-20$
$H_3^n = A_3^2 \oplus_v (\frac{n}{2}-2)B_3^2 \oplus_v C_3^2$	$5n$	$5n$	$5n$

Table 1. Multiplicity of 0s, of 1s and of 2s in the segments A_3^2, B_3^2, C_3^2, $(n/2-2)B_3^2$ and H_3^n.

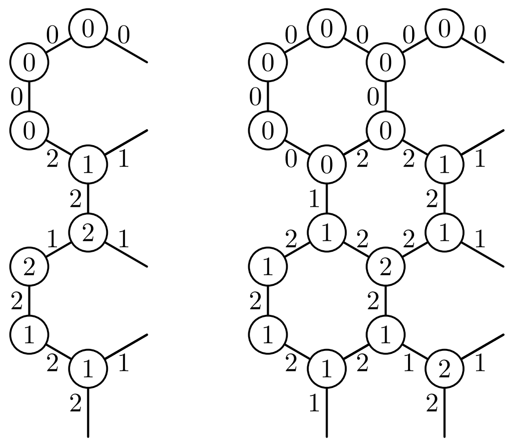

Figure 3. The labeled segments D_1^4 and D_2^4.

Proof. For obtaining a 3-total edge product cordial labeling of $\mathbb{H}_m^4$ we need the next labeled segments D_1^4 and D_2^4 depicted in Figure 3.

For $m=3t$, $t\geq 1$, we glue the segment A_3^2 in the vertical direction to the segment C_3^2 and the resulting segment $A_3^2 \oplus_v C_3^2$ we glue in the horizontal direction t times. Hence

$$H_m^4 = \underbrace{\begin{bmatrix} A_3^2 \\ \oplus_v \\ C_3^2 \end{bmatrix} \oplus_h \begin{bmatrix} A_3^2 \\ \oplus_v \\ C_3^2 \end{bmatrix} \oplus_h \cdots \oplus_h \begin{bmatrix} A_3^2 \\ \oplus_v \\ C_3^2 \end{bmatrix}}_{t}.$$

For $m=3t+1$, $t\geq 1$, we glue the segment $[A_3^2 \oplus_v C_3^2]$ horizontally t times and moreover, we glue horizontally the segment D_1^4 and we obtain

$$H_m^4 = \underbrace{\begin{bmatrix} A_3^2 \\ \oplus_v \\ C_3^2 \end{bmatrix} \oplus_h \begin{bmatrix} A_3^2 \\ \oplus_v \\ C_3^2 \end{bmatrix} \oplus_h \cdots \oplus_h \begin{bmatrix} A_3^2 \\ \oplus_v \\ C_3^2 \end{bmatrix}}_{t} \oplus_h D_1^4.$$

For $m = 3t + 2$, $t \geq 1$, we glue the segment $[A_3^2 \oplus_v C_3^2]$ horizontally t times and moreover, we glue horizontally the segment D_2^4 and we create the segment

$$H_m^4 = \underbrace{\begin{bmatrix} A_3^2 \\ \oplus_v \\ C_3^2 \end{bmatrix} \oplus_h \begin{bmatrix} A_3^2 \\ \oplus_v \\ C_3^2 \end{bmatrix} \oplus_h \cdots \oplus_h \begin{bmatrix} A_3^2 \\ \oplus_v \\ C_3^2 \end{bmatrix}}_{t} \oplus_h D_2^4.$$

Table 2 shows how many times the numbers 0, 1 and 2 are used as edge and vertex labels in the segments D_1^4 and D_2^4 and in the resulting segment H_m^4.

segment	$e_\varphi(0) + v_{\varphi^*}(0)$	$e_\varphi(1) + v_{\varphi^*}(1)$	$e_\varphi(2) + v_{\varphi^*}(2)$
D_1^4	6	7	7
D_2^4	14	13	13
H_m^4, $m = 3t$, $t \geq 1$	$20t$	$20t$	$20t$
H_m^4, $m = 3t + 1$, $t \geq 1$	$20t + 6$	$20t + 7$	$20t + 7$
H_m^4, $m = 3t + 2$, $t \geq 1$	$20t + 14$	$20t + 13$	$20t + 13$

Table 2. Multiplicity of 0s, of 1s and of 2s in the segments D_1^4, D_2^4 and H_m^4.

One can see that the resulting segments H_m^4 in each previous case satisfies the property of having a 3-total edge product cordial labeling. We identify two lateral sides of the segment H_m^4 to form a cylinder and then we identify the top and bottom sides of the cylinder such that we join the open edges in the bottom side of the cylinder to the corresponding vertices in the top side of the cylinder labeled by 0. Since the open edges in the corresponding segments are labeled with number 1 (except for open edges incident to vertices in the top side) it follows that gluing these segments does not have any impact on the vertex labels in the resulting toroidal polyhex $\mathbb{H}_m^4$.

