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Some families of graphs with no nonzero real domination roots

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Abstract

Let G be a simple graph of order n. The domination polynomial of G is the polynomial $D(G,x) = \sum_{i=\gamma(G)}^{n} d(G,i)x^i$, where $d(G,i)$ is the number of dominating sets of G of size i and $\gamma(G)$ is the domination number of G. A root of $D(G,x)$ is called a domination root of G. Obviously, 0 is a domination root of every graph G with multiplicity $\gamma(G)$. In the study of the domination roots of graphs, this naturally raises the question: Which graphs have no nonzero real domination roots? In this paper we present some families of graphs whose have this property.

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1. Introduction

All graphs in this paper are simple of finite orders, i.e., graphs are undirected with no loops or parallel edges and with finite number of vertices. Let $G = (V, E)$ be a simple graph. For any vertex $v \in V(G)$, the open neighborhood of v is the set $N(v) = \{u \in V(G) | uv \in E(G)\}$ and the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the open neighborhood of S is $N(S) = \bigcup_{v\in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. The complement G^c of a graph G is a graph with the same vertex set as G and with the property that two vertices are adjacent in G^c if and only if they are not adjacent in G. A set $S \subseteq V(G)$ is a dominating

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set if $N[S] = V$ or equivalently, every vertex in $V(G)\backslash S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. For a detailed treatment of domination theory, the reader is referred to [22].

Let $\mathcal{D}(G, i)$ be the family of dominating sets of a graph G with cardinality i and let $d(G, i) = |\mathcal{D}(G, i)|$. The domination polynomial $D(G, x)$ of G is defined as $D(G, x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G, i)x^i$ (see [2, 6]). This polynomial is the generating polynomial for the number of dominating sets of each cardinality. Similar to generating polynomials for other combinatorial sequences, such as independents sets in a graph [9, 11, 13, 16, 19, 20, 21], they have attracted recent attention to name but a few references. The algebraic encoding of salient counting sequences allows one to not only develop formulas more easily, but also, often, to prove unimodality results via the nature of the the roots of the associated polynomials (a well known result of Newton states that if a real polynomial with positive coefficients has all real roots, then the coefficients form a unimodal sequence, see, for examples, [14] and [30]). A root of $D(G, x)$ is called a *domination root* of G (see [12]). The set of roots of $D(G, x)$ is denoted by $Z(D(G, x))$. It is known that -1 is not a domination root as the number of dominating sets in a graph is always odd [8]. On the other hand, of course, 0 is a domination root of every graph G with multiplicity $\gamma(G)$. The existing research on the roots of domination polynomials has been restricted to those graphs with exactly two, three or exactly four domination roots [2, 4]. Also in [12] Brown and Tufts studied the location of the roots of domination polynomials for some families of graphs such as bipartite cocktail party graphs and complete bipartite graphs. In particular, they showed that the set of all domination roots is dense in the complex plane. For some very recent developments on domination roots see [28]. Let G be a graph of order n and minimum degree δ. Oboudi in [28] showed that all roots of $D(G, x)$ lie in the set $\{z : |z + 1| \leq \sqrt[\delta+1]{2^n - 1}\}$ and $D(G, x)$ has at least $\delta - 1$ non-real roots.

In the study of the domination roots of graphs, this naturally raises the question: Which graphs have no nonzero real domination roots? In this paper we would like to present some families of graphs with this property. Let $\mathcal{G}$ be the family of all simple finite graphs. We define the subfamily graphs $\mathcal{CG}$ by $\mathcal{CG} = \{G \in \mathcal{G} | Z(D(G, x)) \subseteq \mathbb{C} \setminus \mathbb{R}\}$.

In the next section, we present some families of graphs which are in $\mathcal{CG}$. In Section 3 we consider the complement of the friendship graphs, F_n^c and compute their domination polynomials, exploring the nature and location of their roots. As a consequence we show that $F_n^c \in \mathcal{CG}$.

2. Some families of graphs in $\mathcal{CG}$

In the beginning of the study of domination roots of graphs, one can see that there are graphs with no nonzero real domination roots. As examples, the complete graph K_n for odd n and the complete bipartite graph $K_{n,n}$ for even n, are in $\mathcal{CG}$. With these motivations, in [1] the authors asked the question: "Which graphs have no nonzero real domination roots?" In other words, which graphs lie in $\mathcal{CG}$?.

