Electronic Journal of Graph Theory and Applications **6** (2) (2018), 250–257

Electronic Journal of
Graph Theory and Applications

Total vertex irregularity strength of trees with maximum degree five

Susilawati, Edy Tri Baskoro*, Rinovia Simanjuntak

Combinatorial Mathematics Research Group, Faculty of Mathematics and Natural Sciences
Institut Teknologi Bandung, Jalan Ganesa 10 Bandung, Indonesia

susilawati_nurdin@students.itb.ac.id, ebaskoro@math.itb.ac.id, rino@math.itb.ac.id

Abstract

In 2010, Nurdin, Baskoro, Salman and Gaos conjectured that the total vertex irregularity strength of any tree T is determined only by the number of vertices of degrees 1, 2 and 3 in T. This paper will confirm this conjecture by considering all trees with maximum degree five. Furthermore, we also characterize all such trees having the total vertex irregularity strength either t_1, t_2 or t_3, where $t_i = \lceil (1 + \sum_{j=1}^{i} n_j)/(i+1) \rceil$ and n_i is the number of vertices of degree i.

Keywords: irregularity strength, total vertex irregularity strength, tree
Mathematics Subject Classification: 05C78, 05C05
DOI: 10.5614/ejgta.2018.6.2.5

1. Introduction

In [7], Chartrand et al. proposed the following problem: assign positive integer labels to all the edges of a connected graph of order greater than 2 in such a way that the graph becomes irregular, i.e., the weights (label sums) at all vertices are different. Find the minimum value of the largest label over all such irregular assignments. This value is well known as the *irregularity strength* of the graph.

Motivated by this problem, a survey paper of Gallian [8] and a book of Wallis [16], Baca et al. [5] introduced the total vertex irregularity strength of a graph as follows. Let $G(V, E)$ be a simple graph. For a labeling $\phi : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, k\}$, the *weight* of a vertex x is defined as $wt(x) = \phi(x) + \sum_{xy \in E} \phi(xy)$. A labeling ϕ is called a *vertex irregular total k-labeling* if the

Received: 13 July 2017, Revised: 12 January 2018, Accepted: 27 May 2018.
*Corresponding author.

weights of all vertices are distinct. The minimum k for which the graph G has a vertex irregular total k-labeling is called the *total vertex irregularity strength* of G and it is denoted by $tvs(G)$. In [5], Baca et al. proved that $tvs(C_n) = \lceil \frac{n+2}{3} \rceil, n \geq 3$; $tvs(K_n) = 2$; $tvs(K_{1,n}) = \lceil \frac{n+1}{2} \rceil$; and $tvs(C_n \times P_2) = \lceil \frac{2n+3}{4} \rceil$. For a tree T with m pendant vertices and no vertices of degree 2, they proved that $\lceil \frac{m+1}{2} \rceil \leq tvs(T) \leq m$. They also proved that if G is a (p, q)-graph with minimum degree δ and maximum degree Δ, then $\lceil \frac{p+\delta}{\Delta+1} \rceil \leq tvs(G) \leq p + \Delta - 2\delta + 1$.

In 2010, Nurdin, Baskoro, Salman and Gaos [11] determined the total vertex irregularity strength of trees containing vertices of degree 2, namely a subdivision of a star and a subdivision of a particular caterpillar. They also improved some of the bounds given in [5] and showed that tvs of any tree with n_1 pendant vertices and containing no vertices of degree 2 is $\lceil \frac{n_1+1}{2} \rceil$.

In the same paper, Nurdin et al. also determined the total vertex irregularity strength of trees without vertices of degrees two and three. They also conjectured that the total vertex irregularity strength of any tree is determined by the number of its vertices of degrees $1, 2$, and 3 only. Precisely, they conjectured that $tvs(T) = max\{t_1, t_2, t_3\}$, where $t_i = \lceil (1 + \sum_{j=1}^{i} n_j)/(i + 1) \rceil$ and n_i is the number of vertices of degree $i \in [1, 3]$.

