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Degree equitable restrained double domination in graphs

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Abstract

A subset $D \subseteq V(G)$ is called an equitable dominating set of a graph G if every vertex $v \in V(G)\setminus D$ has a neighbor $u \in D$ such that $|d_G(u) - d_G(v)| \leq 1$. An equitable dominating set D is a degree equitable restrained double dominating set (DERD-dominating set) of G if every vertex of G is dominated by at least two vertices of D, and $\langle V(G) \setminus D\rangle$ has no isolated vertices. The DERD-domination number of G, denoted by $\gamma^e_{cl}(G)$, is the minimum cardinality of a DERD-dominating set of G. We initiate the study of DERD-domination in graphs and we obtain some sharp bounds. Finally, we show that the decision problem for determining $\gamma^e_{cl}(G)$ is NP-complete.

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1. Introduction

Let $G = (V, E)$ be a graph. The number of vertices of G we denote by n and the number of edges we denote by m, thus $|V(G)| = n$ and $|E(G)| = m$. The complement of G, denoted by $\bar{G}$, is a graph which has the same vertices as G, and in which two vertices are adjacent if and

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only if they are not adjacent in G. By the open neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. By the closed neighborhood of a vertex v of G we mean the set $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its open neighborhood. A vertex is called isolated if it has no neighbors, while it is called universal if it is adjacent to all other vertices. Let S be a subset of the set of vertices of G, and let $u \in S$. A vertex v is a private neighbor of u with respect to S if $N_G[v] \cap S = \{u\}$. The set of private neighbors of u with respect to S is the set $pn[u, S] = \{v \colon N_G[v] \cap S = \{u\}\}$. If $u \in pn[u, S]$ and u is an isolated vertex in $\langle S \rangle$, then u is called its own private neighbor. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is weak if it is adjacent to exactly one leaf. We say that a vertex is isolated if it has no neighbor. Let $\Delta(G)$ mean the maximum degree among all vertices of G. The path (cycle, respectively) on n vertices we denote by P_n (C_n, respectively). A wheel W_n, where $n \geq 4$, is a graph with n vertices, formed by connecting a vertex to all vertices of a cycle C_{n-1}. The distance between two vertices of a graph is the number of edges in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The diameter of a graph G, denoted by $\text{diam}(G)$, is the maximum eccentricity among all vertices of G. By $K_{p,q}$ we denote a complete bipartite graph with partite sets of cardinalities p and q. By a star we mean the graph $K_{1,q}$. By a double star we mean a graph obtained from a star by joining a positive number of vertices to one of its leaves. Generally, let $K_{t_1,t_2,\ldots,t_k}$ denote the complete multipartite graph with vertex set $S_1 \cup S_2 \cup \ldots \cup S_k$, where $|S_i| = t_i$ for positive integers $i \leq t$.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. For a comprehensive survey of domination in graphs, see [4, 5].

A subset $D \subseteq V(G)$ is a restrained dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D as well as a neighbor in $V(G) \setminus D$. The restrained domination number of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set of G. A restrained dominating set of G of minimum cardinality is called a $\gamma_r(G)$-set.

A dominating set D of a graph G is said to be a cototal dominating set of G if the induced subgraph $\langle V(G) \setminus D \rangle$ has no isolated vertices. The cototal domination number of G, denoted by $\gamma_{cl}(G)$, is the minimum cardinality of a cototal dominating set of G. Restrained domination in graphs was introduced by Domke et. al [1]. Independently, Kulli et. al [9] initiated the study of cototal domination in graphs. The concepts of restrained domination and cototal domination are equivalent.

A subset $D \subseteq V(G)$ is a double dominating set of G if every vertex of G is dominated by at least two vertices of D. The double domination number of G, denoted by $\gamma_d(G)$, is the minimum cardinality of a double dominating set of G. The study of double domination in graphs was initiated by Harary and Haynes [3].

A subset $D \subseteq V(G)$ is a restrained double dominating set of G if every vertex of G is dominated by at least two vertices of D, and no vertex of $\langle V(G) \setminus D \rangle$ is isolated. The restrained double domination number of G, denoted by $\gamma_{dcl}(G)$, is the minimum cardinality of a restrained double dominating set of G. The study of restrained double domination in graphs was initiated by in [8].