The graph $\mathbb{H}_m^n$ with a 3-total edge product cordial labeling for n even, $n \geq 6$, we obtain by using the segment $H_3^n = A_3^2 \oplus_v (n/2-2) B_3^2 \oplus_v C_3^2$ from Theorem 2.1 such that the certain multiple of the segment H_3^n we glue in horizontal direction with a special segment which depends on n and m.

For next operations we need new segments A_1^4, C_2^4, C_1^2 and C_1^4 depicted in Figure 4 and the segments B_1^6, C_1^6 and C_2^6 shown in Figure 5.

Table 3 gives multiplicity of numbers 0, 1 and 2 used in the segments A_1^4, C_2^4, C_1^2, C_1^4, B_1^6, C_1^6 and C_2^6.

Let us consider the following 7 cases.

Case 1. For $m = 3t$, $t \geq 1$, and every n even, $n \geq 6$, we get

$$H_m^n = \underbrace{H_3^n \oplus_h H_3^n \oplus_h \cdots \oplus_h H_3^n}_{t}.$$

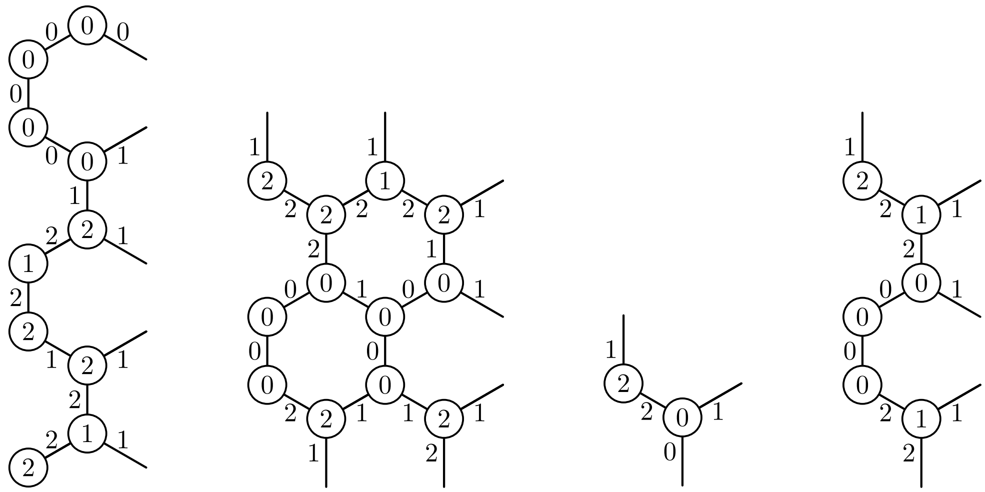

Figure 4. The labeled segments A_1^4, C_2^4, C_1^2 and C_1^4.

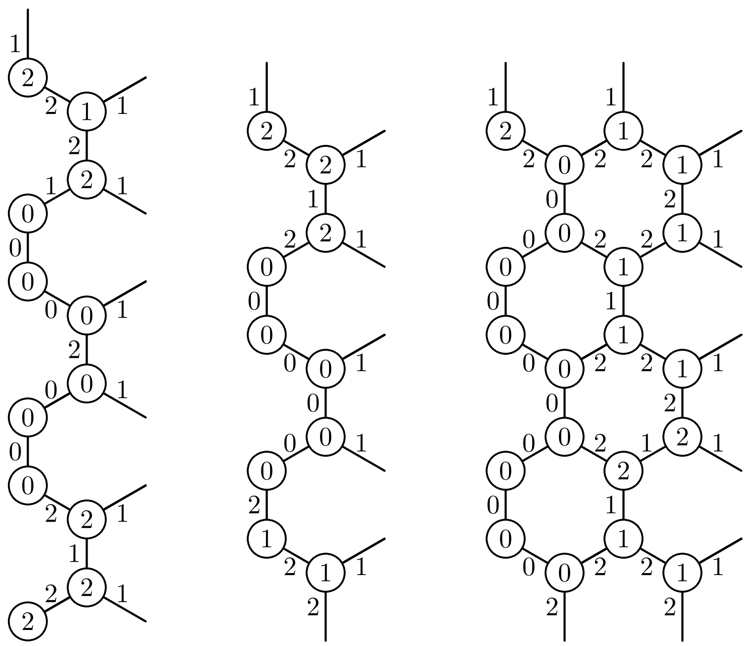

Figure 5. The labeled segments B_1^6, C_1^6 and C_2^6.