Many paper studied about the total vertex irregularity strengths of graphs, see [1, 2, 3, 4, 6, 10, 12]. Susilawati, Baskoro and Simanjuntak [13] determined the total vertex irregularity strength for the subdivision of several classes of trees, including the subdivision of a caterpillar, the subdivision of a fire cracker, and the subdivision of an amalgamation of stars. In other paper [14], they also gave the total vertex irregularity strength of any tree with maximum degree four. Recently, they studied about total vertex irregularity strength for subdivision of trees [15]

In this paper, we show that the total vertex irregularity strength of any tree T with maximum degree five is either t_1, t_2 or t_3. This fact strengthens the conjecture of Nurdin et al. [11]. Furthermore, we also characterize all such trees T with the total vertex irregularity strength t_1, t_2 or t_3.

2. Main Results

In this section, we show that the total vertex irregularity strength of any tree with maximum degree five is $\max\{t_1, t_2, t_3\}$. This result enhances the conjecture of Nurdin et al. (2010). We also characterize all trees with maximum degree five having the total vertex irregularity strength t_1, t_2 or t_3.

To start with, we present the well-known fact regarding the relationship between the number n_i of vertices degree i in any tree T, namely $n_1 = 2 + \sum_{i \geq 3}(i-2)n_i$ [9].

Theorem 2.1. *[11] Let T be a tree with maximum degree Δ. Let n_i be the number of vertices of degree i. Then,* $tvs(T) \geq max\left\{ \lceil \frac{1+n_1}{2} \rceil, \lceil \frac{1+n_1+n_2}{3} \rceil, \dots, \lceil \frac{1+n_1+n_2+\cdots+n_\Delta}{\Delta+1} \rceil \right\}$.

From now on, we will only consider trees with maximum degree five. For $1 \leq i \leq 5$, let n_i be the number of vertices of degree i and define $t_i = \lceil (1+\sum_{j=1}^{i} n_j)/(i+1) \rceil$. By substituting $n_1 = 2 + \sum_{i \geq 3}(i-2)n_i$ into t_i, we obtain that $t_1 = \lceil \frac{90+30n_3+60n_4+90n_5}{60} \rceil$, $t_2 = \lceil \frac{60+20n_2+20n_3+40n_4+60n_5}{60} \rceil$, $t_3 = \lceil \frac{45+15n_2+30n_3+30n_4+45n_5}{60} \rceil$, $t_4 = \lceil \frac{36+12n_2+24n_3+36n_4+36n_5}{60} \rceil$, and $t_5 = \lceil \frac{30+10n_2+20n_3+30n_4+40n_5}{60} \rceil$. Let

$t_i = \lceil \frac{q_i}{60} \rceil$ for $i = 1, 2, 3, 4, 5$. Thus, $q_1 = 90 + 30n_3 + 60n_4 + 90n_5,\ q_2 = 60 + 20n_2 + 20n_3 + 40n_4 + 60n_5,\ q_3 = 45 + 15n_2 + 30n_3 + 30n_4 + 45n_5$, $q_4 = 36 + 12n_2 + 24n_3 + 36n_4 + 36n_5$, and $q_5 = 30 + 10n_2 + 20n_3 + 30n_4 + 40n_5$.

Now, we start with proving that t_4 or t_5 cannot be the maximum value of all the values t_is.