A subset $D \subseteq V(G)$ is called an equitable dominating set of G if every vertex $v \in V(G) \setminus D$ has a neighbor $u \in D$ such that $|d_G(u) - d_G(v)| \leq 1$. The equitable domination number of G, denoted by $\gamma^e(G)$, is the minimum cardinality of an equitable dominating set of G. The concept of equitable domination in graphs was introduced by V. Swaminathan and K. Dharmalingam [11] by considering the following real world situation. In a network, nodes with nearly equal capacity may interact with each other in a better way. In societies, persons with nearly equal statuses tend to be friendly. For more details on the domination refer [6, 7, 10, 12].

We introduce a new variant of equitable domination, namely the degree equitable restrained double domination (DERD-domination), and we initiate the study of this parameter. An equitable dominating set D of a graph G is said to be a DERD-dominating set of G if every vertex of G is dominated by at least two vertices of D, and $\langle V(G) \setminus D \rangle$ has no isolated vertices. The DERD-domination number of G, denoted by $\gamma^e_{cl}(G)$, is the minimum cardinality of a DERD-dominating set of G.

2. Results

Since the one-vertex graph, as well as all graphs with an isolated vertex, does not have a DERD-dominating set, in this paper we consider only graphs without isolated vertices.

We begin with the following straightforward observations.

Observation 1. *Let G be a graph without isolated vertices. Then every DERD-dominating set of G contains all leaves and support vertices of G.*

Observation 2. *There is no graph G such that $\gamma^e_{cl}(G) = n - 1$.*

Observation 3. *For every positive integer n we have*

$$\gamma^e_{cl}(K_n) = \begin{cases} 3, & \text{if } n = 3; \\ 2, & \text{otherwise.} \end{cases}$$

Observation 4. *For every integer $n \geq 2$ we have $\gamma^e_{cl}(P_n) = n$.*

Observation 5. *If $n \geq 3$ is an integer, then $\gamma^e_{cl}(C_n) = n$.*

Observation 6. *For every integer $n \geq 4$ we have $\gamma^e_{cl}(W_n) = \lfloor n/2 \rfloor$.*

Observation 7. *If m and n are positive integers, then*

$$\gamma^e_{cl}(K_{m,n}) = \begin{cases} 4, & \text{if } |m - n| \leq 1 \text{ and } 3 \leq m \leq n; \\ m + n, & \text{otherwise.} \end{cases}$$

We have the following property of regular and $(k, k+1)$-biregular graphs.

Theorem 8. *If a graph G is regular or $(k, k+1)$-biregular, for any integer k, then $\gamma^e_{cl}(G) = \gamma_{dcl}(G)$.*

Proof. Let D be a minimum restrained double dominating set of G. Let $u \in V(G) \setminus D$. Thus there exist vertices $w, v \in D$ such that $uw, uv \in E(G)$. We have $|d_G(u) - d_G(v)| \leq 1$ and $|d_G(u) - d_G(w)| \leq 1$. Therefore D is a DERD-dominating set of G. Consequently, $\gamma_{cl}^{e}(G) \leq |D| = \gamma_{dcl}(G)$. Obviously, $\gamma_{dcl}(G) \leq \gamma_{cl}^{e}(G)$. This implies that $\gamma_{cl}^{e}(G) = \gamma_{dcl}(G)$. □

Theorem 9. *For every graph G we have $2 \leq \gamma_{cl}^{e}(G) \leq n$. Further, the lower bound is attained if and only if $G = K_2$ or $G = K_n - \{x\}$ where x is any vertex in K_n; $n \geq 5$ and the upper bound is attained if and only if G does not contain an edge $uv \in E(G)$ which satisfies the following conditions:*

(i) there are vertices $w \in N_G(u)$ and $z \in N_G(v)$ such that $|N_G(u)| \geq 3$ and $|N_G(v)| \geq 3$;

(ii) there are vertices $w \in N_G(u)$ and $z \in N_G(v)$ such that $|d_G(u) - d_G(w)| \leq 1$ and $|d_G(v) - d_G(z)| \leq 1$.