Case 2. For $m = 3t + 1$, $t \geq 1$, and $n = 6s$, $s \geq 1$, the special segment we obtain by gluing vertically the segment A_1^4 with vertically $(s-1)$ times of the segment B_1^6 and the resulting segment $A_1^4 \oplus_v (s-1)B_1^6$ we glue in the vertical direction with the segment C_1^2. Then we have

$$H_m^n = \left[\underbrace{H_3^n \oplus_h H_3^n \oplus_h \cdots \oplus_h H_3^n}_{t} \right] \oplus_h \begin{bmatrix} A_1^4 \\ \oplus_v \\ (s-1)B_1^6 \\ \oplus_v \\ C_1^2 \end{bmatrix}.$$

segment	$e_\varphi(0) + v_{\varphi^*}(0)$	$e_\varphi(1) + v_{\varphi^*}(1)$	$e_\varphi(2) + v_{\varphi^*}(2)$
A_1^4	8	8	8
C_2^4	10	11	11
C_1^2	2	2	2
C_1^4	5	6	5
B_1^6	10	10	10
C_1^6	9	9	8
C_2^6	17	18	17

Table 3. Multiplicity of 0s, of 1s and of 2s in the segments A_1^4, C_2^4, C_1^2, C_1^4, B_1^6, C_1^6 and C_2^6.

Case 3. For $m = 3t + 2$, $t \geq 1$, and $n = 6s$, $s \geq 1$, the segment H_m^n we obtain as follows

$$H_m^n = \left[\underbrace{H_3^n \oplus_h H_3^n \oplus_h \cdots \oplus_h H_3^n}_{t}\right] \oplus_h \begin{bmatrix} A_1^4 \\ \oplus_v \\ (s-1)B_1^6 \\ \oplus_v \\ C_1^2 \end{bmatrix} \oplus_h \begin{bmatrix} A_1^4 \\ \oplus_v \\ (s-1)B_1^6 \\ \oplus_v \\ C_1^2 \end{bmatrix}.$$

Case 4. For $m = 3t + 1$, $t \geq 1$, and $n = 6s + 2$, $s \geq 1$, the special segment we create by gluing the segment A_1^4 in the vertical direction with vertically $(s-1)$ times of the segment B_1^6 and the resulting segment $A_1^4 \oplus_v (s-1)B_1^6$ we glue in the vertical direction with the segment C_1^4. Then we get

$$H_m^n = \left[\underbrace{H_3^n \oplus_h H_3^n \oplus_h \cdots \oplus_h H_3^n}_{t}\right] \oplus_h \begin{bmatrix} A_1^4 \\ \oplus_v \\ (s-1)B_1^6 \\ \oplus_v \\ C_1^4 \end{bmatrix}.$$

Case 5. For $m = 3t + 2$, $t \geq 1$, and $n = 6s + 2$, $s \geq 1$, the special segment we obtain by gluing the segment $[A_1^4 \oplus_h A_1^4]$ in the vertical direction with vertically $(s-1)$ times of the segment $[B_1^6 \oplus_h B_1^6]$ and the resulting segment $[A_1^4 \oplus_h A_1^4] \oplus_v (s-1)[B_1^6 \oplus_h B_1^6]$ we glue in the vertical direction with the segment C_2^4. Then we have

$$H_m^n = \left[\underbrace{H_3^n \oplus_h H_3^n \oplus_h \cdots \oplus_h H_3^n}_{t}\right] \oplus_h \begin{bmatrix} [A_1^4 \oplus_h A_1^4] \\ \oplus_v \\ (s-1)[B_1^6 \oplus_h B_1^6] \\ \oplus_v \\ C_2^4 \end{bmatrix}.$$

Case 6. For $m = 3t + 1$, $t \geq 1$, and $n = 6s + 4$, $s \geq 1$, the special segment we generate by gluing the segment A_1^4 in the vertical direction with vertically $(s-1)$ times of the segment B_1^6 and the resulting segment $A_1^4 \oplus_v (s-1)B_1^6$ we glue in the vertical direction with the segment C_1^6. The expected resulting segment will be as follows

$$H_m^n = \left[\underbrace{H_3^n \oplus_h H_3^n \oplus_h \cdots \oplus_h H_3^n}_{t} \right] \oplus_h \begin{bmatrix} A_1^4 \\ \oplus_v \\ (s-1)B_1^6 \\ \oplus_v \\ C_1^6 \end{bmatrix}.$$