Lemma 2.1. *Let T be a tree with maximum degree five. Then, $t_5 \le t_2, t_5 \le t_3$ and there exists some $i \in \{1, 4\}$ such that $t_5 \le t_i$.*

Proof. Consider $t_i = \lceil \frac{q_i}{60} \rceil$ for $i = 1, 2, 3, 4, 5$. Then, we have $q_1 - q_5 = 60 - 10n_2 + 10n_3 + 30n_4 + 50n_5, q_2 - q_5 = 30 + 10n_2 + 10n_4 + 20n_5, q_3 - q_5 = 15 + 5n_2 + 10n_3 + 5n_5$, and $q_4 - q_5 = 6 + 2n_2 + 4n_3 + 6n_4 - 4n_5$. Since $q_2 - q_5$ and $q_3 - q_5$ are positive, then $t_5 \le t_2$ and $t_5 \le t_3$. Now, we need only to show that either $q_1 - q_5$ or $q_4 - q_5$ is non negative. If $q_1 - q_5 \ge 0$ then the proof concludes, otherwise $60 - 10n_2 + 10n_3 + 30n_4 + 50n_5 < 0$. This implies that $n_2 > 6 + n_3 + 3n_4 + 5n_5$. Thus, $q_4 - q_5 = 6 + 2n_2 + 4n_3 + 6n_4 - 4n_5 > 6 + 2(6 + n_3 + 3n_4 + 5n_5) + 4n_3 + 6n_4 - 4n_5 = 18 + 6n_3 + 12n_4 + 6n_5 > 0$. □

Lemma 2.2. *Let T be a tree with maximum degree five. Then, there exists some $i \in \{1, 2\}$ such that $t_4 \le t_i$.*

Proof. Consider $t_i = \lceil \frac{q_i}{60} \rceil$ for $i = 1, 2, 3, 4$. Then, we have $q_1 - q_4 = 54 - 12n_2 + 6n_3 + 24n_4 + 54n_5$ and $q_2 - q_4 = 24 + 8n_2 - 4n_3 + 4n_4 + 24n_5$. Thus, we need only to show that either $q_1 - q_4$ or $q_2 - q_4$ is non negative. If $q_1 - q_4 \ge 0$ then the proof concludes. Otherwise, $54 - 12n_2 + 6n_3 + 24n_4 + 54n_5 < 0$, and so $n_2 > \frac{9}{2} + \frac{n_3}{2} + 2n_4 + \frac{9n_5}{2}$. Furthermore, $q_2 - q_4 = 24 + 8n_2 - 4n_3 + 4n_4 + 24n_5 > 24 + 8(\frac{9}{2} + \frac{n_3}{2} + 2n_4 + \frac{9n_5}{2}) - 4n_3 + 4n_4 + 24n_5 = 60 + 20n_4 + 60n_5 > 0$. □

From two above lemmas, we can conclude that $\max\{t_1, t_2, t_3, t_4, t_5\} = \max\{t_1, t_2, t_3\}$. The following three lemmas will give the necessary and sufficient conditions for t_1, t_2 or t_3 to be the maximum value.

Lemma 2.3. *Let T be a tree with maximum degree five.* $\max\{t_1, t_2, t_3\} = t_1$ *if and only if* $(2n_3 - 2n_4 - 3n_5 - 3 \le n_2 \le \frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2})$ *or* $(\frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2} \le n_4 < n_3 - \frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2})$.

Proof. Consider the following two cases.

Case 1. $t_2 \ge t_3$.

Consider $t_i = \lceil \frac{q_i}{60} \rceil$ for $i = 1, 2, 3$. Since $q_1 = 90 + 30n_3 + 60n_4 + 90n_5, q_2 = 60 + 20n_2 + 20n_3 + 40n_4 + 60n_5, q_3 = 45 + 15n_2 + 30n_3 + 30n_4 + 45n_5$ and $t_2 - t_3 \ge 0$, then $n_2 \ge 2n_3 - 2n_4 - 3n_5 - 3$. The fact of $\max\{t_1, t_2, t_3\} = t_1$ implies that $t_1 \ge t_2$, that is $90 + 30n_3 + 60n_4 + 90n_5 \ge 60 + 20n_2 + 20n_3 + 40n_4 + 60n_5$. This yields that $n_2 \le \frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2}$. Then, we have $2n_3 - 2n_4 - 3n_5 - 3 \le n_2 \le \frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2}$.

Case 2. $t_2 < t_3$.