Proof. Lower bound follows from the definition of DERD-set. Now consider the equality of lower bound. Suppose $\gamma_{cl}^{e}(G) = 2$ and $G \neq K_n$ or $K_n - \{x\}$. Then G contains at least two vertices $u, v \in V(G)$ such that $\langle\{u, v\}\rangle$ contains no edge. Let D be DERD-set of G such that $u, v \notin D$. Let $w, x \in D$. Since u and v are independent vertices in G, therefore w and x must be adjacent to both u and v also by the definition of DERD-set $\langle V - D\rangle$ contains no isolated vertices. Therefore, we need at least one more vertex to compliance the necessary conditions required to define DERD-set in G. Hence $|D| \geq 3$, a contradiction.

Conversely, suppose $G = K_n$, then by Observation 3, $\gamma_{cl}^{e}(G) = 2$ and if $G = K_n - \{x\}$; $n \geq 5$, then any two adjacent vertices will form a DERD-set for G. Hence $\gamma_{cl}^{e}(G) = 2$.
Now consider the upper bound. Suppose $\gamma_{cl}^{e}(G) = n$ and G contains an edge which satisfied the conditions in the hypothesis of the theorem, then $V - \{w, z\}$ will form a DERD-set for G. Hence $\gamma_{cl}^{e}(G) = |V - \{w, z\}| = n - 2$. Hence G must not contain an edge as stated in the hypothesis of the theorem. □

We now characterize the trees T such that $\gamma_{cl}^{e}(T) = n$.

Theorem 10. *Let T be a tree. We have $\gamma_{cl}^{e}(T) = n$ if and only if T does not contain an edge $uv \in E(T)$ which is incident to exactly four weak support vertices x, y, z, w such that $N(x) \cap N(y) = \{u\}$ and $N(z) \cap N(w) = \{v\}$.*

Proof. Let T be a tree and $\gamma_{cl}^{e}(T) = n$. Suppose T does not satisfies the hypothesis of the theorem, then there exist at least an edge $uv \in E(T)$ incident to exactly four support vertices x, y, z, w such that $N(x) \cap N(y) = \{u\}$ and $N(z) \cap N(w) = \{v\}$ which implies that $V - \{u, v\}$ is isomorphic to K_2. Therefore $|D| = n - 2$. Hence $\gamma_{cl}^{e}(T) = |D| = n - 2$, a contradiction.

Conversely, suppose G does not contain an edge $uv \in E(T)$ as stated in the hypothesis of the theorem, then $\langle V - D\rangle = \pi$, which implies that $|D| = n$. Hence $\gamma_{cl}^{e}(T) = |D| = n$. □

By Observation 2, there exists no graph with $\gamma_{cl}^{e}(T) = n - 1$.

We now consider trees T such that $\gamma_{cl}^{e}(T) \leq n - 2$.

Let $S(n,k)$-star (where $n \geq 2$ and $k \geq 1$) be a tree obtained from a path P_n making each vertex $v_i \in V(P_n)$ $(2 \leq i \leq n)$ adjacent to least k new leaves. We have $|V(S(n,k))| = n + k$ and $|E(S(n,k))| = n + k - 1$.

Operation $\mathcal{O}$: Let v be a support vertex of a tree T. Attach $|d_G(v) - 1|$ or $|d_G(v) - 2|$ leaves to at least one leaf adjacent to v, and attach exactly one leaf to other leaves adjacent to v.

Let $\mathcal{T}$ be the family of trees such that
$\mathcal{T} = \{T \colon T$ is obtained from a star by a finite sequence of operations $\mathcal{O}\}$.

We now characterize the trees with $\gamma^e_{cl}(T) = n - 2$.

Theorem 11. *If T is a tree with at least six vertices, then $\gamma^e_{cl}(T) = n - 2$ if and only if $T \in \mathcal{T}$ and T is obtained from a $S(2,k)$-star ($k \geq 2$) by a finite sequence of operations $\mathcal{O}$.*

Similarly, we can characterize the trees with $\gamma^e_{cl}(T) = k$ ($k \geq 3$) by $S(n, n-k)$-star by finite sequence of operations $\mathcal{O}$.