Case 7. For $m = 3t+2$, $t \geq 1$, and $n = 6s+4$, $s \geq 1$, the special segment we create by gluing the segment $[A_1^4 \oplus_h A_1^4]$ in the vertical direction with vertically $(s-1)$ times of the segment $[B_1^6 \oplus_h B_1^6]$ and the resulting segment $[A_1^4 \oplus_h A_1^4] \oplus_v (s-1)[B_1^6 \oplus_h B_1^6]$ we glue in the vertical direction with the segment C_2^6. The required segment has the following form

$$H_m^n = \left[\underbrace{H_3^n \oplus_h H_3^n \oplus_h \cdots \oplus_h H_3^n}_{t} \right] \oplus_h \begin{bmatrix} [A_1^4 \oplus_h A_1^4] \\ \oplus_v \\ (s-1)[B_1^6 \oplus_h B_1^6] \\ \oplus_v \\ C_2^6 \end{bmatrix}.$$

All possible cases for obtaining the segment H_m^n for n even, $n \geq 6$ and $m \geq 3$, are described in Table 4, where it is shown how many times the numbers 0, 1 and 2 are used as edge and vertex labels.

Identifying two lateral sides of the segment H_m^n we form a cylinder and then identifying the top and bottom sides of the cylinder such that the open edges in the bottom side of the cylinder are joined to the corresponding vertices in the top side of the cylinder, we obtain the toroidal polyhex $\mathbb{H}_m^n$. We can see that the labels of the open edges used to glue the corresponding segments has no effect on the vertex labels in the resulting segments/graphs. From Table 4 it follows that for every case the resulting toroidal polyhex $\mathbb{H}_m^n$ is 3-total edge product cordial.

□

A *helical torus* decorated with graphite can be formed by joining the opposite ends of a chiral or helical nanotube, such that the cycle of one end of nanotube is rotated relative to the cycle of the second end of nanotube, see [16]. More precisely, we suppose that a nanotube is our cylinder created identifying two lateral sides of P_m^n in Figure 1. To create the helical torus we identify the top and bottom sides of the cylinder/nanotube such that we identify the vertices u_i^0 and u_{i+j}^n and the vertices v_i^0 and v_{i+j}^n, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m-1$, where $i + j$ is taken modulo m.

Since in the proof of Theorem 2.2 the segment H_m^n, for n even, $n \geq 4$ and $m \geq 3$, contains in the top side only the vertices labeled by 0, it follows that any gluing of the open edges in the bottom side of H_m^n to the vertices in the top side is possible and does not have any impact on the vertex labels in the resulting graph of the helical torus $\mathbb{H}_m^n$. It means that the gluing operation satisfies the property of having a 3-total edge product cordial labeling. Hence we have the following theorem.

Theorem 2.3. *For n even, $n \geq 4$ and $m \geq 3$, the helical torus $\mathbb{H}_m^n$ admits a 3-total edge product cordial labeling.*

Case	segment H_m^n	$e_\varphi(0) + v_{\varphi^*}(0)$	$e_\varphi(1) + v_{\varphi^*}(1)$	$e_\varphi(2) + v_{\varphi^*}(2)$
1.	$m = 3t, t \geq 1$ $n \geq 6$, n even	$5nt$	$5nt$	$5nt$
2.	$m = 3t+1, t \geq 1$ $n = 6s, s \geq 1$	$5nt + 10s$	$5nt + 10s$	$5nt + 10s$
3.	$m = 3t+2, t \geq 1$ $n = 6s, s \geq 1$	$5nt + 20s$	$5nt + 20s$	$5nt + 20s$
4.	$m = 3t+1, t \geq 1$ $n = 6s+2, s \geq 1$	$5nt + 10s + 3$	$5nt + 10s + 4$	$5nt + 10s + 3$
5.	$m = 3t+2, t \geq 1$ $n = 6s+2, s \geq 1$	$5nt + 20s + 6$	$5nt + 20s + 7$	$5nt + 20s + 7$
6.	$m = 3t+1, t \geq 1$ $n = 6s+4, s \geq 1$	$5nt + 10s + 7$	$5nt + 10s + 7$	$5nt + 10s + 6$
7.	$m = 3t+2, t \geq 1$ $n = 6s+4, s \geq 1$	$5nt + 20s + 13$	$5nt + 20s + 14$	$5nt + 20s + 13$

Table 4. Multiplicity of 0s, of 1s and of 2s in the segment H_m^n for n even, $n \geq 6$ and $m \geq 3$.