Consider $t_i = \left\lceil \frac{q_i}{60} \right\rceil$ for $i = 1, 2, 3$. Since $q_1 = 90+30n_3+60n_4+90n_5, q_2 = 60+20n_2+20n_3+40n_4+60n_5, q_3 = 45+15n_2+30n_3+30n_4+45n_5$ and $t_3 - t_2 > 0$, then $n_3 > \frac{3}{2} + \frac{n_2}{2} + n_4 + \frac{3n_5}{2}$. The fact of $\max\{t_1, t_2, t_3\} = t_1$ implies that $t_1 \geq t_3$, that is $90 + 30n_3 + 60n_4 + 90n_5 \geq 45 + 15n_2 + 30n_3 + 30n_4 + 45n_5$. This yields that $n_2 \leq 3 + 2n_4 + 3n_5$. Then, we have $\frac{n_2}{2} - \frac{3n_5}{2} - 3 \leq n_4 < n_3 - \frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2}$.

Then, if $\max\{t_1, t_2, t_3\} = t_1$ we have $(2n_3 - 2n_4 - 3n_5 - 3 \leq n_2 \leq \frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2})$ or $(\frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2} \leq n_4 < n_3 - \frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2})$.

Conversely, by substituting the conditions $(2n_3 - 2n_4 - 3n_5 - 3 \leq n_2 \leq \frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2})$ or $(\frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2} \leq n_4 < n_3 - \frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2})$ into $t_i = \lceil (1 + \sum_{j=1}^{i} n_j)/(i+1) \rceil$ where $i \in \{1, 2, 3\}$ we could obtain $\max\{t_1, t_2, t_3\} = t_1$. □

Lemma 2.4. *Let T be a tree with maximum degree five.* $\max\{t_1, t_2, t_3\} = t_2$ *if and only if* $(\frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2} \leq n_2 \leq 3 + 2n_4 + 3n_5)$ *or* $(n_3 - \frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2} \leq n_4 < \frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2})$.

Proof. Consider the following two cases.

Case 1. $t_1 \geq t_3$

Consider $t_i = \left\lceil \frac{q_i}{60} \right\rceil$ for $i = 1, 2, 3$. Since $q_1 = 90+30n_3+60n_4+90n_5, q_2 = 60+20n_2+20n_3+40n_4+60n_5, q_3 = 45+15n_2+30n_3+30n_4+45n_5$, and $t_1 - t_3 \geq 0$ then $n_2 \leq 3+2n_4+3n_5$. Since $\max\{t_1, t_2, t_3\} = t_2$, then $t_2 \geq t_1$, that is $60+20n_2+20n_3+40n_4+60n_5 \geq 90+30n_3+60n_4+90n_5$. This yields that $n_2 \geq \frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2}$. Then, we have $\frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2} \leq n_2 \leq 3 + 2n_4 + 3n_5$.

Case 2. $t_1 < t_3$

Consider $t_i = \left\lceil \frac{q_i}{60} \right\rceil$ for $i = 1, 2, 3$. Since $q_1 = 90 + 30n_3 + 60n_4 + 90n_5, q_2 = 60 + 20n_2 + 20n_3 + 40n_4 + 60n_5, q_3 = 45 + 15n_2 + 30n_3 + 30n_4 + 45n_5$, and $t_3 - t_1 > 0$ then $n_2 > 3 + 2n_4 + 3n_5$. Since $\max\{t_1, t_2, t_3\} = t_2$, then $t_2 \geq t_3$, that is $60 + 20n_2 + 20n_3 + 40n_4 + 60n_5 \geq 45 + 15n_2 + 30n_3 + 30n_4 + 45n_5$. This yields that $n_3 \leq \frac{3}{2} + \frac{n_2}{2} + n_4 + \frac{3n_5}{2}$. Then, we have $n_3 - \frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2} \leq n_4 < \frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2}$.