In this section we use the existing results on domination polynomials to find some families of graphs that belong to $\mathcal{CG}$. We need some preliminaries.

The *join* $G = G_1 + G_2$ of two graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2. The following theorem gives a formula for the domination polynomial of a join of two graphs.

Theorem 2.1. [2] *Let G and H be nonempty graphs of order n and m, respectively. Then,*

$$D(G+H,x) = ((1+x)^n - 1)((1+x)^m - 1) + D(G,x) + D(H,x).$$

For two graphs $G = (V,E)$ and $H = (W,F)$, the *corona* $G \circ H$ is the graph arising from the disjoint union of G with $|V|$ copies of H, by adding edges between the ith vertex of G and all vertices of ith copy of H [17]. We need the following theorem which is for computing the domination polynomial of the corona products of two graphs.

Theorem 2.2. [3] *Let $G = (V,E)$ and $H = (W,F)$ be nonempty graphs of order n and m, respectively. Then*

$$D(G \circ H, x) = (x(1+x)^m + D(H,x))^n.$$

Let K_k be a complete graph on k vertices and S be an independent set of $n-k$ vertices. A (k,n)-*star*, denoted by $S_{k,n-k}$, is defined as $S_{k,n-k} = K_k + S$. The *book graph* B_n can be constructed by bonding n copies of the cycle graph C_4 along a common edge $\{u,v\}$. In [5] it was proved that, for every $n \in \mathbb{N}$,

$$D(B_n,x) = (x^2+2x)^n(2x+1) + x^2(x+1)^{2n} - 2x^n.$$

The following theorem gives some families of graphs that are in $\mathcal{CG}$.

Theorem 2.3. *(i)* [5] *Every graph H in the family*

$$\{G \circ K_{2n}, (G \circ K_{2n}) \circ K_{2n}, ((G \circ K_{2n}) \circ K_{2n}) \circ K_{2n}, \cdots\}$$

lies in $\mathcal{CG}$.

(ii) [23] *For odd n and even k, the k-star $S_{k,n-k}$ is in $\mathcal{CG}$.*

(iii) [23] *For odd n and odd k, every graph H in the family*

$$\{G \circ S_{k,n-k}, (G \circ S_{k,n-k}) \circ S_{k,n-k}, ((G \circ S_{k,n-k}) \circ S_{k,n-k}) \circ S_{k,n-k}, \cdots\}$$

lies in $\mathcal{CG}$.

(iv) [5] Every graph H in the family $\{G \circ B_2, (G \circ B_2) \circ B_2, ((G \circ B_2) \circ B_2) \circ B_2, \cdots\}$ lies in $\mathcal{CG}$.

In [27], Levit and Mandrescu constructed a family of graphs H_n from the path P_n by the "clique cover construction", as shown in Figure 1. By H_0 we mean the null graph. To compute the domination polynomial of H_n, we need some preliminaries and well known results.

An *irrelevant* edge is an edge $e \in E(G)$, such that $D(G,x) = D(G-e,x)$, and a vertex $v \in V(G)$ is *domination-covered*, if every dominating set of $G - v$ includes at least one vertex adjacent to v in G [26].

Theorem 2.4. [26] *Let $G = (V,E)$ be a graph. A vertex $v \in V$ is domination-covered if and only if there is a vertex $u \in N[v]$ such that $N[u] \subseteq N[v]$.*

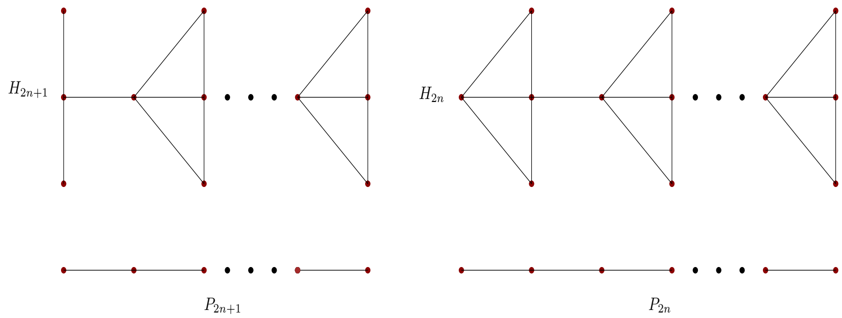

Figure 1. Graphs H_{2n+1} and H_{2n}, respectively.