By joining the opposite ends of a nanotube we can obtain the *Klein-bottle polyhex*. More precisely, to create the Klein-bottle polyhex $\mathbb{KB}_m^n$ we identify the top and bottom sides of the cylinder/nanotube (see Figure 1) such that we identify the vertices u_1^0 and u_1^n, the vertices u_i^0 and u_{m+2-i}^n, for $i = 2, 3, \ldots, m$, and the vertices v_i^0 and v_{m+1-i}^n, for $i = 1, 2, \ldots, m$.

With respect to the fact that the gluing operation does not have any impact on the vertex labels in the resulting graph of the Klein-bottle polyhex $\mathbb{KB}_m^n$, therefore the property of admitting a 3-total edge product cordial labeling also holds. Consequently we get the following theorem.

Theorem 2.4. *For n even, $n \geq 4$ and $m \geq 3$, the Klein-bottle polyhex $\mathbb{KB}_m^n$ admits a 3-total edge product cordial labeling.*

3. Conclusion

In this paper we proved the existence of the 3-total edge product cordial labeling for the toroidal polyhex, respectively helical torus, $\mathbb{H}_m^n$, and for the Klein-bottle polyhex $\mathbb{KB}_m^n$, for n even, $n \geq 4$ and $m \geq 3$. In this case the Klein-bottle polyhex $\mathbb{KB}_m^n$ is a cubic bipartite graph. The next construction describes the existence of the Klein-bottle polyhex $\mathbb{KB}_{m+1/2}^n$ as a cubic non-bipartite graph.

Let L be a regular hexagonal lattice and let $P^n_{m+1/2}$ be a quadrilateral section (with $m + 1/2$ hexagons on the top and bottom sides, $m \geq 1$, and $n \geq 2$ hexagons on the lateral sides, n is even) cut from the regular hexagonal lattice L, (see Figure 6).

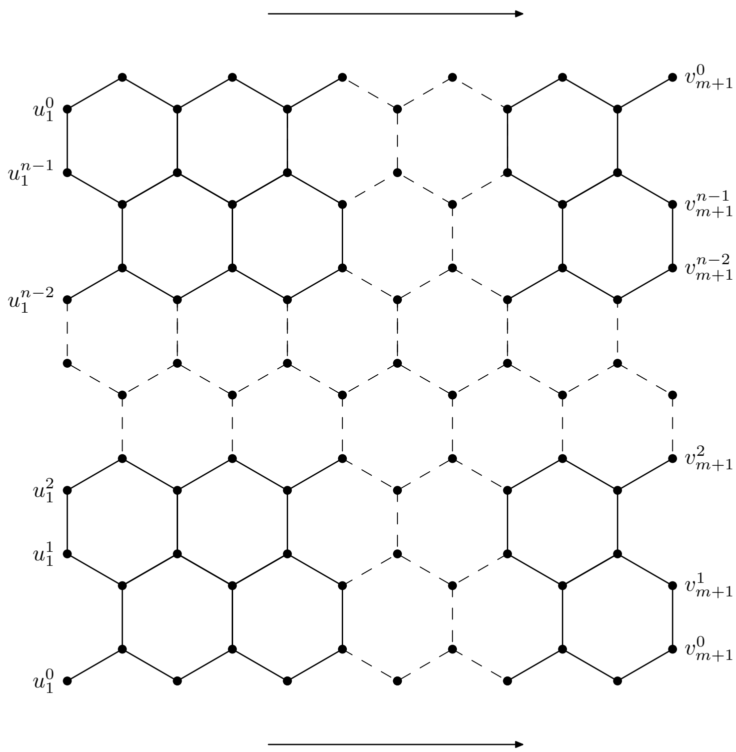

Figure 6. Quadrilateral section $P^n_{m+1/2}$ cut from the regular hexagonal lattice L.

We identify the top and bottom sides of $P^n_{m+1/2}$ to form a cylinder. Then we identify the lateral sides of the cylinder such that we identify the vertices u_1^0 and v_{m+1}^0, and the vertices u_1^j and v_{m+1}^{n-j}, for $j = 1, 2, \dots, n-1$, to obtain the Klein-bottle polyhex $\mathbb{KB}^n_{m+1/2}$. We can see that $\mathbb{KB}^n_{m+1/2}$ is a cubic non-bipartite graph of order $2n(m + 1/2)$ and size $3n(m + 1/2)$ embedded on the Klein-bottle and contains $n(m + 1/2)$ hexagons.

Let us suggest the following open problem.

Problem 1. *Find a* 3*-total edge product cordial labeling for the Klein-bottle polyhex* $\mathbb{KB}^n_{m+1/2}$*, for* n *even,* $n \geq 2$ *and* $m \geq 3$*.*

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