Conversely, by substituting the conditions $\frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2} \leq n_2 \leq 3 + 2n_4 + 3n_5$ or $n_3 - \frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2} \leq n_4 < \frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2}$ into $t_i = \lceil (1 + \sum_{j=1}^{i} n_j)/(i+1) \rceil$ where $i \in \{1, 2, 3\}$ we could obtain $\max\{t_1, t_2, t_3\} = t_2$. □

Lemma 2.5. *Let T be a tree with maximum degree five.* $\max\{t_1, t_2, t_3\} = t_3$ *if and only if* $(3 + 2n_4 + 3n_5 \leq n_2 \leq \frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2})$ *or* $(\frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2} < n_2 \leq 2n_3 - 2n_4 - 3n_5 - 3)$.

Proof. Consider the following two cases.

Case 1. $t_1 \geq t_2$.

Consider $t_i = \left\lceil \frac{q_i}{60} \right\rceil$ for $i = 1, 2, 3$. Since $q_1 = 90+30n_3+60n_4+90n_5, q_2 = 60+20n_2+20n_3+40n_4+60n_5, q_3 = 45+15n_2+30n_3+30n_4+45n_5$, and $t_1 - t_2 \geq 0$ then $n_2 \leq \frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2}$. Since

$\max\{t_1, t_2, t_3\} = t_3$, then $t_3 \geq t_1$, that is $45+15n_2+30n_3+30n_4+45n_5 \geq 90+30n_3+60n_4+90n_5$. This yields that $n_2 \geq 3 + 2n_4 + 3n_5$. Then, we have $3 + 2n_4 + 3n_5 \leq n_2 \leq \frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2}$.

Case 2. $t_1 < t_2$.

Consider $t_i = \lceil \frac{q_i}{60} \rceil$ for $i = 1, 2, 3$. Since $q_1 = 90 + 30n_3 + 60n_4 + 90n_5, q_2 = 60 + 20n_2 + 20n_3 + 40n_4 + 60n_5, q_3 = 45 + 15n_2 + 30n_3 + 30n_4 + 45n_5$, and $t_2 - t_1 > 0$ then $n_2 > \frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2}$. Since $\max\{t_1, t_2, t_3\} = t_3$, then $t_3 \geq t_2$, that is $45 + 15n_2 + 30n_3 + 30n_4 + 45n_5 \geq 60 + 20n_2 + 20n_3 + 40n_4 + 60n_5$. This yields that $n_2 \leq 2n_3 - 2n_4 - 3n_5 - 3$. Then, we have $\frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2} < n_2 \leq 2n_3 - 2n_4 - 3n_5 - 3$.

Conversely, by substituting $3 + 2n_4 + 3n_5 \leq n_2 \leq \frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2}$ or $\frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2} < n_2 \leq 2n_3 - 2n_4 - 3n_5 - 3$ into $t_i = \lceil (1 + \sum_{j=1}^{i} n_j)/(i+1) \rceil$ where $i \in \{1, 2, 3\}$. Then, we could obtain $\max\{t_1, t_2, t_3\} = t_3$. $\square$

Next, in Theorem 2.2, we characterize all trees with maximum degree five such that $tvs(T) = t_1$. In Theorems 2.3 and 2.4, we show a similar characterization for all trees T with $tvs(T) = t_2$ and $tvs(T) = t_3$, respectively. We call a vertex $v \in T$ an *exterior vertex* if there exists a pendant vertex in T which is adjacent to v. The vertices other than exterior and pendant vertices are called *interior vertices*.

Theorem 2.2. *Let T be a tree with maximum degree five. $tvs(T) = t_1$ if and only if* $(2n_3 - 2n_4 - 3n_5 - 3 \leq n_2 \leq \frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2})$ *or* $(\frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2} \leq n_4 < n_3 - \frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2})$.