Theorem 2.5. [26] *Let $G = (V, E)$ be a graph. An edge $e = \{u, v\} \in E$ is an irrelevant edge in G, if and only if u and v are domination-covered in $G - e$.*

The following theorem yields formula for the domination polynomials of H_n. Other families of graphs with the same domination polynomial have been studied in [25].

Theorem 2.6. *Let H_n be the graphs in Figure 1.*

(i) For every $n \in \mathbb{N}$, $D(H_{2n}, x) = (x^4 + 4x^3 + 6x^2 + 2x)^n$.
(ii) For every $n \in \mathbb{N}$, $D(H_{2n+1}, x) = (x^3 + 3x^2 + x)(x^4 + 4x^3 + 6x^2 + 2x)^n$.

Proof. (i) Let G be a graph of order 4 as in Figure 2 and $e_1, \ldots, e_n$ be the edges with end-vertices of degree 4, which connect each two G in H_{2n}. By Theorem 2.4, the two end-vertices of every edge e_i are domination-covered in H_{2n}, and so by Theorem 2.5, every edge e_i is an irrelevant edge of H_{2n}. Since $D(G, x) = x^4 + 4x^3 + 6x^2 + 2x$, by induction we have

$$D(H_{2n}, x) = (x^4 + 4x^3 + 6x^2 + 2x)^n.$$

(ii) Let e be an edge joining H_{2n} and the leftmost vertical P_3 in H_{2n+1}. By Theorem 2.4, the two end-vertices of edge e are domination-covered in H_{2n+1}. So, by Theorem 2.5, the edge e is an irrelevant edge of H_{2n+1}. So $D(H_{2n+1}, x) = D(P_3 \cup H_{2n}, x)$ and therefore by part (i) we have the result. □

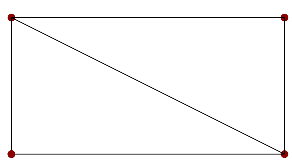

Figure 2. The graph in the proof of Theorem 2.6 (i)

Here using Theorem 2.6 we present other families of graphs in $\mathcal{CG}$.

Theorem 2.7. (i) *The graphs of the form $H_n + H_n$, $H_{n+1} + B_n$, for $n \geq 3$, and the graphs of the form $B_n + B_n$, for odd n are in $\mathcal{CG}$.*

(ii) *The graphs of the form $B_{n+1} + B_n$, for even n, and $B_{n+1} + H_n$, for $n \geq 4$ are in $\mathcal{CG}$.*

Proof. Since the coefficients of domination polynomials are positive integers, we investigate real domination roots for $x \leq 0$.

(i) By Theorem 2.1, we can deduce that for each natural number $n \geq 3$,

$$D(H_n + H_n, x) = ((1+x)^{|V(H_n)|} - 1)^2 + 2D(H_n, x).$$

To obtain the domination roots of $H_n + H_n$, we shall solve the following equation:

$$((1+x)^{|V(H_n)|} - 1)^2 = -2D(H_n, x). \quad (1)$$

We consider two cases, and show that in each there is no nonzero solution.

- If $n \geq 3$ is even, i.e., $n = 2k$ for some natural $k \geq 2$, then the equation (1) is equivalent to the following equation

$$\begin{aligned} ((1+x)^{4k} - 1)^2 &= -2(x^4 + 4x^3 + 6x^2 + 2x)^k \\ &= -2((1+x)^4 - 2x - 1)^k. \end{aligned} \quad (2)$$

  For $x \leq 0$ and even k, the equation (2) is true just for real number $x = 0$, because for nonzero real number x, the left side of the equality (2) is positive and the right side is negative. To investigate other cases, we draw the diagram of both sides of the equation (2) for $k = 1$ in Figure 3 and consider the two points -0.2133340651 and -0.4563109873. Observe that for $x \leq -0.4563109873$ and odd k, the left side of equality (2) is positive but the right side is negative. If $k \geq 3$ is odd and $-0.4563109873 < x \leq 0$, then the left side of equality (2) is greater than the right side. If $k = 1$ and $-0.2133340651 < x \leq 0$, then the right side of equality (2) is greater than the left side, and the left side of equality (2) is greater than the right side when $-0.4563109873 < x \leq -0.2133340651$.