Proof. According to Lemma 2.3 and Theorem 2.1, we have $tvs(T) \geq t_1$ if and only if $(2n_3 - 2n_4 - 3n_5 - 3 \leq n_2 \leq \frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2})$ or $(\frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2} \leq n_4 < n_3 - \frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2})$. Now, we need to show that $tvs(T) \leq t_1$. Define a total labeling $\phi : V(T) \cup E(T) \to \{1, 2, \ldots, t_1\}$ in T by using the following algorithm.

Labeling Algorithm 1

Label all edges $e \in E(T)$ by the following steps.

(a). Let $W = \{w_1, w_2, \ldots, w_k\}$ be the set of exterior vertices, where $d(w_i) \geq d(w_{i+1})$ and $|E(w_i)| \geq |E(w_{i+1})|$ where $E(w_i)$ is the set of pendant edges incident to w_i, and $d(w_i)$ is the degree of vertices w_i, for each i.
Let $E_1 = \bigcup_{i=1}^{k} E(w_i)$ be an ordered set of all pendant edges in T.

(b). Label the first t_1 pendant edges in E_1 with $\{1, 2, 3, \ldots t_1\}$, respectively.

(c). Label $(n_1 - t_1)$ remaining pendant edges with t_1.

(d). For all the remaining edges $e \in E(T) - E_1$, we define $\phi(e) = t_1$.

Label all vertices $v \in V(T)$ by the following steps.

(a). Let $V_1 = \{w_{ij} | i = 1, 2, 3, \ldots, k \text{ and } j = 1, 2, 3, \ldots, j_i\}$ be an ordered set of pendant vertices adjacent to w_i, where j_i is the number of pendant vertices adjacent to w_i.

(b). Label the first t_1 pendant vertices in V_1 with 1.

(c). Label $(n_1 - t_1)$ remaining pendant vertices with $2, 3, \ldots, n_1 - t_1 + 1$.

(d). Denote all non-pendant vertices by $y_1, y_2, y_3, \ldots, y_N$, where $N = n_2 + n_3 + n_4 + \cdots + n_\Delta$ such that $s(y_1) \leq s(y_2) \leq s(y_3) \leq \cdots \leq s(y_N)$, with $s(y) = \sum_{yz \in E(T)} \phi(yz)$, which can be considered as a temporary weight of y_i.

(e). Now, define $\phi(y_i)$ recursively as follows:

$$\phi(y_1) = n_1 + 2 - s(y_1),$$

which implies $wt(y_1) = \phi(y_1) + s(y_1)$. For $2 \leq i \leq N$, we define

$$\phi(y_i) = \max\{1, wt(y_{i-1}) + 1 - s(y_i)\}.$$

We observe that ϕ is a labeling from $V(T) \cup E(T)$ into $\{1, 2, 3, \ldots, t_1\}$, the weights of n_1 constitute the set $\{2, 3, 4, \ldots, n_1 + 1\}$, and the weights of all remaining vertices form a sequence $n_1 + 2 = wt(y_1) < wt(y_2) < wt(y_3) < \cdots < wt(y_N)$. Therefore, $tvs(T) \leq t_1$. $\square$

Theorem 2.3. *Let T be a tree with maximum degree five. $tvs(T) = t_2$ if and only if $(\frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2} \leq n_2 \leq 3 + 2n_4 + 3n_5)$ or $(n_3 - \frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2} \leq n_4 < \frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2})$.*

Proof. According to Lemma 2.4 and Theorem 2.1, we have $tvs(T) \geq t_2$ if and only if $(\frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2} \leq n_2 \leq 3 + 2n_4 + 3n_5)$ or $(n_3 - \frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2} \leq n_4 < \frac{n_2}{2} - \frac{3n_5}{2} - \frac{3}{2})$. Now, we will show that $tvs(T) \leq t_2$. Let us define a total labeling $\phi : V(T) \cup E(T) \rightarrow \{1, 2, \ldots, t_2\}$ in T by using the Labeling Algorithm 2 for $i = 2$ as follow.

Labeling Algorithm 2

Label all edges $e \in E(T)$ by the following steps.

(a). Let $W = \{w_1, w_2, \ldots, w_k\}$ be the set of exterior vertices, where $d(w_i) \geq d(w_{i+1})$ and $|E(w_i)| \geq |E(w_{i+1})|$ for $i = 1, 2, \cdots, k-1$, where $E(w_i)$ is the set of all pendant edges incident to vertex w_i. Let $E_1 = \bigcup_{i=1}^{k} E(w_i)$ be an ordered set of all pendant edges in T.
(b). Label the first t_i pendant edges in E_1 with $\{1, 2, 3, \ldots, t_i\}$, respectively.
(c). Label $(n_1 - t_i)$ remaining pendant edges $e \in E_1$ with t_i.
(d). Let E_2 be the ordered set of non-pendant edges where at least one of the end-vertices of degree 2. Denote by e_i where $i = 1, 2, \ldots, n_2$, the edges in E_2 and we define $\phi(e_i) = \lceil \frac{1+n_1+i}{3} \rceil$.
(e). Label all remaining edges $e \in E(T) - E_1(T) - E_2(T)$ with t_i.

Label all vertices $v \in V(T)$ by the following steps.

(a). Let $V_1 = \{w_{ij} | i = 1, 2, 3, \ldots, k$ and $j = 1, 2, 3, \ldots, j_i\}$ be an ordered set of pendant vertices adjacent to w_i, where j_i is the number of pendant vertices adjacent to w_i.

(b). Label the first t_i pendant vertices in V_1 with 1.

(c). Label the $(n_1 - t_i)$ remaining pendant vertices with $2, 3, 4, \ldots, n_1 - t_i + 1$.

(d). Denote all non-pendant vertices by $y_1, y_2, y_3, \ldots, y_N$, where $N = n_2 + n_3 + n_4 + \cdots + n_\Delta$, such that $s(y_1) \leq s(y_2) \leq s(y_3) \leq \cdots \leq s(y_N)$, with $s(y) = \sum_{yz \in E(T)} \phi(yz)$, which can be considered as a temporary weight of y_i.

(e). We define $\phi(y_i)$ recursively as follows, $\phi(y_1) = n_1 + 2 - s(y_1)$, which implies $wt(y_1) = \phi(y_1) + s(y_1)$. For $2 \leq i \leq N$, $\phi(y_i) = \max\{1, wt(y_{i-1}) + 1 - s(y_i)\}$.

We conclude that ϕ is a labeling from $V(T) \cup E(T)$ into $\{1, 2, \ldots, t_2\}$, and the weights of all pendant vertices constitute the set $\{2, 3, 4, \ldots, n_1 + 1\}$, and the weights of all remaining vertices form a sequence $n_1 + 2 = wt(y_1) < wt(y_2) < \cdots < wt(y_N)$. Therefore, $tvs(T) \leq t_2$. $\square$

Theorem 2.4. *Let T be a tree with maximum degree five. $tvs(T) = t_3$ if and only if* $(3+2n_4+3n_5 \leq n_2 \leq \frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2})$ *or* $(\frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2} < n_2 \leq 2n_3 - 2n_4 - 3n_5 - 3)$.

Proof. According to Lemma 2.5 and Theorem 2.1, we have $tvs(T) \geq t_3$ if and only if $(3 + 2n_4 + 3n_5 \leq n_2 \leq \frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2})$ or $(\frac{3}{2} + \frac{n_3}{2} + n_4 + \frac{3n_5}{2} < n_2 \leq 2n_3 - 2n_4 - 3n_5 - 3)$. Now, we will show that $tvs(T) \leq t_3$, by defining a total labeling $\phi : V(T) \cup E(T) \to \{1, 2, 3, \ldots, t_3\}$ in T and using the Labeling Algorithm 2 for $i = 3$.