- If $n \geq 3$ is odd, i.e., $n = 2k+1$, for some $k \in \mathbb{N}$, then the equation (1) is equivalent to the following equation

$$\begin{aligned} ((1+x)^{4k+3} - 1)^2 &= -2(x^3 + 3x^2 + x)(x^4 + 4x^3 + 6x^2 + 2x)^k \\ &= -2((1+x)^3 - 2x - 1)((1+x)^4 - 2x - 1)^k. \end{aligned} \quad (3)$$

  We consider the following different cases, and show in each there is no nonzero real solution. If $x \leq -1$, there are no real solutions x, because for $-2 \leq x \leq -1$, the left side of equation (3) is positive but its right side is negative. Also for $x < -2$, the left side of equality (3) is greater than the right side. Assuming that $-1 < x < 0$.

  (a) If k is even and $-\frac{1}{2} \leq x < 0$, the left side of equality (3) is greater than the right side, a contradiction.
  (b) If k is odd and $-\frac{1}{2} \leq x < 0$, the left side of equality (3) is positive but the right side is negative, a contradiction.
  (c) For every k and $-1 < x < -\frac{1}{2}$, there are no real solutions x, because the left side of equality (3) is positive but the right side is negative.

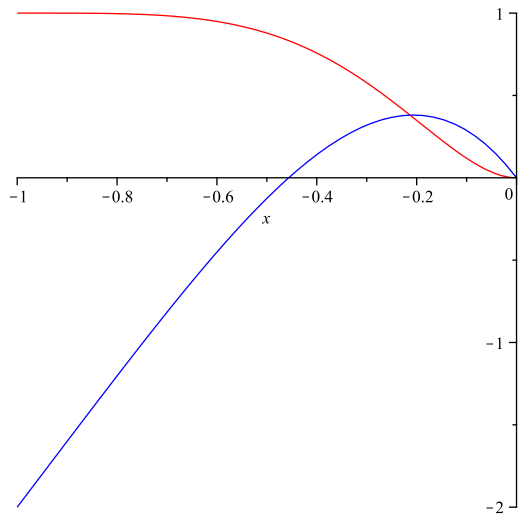

Figure 3. $y = ((1+x)^4 - 1)^2$ (red curve) and $y = -2((1+x)^4 - 2x - 1)$ (blue curve).

To obtain the domination roots of $B_n + B_n$ for each odd natural number n similar to first case, we shall solve the following equation:

$$((1+x)^{2n+2} - 1)^2 = -2\big((x^2+2x)^n(2x+1) + x^2(x+1)^{2n} - 2x^n\big). \qquad (4)$$

We consider three cases and show that in each there is no nonzero real solution.
For $-\frac{1}{2} < x \leq 0$, the equation (4) is true just for real number $x = 0$, because for nonzero real number, the left side of equality (4) is greater than the right side. Now suppose that $-2 \leq x \leq -\frac{1}{2}$, the left side of equality (4) is positive but the right side is negative, a contradiction. Also for $x < -2$, the left side of equality (4) is positive but the right side is negative.
It rests to show that for $n \geq 3$ the graphs of the form $H_{n+1} + B_n$ are in $\mathcal{CG}$. To obtain the domination roots of $H_{n+1} + B_n$, we shall solve the following equation:

$$((1+x)^{|V(H_{n+1})|} - 1)((1+x)^{|V(B_n)|} - 1) = -D(H_{n+1}, x) - D(B_n, x). \qquad (5)$$

We consider two cases, and show that in each there is no nonzero solution.

- If $n \geq 3$ is even, then the equation (5) is equivalent to the following equation

$$((1+x)^{2n+3} - 1)((1+x)^{2n+2} - 1) = -\big((x^3+3x^2+x)((1+x)^4 - 2x - 1)^{\frac{n}{2}} + (x^2+2x)^n(2x+1) + x^2(x+1)^{2n} - 2x^n\big). \qquad (6)$$

  We consider the following different cases and show in each there is no nonzero real solution. For $-2 \leq x \leq 0$, the equation (6) is true just for real number $x = 0$, because for nonzero real number, the left side of the equality (6) is greater than the right side. Observe that for $x < -2$, the both sides of equality (6) are negative, but the right side of it is greater than the left side.

- If $n \geq 3$ is odd, then the equation (5) is equivalent to the following equation

$$((1+x)^{2n+2}-1)^2 = -\big(((1+x)^4-2x-1)^{\lfloor \frac{n}{2} \rfloor +1} + (x^2+2x)^n (2x+1) + x^2(x+1)^{2n} - 2x^n\big). \quad (7)$$

We consider the following different cases, and show in each there is no nonzero real solution. For $-\frac{1}{2} < x \leq 0$, the equation (7) is true just for real number $x = 0$, because for nonzero real number, the left side of the equality (7) is greater than the right side. Observe that for $x \leq -\frac{1}{2}$, the left side of equality (7) is positive but the right side is negative.

(ii) To obtain the domination roots of $B_{n+1} + B_n$ for each even natural number n similar to first case, we solve the following equation:

$$\begin{aligned}((1+x)^{2n+4}-1)((1+x)^{2n+2}-1) &= -D(B_{n+1},x) - D(B_n,x) \\ &= -\big((x^2+2x)^n(2x+1)(1+x)^2 \\ &\quad +x^2(x+1)^{2n}(1+(1+x)^2) \\ &\quad -2x^n(1+x)\big). \end{aligned} \quad (8)$$

We consider three cases, and show that in each there is no nonzero real solution.
For $-1 \leq x \leq 0$, the equation (8) is true just for real number $x = 0$, because for nonzero real number, the left side of equality (8) is greater than the right side. Also for $x < -2$, the left side of equality (8) is greater than the right side. Assuming that $-2 \leq x < -1$, the right side of equality (8) is greater than the left side.
To obtain the domination roots of $B_{n+1} + H_n$, we solve the following equation:

$$((1+x)^{|V(B_{n+1})|}-1)((1+x)^{|V(H_n)|}-1) = -D(B_{n+1},x) - D(H_n,x). \quad (9)$$

This case is similar to the last case in proof of Part (i). □

Domination roots of the graphs $H_n + H_n$ and $B_n + B_n$ in Theorem 2.7 (i), for $3 \leq n \leq 20$ are shown in Figure 4.

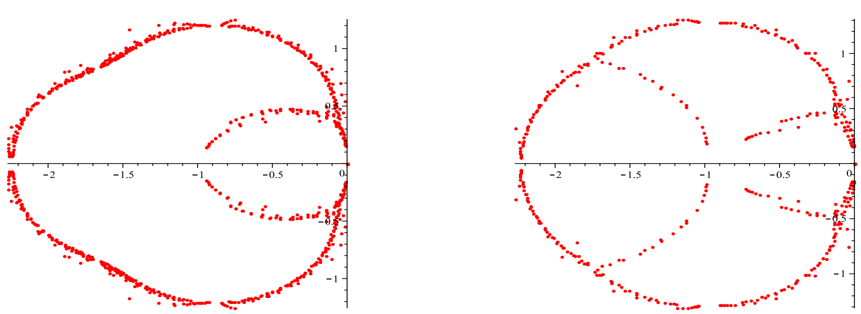

Figure 4. Domination roots of $H_n + H_n$ and $B_n + B_n$ in Theorem 2.7 (i), for $3 \leq n \leq 20$, respectively.

3. Domination roots of the complement of the friendship graphs

The friendship (or Dutch-Windmill) graph F_n is a graph that can be constructed by coalescing n copies of the cycle graph C_3 of length 3 with a common vertex. The Friendship Theorem of Paul Erdös, Alfred Rényi and Vera T. Sós [15], states that graphs with the property that every two vertices have exactly one neighbour in common are exactly the friendship graphs. Figure 5 shows some examples of friendship graphs. The following theorem states that for each odd n, the friendship graph F_n lie in $\mathcal{CG}$.