We conclude that ϕ is a labeling from $V(T) \cup E(T)$ into $\{1, 2, \ldots, t_3\}$, and the weights of all pendant vertices constitute the set $\{2, 3, 4, \ldots, n_1 + 1\}$, and the weights of all remaining vertices form a sequence $n_1 + 2 = wt(y_1) < wt(y_2) < \cdots < wt(y_N)$. Therefore, $tvs(T) \leq t_3$. $\square$

3. Conclusion

In this paper, we prove that for any tree T with maximum degree five, the total vertex irregularity strength of this tree T is $\max\{t_1, t_2, t_3\}$. This fact strengthens the conjecture of Nurdin *et al.* (2010). Moreover, we give necessary and sufficient conditions for all trees T with maximum degree five such that the total vertex irregularity strength is either t_1, t_2 or t_3. To conclude this paper, we give an open problem below.

Open Problem 3.1. *Find the total vertex irregularity strength of a tree with maximum degree at least 6.*

Acknowledgement

This research was supported by the Research Grant "Program Riset Inovasi KK ITB" and "Program PMDSU", Ministry of Research, Technology and Higher Education, Indonesia.

References

[1] A. Ahmad, K.M. Awan, I. Javaid and Slamin, Total vertex irregularity strength of wheel related graphs, *Australas. J. Combin.* **51** (2011), 147–156.

[2] A. Ahmad, M. Bača and Y. Bashir, Total vertex irregularity strength of certain classes of unicyclic graphs, *Bull. Math. Soc. Sci. Math. Roumanie* **2** (2014), 147–152.

[3] O. Al-Mushayt, A. Arshad and M.K. Siddiqui, Total vertex irregularity strength of convex polytope graphs, *Acta Math. Univ. Comenianae* **LXXXII** (1) (2013), 29–37.

[4] M. Anholcer, M. Karoński and F. Pfender, Total vertex irregularity strength of forest, *arXiv:1103.2087* (2011).

[5] M. Bača, S. Jendrol', M. Miller and J. Ryan, On irregular total labellings, *Discrete Math.* **307** (2007), 1378–1388.

[6] T. Bohman and D. Kravitz, On the irregularity strength of trees, *J. Graph Theory* **45** (4) (2004), 241–254.

[7] G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz and F. Saba, Irregular network, *Congr. Numer.* **64** (1988), 187–192.

[8] J.A. Gallian, A Dynamic survey of graph labeling, *Electron. J. Combin* (2017), #DS6.

[9] N. Hartsfield and G. Ringel, *Pearls in Graph Theory: A Comprehensive Introduction*, Academic Press, INC. (1990).

[10] D. Indriati, W.I.E. Wijayanti, K.A. Sugeng, M. Bača and A. Semaničová-Feňovčíková, The total vertex irregularity strength of generalized helm graphs and prism with outer pendant edges, *Australas. J. Combin.* **65** (1) (2016), 14–26.

[11] Nurdin, E.T. Baskoro, A.N.M. Salman and N.N. Gaos, On total vertex irregularity strength of trees, *Discrete Math.* **310** (2010), 3043–3048.

[12] Slamin, Dafik and W. Winnona, Total vertex irregularity strength of the disjoint union of sun graphs, *International Journal of Combinatorics* **2012** (2012), 1–9.

[13] Susilawati, E.T. Baskoro and R. Simanjuntak, Total vertex irregularity labelings for subdivision of several classes of tree, *Procedia Computer Science* **74** (2015), 112–117.

[14] Susilawati, E.T. Baskoro and R. Simanjuntak, Total vertex irregularity strength of tree with maximum degree four, *AIP Conference Proceedings* **1707**, 020022 (2016); DOI: 10.1063/1.4940823.

[15] Susilawati, E.T. Baskoro and R. Simanjuntak, On the Vertex Irregular Total Labeling for Subdivision of Trees, *Australas. J. Combin.* **71** (2) (2018), 293-302.

[16] W.D. Wallis, *Magic Graphs*, Birkhäuser, Boston (2001).