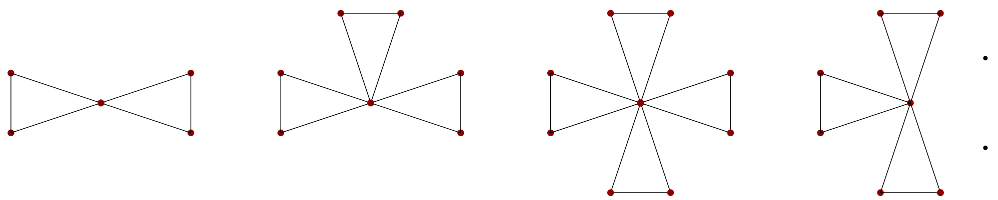

Figure 5. Friendship graphs F_2, F_3, F_4 and F_n, respectively.

Theorem 3.1. [5] (i) *For every* $n \in \mathbb{N}$, $D(F_n, x) = (2x + x^2)^n + x(1 + x)^{2n}$.
(ii) *For odd* n, $F_n \in \mathcal{CG}$.

The nature and location of domination roots of friendship graphs have been studied in [5]. It is natural to ask about the domination polynomial and the domination roots of the complement of the friendship graphs. The Turán graph $T(n, r)$ is a complete multipartite graph formed by partitioning a set of n vertices into r subsets, with sizes as equal as possible, and connecting two vertices by an edge whenever they belong to different subsets. The graph will have $(n \; mod \; r)$ subsets of size $\lceil \frac{n}{r} \rceil$, and $r - (n \; mod \; r)$ subsets of size $\lfloor \frac{n}{r} \rfloor$. That is, it is a complete r-partite graph

$$K_{\lceil \frac{n}{r} \rceil, \lceil \frac{n}{r} \rceil, \ldots, \lfloor \frac{n}{r} \rfloor, \lfloor \frac{n}{r} \rfloor}.$$

The Turán graph $T(2n, n)$ can be formed by removing a perfect matching composed by n edges no two of which are adjacent, from a complete graph K_{2n}. As Roberts (1969) showed, this graph has boxicity exactly n; it is sometimes known as the Roberts graph [29]. If n couples go to a party, and each person shakes hands with every person except his or her partner, then this graph describes the set of handshakes that take place; for this reason it is also called the cocktail party graph. So, the cocktail party graph $CP(t)$ of order $2t$ is the graph with vertices $b_1, b_2, \cdots, b_{2t}$ in which each pair of distinct vertices form an edge with the exception of the pairs $\{b_1, b_2\}, \{b_3, b_4\}, \ldots, \{b_{2t-1}, b_{2t}\}$. It is easy to check that the complement of the friendship graph F_n is $CP(n) \cup K_1$.

Theorem 3.2. *For every* $n \in \mathbb{N}$, $D(F_n^c, x) = \big((1 + x)^{2n} - (1 + 2nx)\big)x$.

Proof. An elementary observation is that if G_1 and G_2 are graphs of orders n_1 and n_2, respectively, then $D(G_1 \cup G_2, x) = D(G_1, x)D(G_2, x)$. Clearly $D(K_1, x) = x$ and there are no dominating sets of size 1 in $CP(n)$. So $D(CP(n), x) = (1 + x)^{2n} - (1 + 2nx)$ and we have the result. □

In [12] a family of graphs was produced with roots just barely in the right-half plane, showing that not all domination polynomials are stable (a polynomial f over the complex field is stable, if

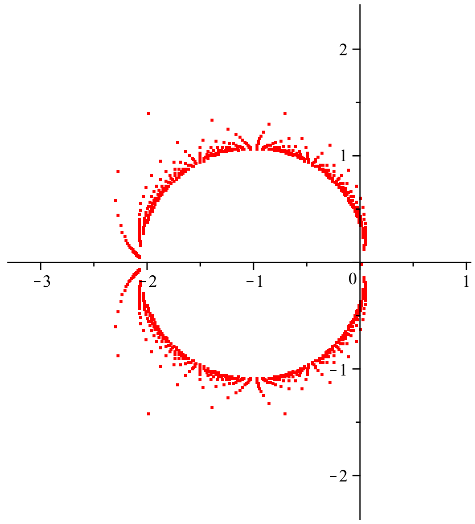

Figure 6. Domination roots of graphs F_n^c, for $1 \leq n \leq 30$.

and only if all of its zeros have non-positive real part), but Figure 6 provides an explicit family (namely the F_n^c) which domination roots have positive real part. The domination roots of the complements of the friendship graphs exhibit a number of interesting properties (see Figure 6). Even though we cannot find the roots explicitly, there is much we can say about them. Here we prove that for each natural number n, the complement of the friendship graphs F_n^c lie in $\mathcal{CG}$.

Theorem 3.3. *For every natural number n, the complement of the friendship graphs F_n^c lie in $\mathcal{CG}$.*

Proof. It suffices to show that for each natural n, the cocktail party graph $CP(n)$ is in $\mathcal{CG}$. By Theorem 3.2, for every $n \in \mathbb{N}$, $D(CP(n), x) = (1 + x)^{2n} - (1 + 2nx)$. If $D(CP(n), x) = 0$ then for $x \neq 0$, we have

$$(1 + x)^{2n} = 1 + 2nx.$$

We consider three cases, and show in each there is no nonzero solution.

- $x > 0$: Obviously the above equality is true just for real number $x = 0$, since for nonzero real number the left side of the equality is greater than the right side.
- $x \leq -1$: In this case the left side is greater than 0 and the right side $1 + 2nx$ is less than -1, a contradiction.
- $-1 < x < 0$: In this case obviously there are no real solutions x, since the left side of equality is greater than the right side.

Thus in any event, there are no nonzero real domination roots of the cocktail party graph. □

The plot in Figure 6 suggests that the roots tend to lie on a curve. In order to find the limiting curve, we will need a definition and a well known result.

Definition 3.4. *If $f_n(x)$ is a family of (complex) polynomials, we say that a number $z \in \mathbb{C}$ is a limit of roots of $f_n(x)$ if either $f_n(z) = 0$ for all sufficiently large n or z is a limit point of the set $\mathbb{R}(f_n(x))$, where $\mathbb{R}(f_n(x))$ is the union of the roots of the $f_n(x)$.*

The following restatement of the Beraha-Kahane-Weiss theorem [7] can be found in [10].

Theorem 3.5. *Suppose $f_n(x)$ is a family of polynomials such that*

$$f_n(x) = \alpha_1(x)\lambda_1(x)^n + \alpha_2(x)\lambda_2(x)^n + ... + \alpha_k(x)\lambda_k(x)^n \tag{10}$$

where the $\alpha_i(x)$ and the $\lambda_i(x)$ are fixed non-zero polynomials, such that for no pair $i \neq j$ is $\lambda_i(x) \equiv \omega\lambda_j(x)$ for some $\omega \in \mathbb{C}$ of unit modulus. Then $z \in \mathbb{C}$ is a limit of roots of $f_n(x)$ if and only if either

(i) two or more of the $\lambda_i(z)$ are of equal modulus, and strictly greater (in modulus) than the others; or

(ii) for some j, $\lambda_j(z)$ has modulus strictly greater than all the other $\lambda_i(z)$, and $\alpha_j(z) = 0$.

The following Theorem gives the limits of the domination roots of F_n^c.

Theorem 3.6. *The limit of domination roots of F_n^c is the unit circle with center -1.*

Proof. By Theorem 3.2, the domination polynomial of F_n^c is,

$$\begin{aligned} D(F_n^c, x) &= x((x+1)^2)^n - x(1+2nx) \\ &= \alpha_1(x)\lambda_1^n(x) + \alpha_2(x)\lambda_2^n(x), \end{aligned}$$

where

$$\alpha_1(x) = x, \ \ \lambda_1(x) = (x+1)^2,$$

and

$$\alpha_2(x) = x + 2nx^2, \ \ \lambda_2(x) = 1.$$

Clearly there is no $\omega \in \mathbb{C}$ of modulus 1 for which $\lambda_1 = \omega\lambda_2$ (or vice versa). Also, $\alpha_1,$ and α_2 are not identically zero. Therefore, the initial conditions of Theorem 3.5 are satisfied.
Now, $|x - (-1)|^2 = 1$ implies that x lies on the circle centred at -1. □

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