We need the following theorem to prove our further results.

Theorem 12 ([4])**.** *Let G be a graph without isolated vertices. Then $\gamma(G) = n/2$ if and only if each component of G is a cycle C_4 or $G = H \circ K_1$, for any connected graph H.*

Next we characterize the class of graphs with $\gamma^e_{cl}(G) = 2\gamma(G)$.

Theorem 13. *Let G be a graph without isolated vertices, and which is not a tree. Then $\gamma^e_{cl}(G) = 2\gamma(G)$ if and only if each component of G is a cycle C_4 or $G = H \circ K_1$, for any connected graph H.*

Proof. Let G be a graph without isolated vertices. Let D be a DERD-dominating set of G. If each component of G is a cycle C_4, then by Theorem 12, $\gamma(G) = \frac{n}{2}$ and by Observation 4, we have $\gamma^e_{cl}(G) = n$. If $G = H \circ K_1$, then $\gamma^e_{cl}(G) = n$ as every vertex of $H \circ K_1$ is a leaf or a support vertex. By Theorem 12 we have $\gamma(G) = n/2$. Hence $\gamma^e_{cl}(G) = n = n/2 + n/2 = \gamma(G) + \gamma(G) = 2\gamma(G)$. □

3. Complexity issues for $\gamma^e_{cl}(G)$

To show that the DERD-domination decision problem for arbitrary graphs is NP-complete, we shall use a well known NP-completeness result called Exact Three Cover ($X3C$), which is defined as follows.

EXACT COVER BY 3-SETS ($X3C$).
Instance: A finite set X with $|X| = 3m$ and a collection C of 3-element subsets of X.
Question: Does C contain an exact cover for X, that is, a subcollection $C' \subseteq C$ such that every element of X occurs in exactly one member of C'? Note that if C' exists, then its cardinality is precisely m.

Theorem 14 ([2])**.** *$X3C$ is NP-complete.*

DEGREE EQUITABLE RESTRAINED DOUBLE DOMINATING SET (DERD-dominating set).

Instance: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.
Question: Is there a DERD-dominating set of cardinality at most k?

Theorem 15. *DERD-dominating set problem is NP-complete, even for bipartite graphs.*

Proof. It is clear that the DERD-dominating set problem is NP. To show that it is NP-complete, we establish a polynomial transformation from $X3C$. Let $X = \{x_1, x_2, \ldots, x_{3m}\}$ and $C = \{c_1, c_2, \ldots, c_m\}$ be an arbitrary instance of $X3C$. We construct a bipartite graph G and a positive integer k such that this instance of $X3C$ will have an exact 3-cover if and only if G has a DERD-dominating set of cardinality at most k. With each edge $x_i \in X$, associate a path P_4 with vertices x_i, y_i, z_i, t_i, with each c_j associate a path P_3 with vertices c_j, d_j, s_j. Then add new vertices $u_1, u_2, \ldots, u_{2m}$, and make them adjacent to all x_j's. The construction of a bipartite graph G is completed by joining x_i and c_j if and only if $x_i \in c_j$. Finally, set $k = 2m + 9m$.

Assume that C has an exact 3-cover, say c'. Then

$$\bigcup_{1 \leq i \leq 3m} \{z_i, t_i\} \cup \bigcup_{1 \leq j \leq m} \{d_j, s_j\} \cup \{c_j; c_j \in c'\} \cup \bigcup_{1 \leq j \leq 2m} u_j$$

is a DERD-dominating set of G of cardinality $2m + 9m$. This construction can clearly be determined in polynomial time.

Now assume that D is a DERD-dominating set of cardinality at most $2m + 9m$. Then the vertices in the set L, defined by

$$\bigcup_{1 \leq i \leq 3m} \{z_i, t_i\} \cup \bigcup_{1 \leq j \leq m} \{d_j, s_j\}$$

are all leaves, and their neighbors have to be in D. Hence $|D| - |L| \leq (2m + 9m) - (2m + 6m) = 3m$. Let $I = \{i \in (1, 2, \ldots, 3m) \colon x_i \in D \text{ or } y_i \in D\}$ and let $J = \{j \in (1, 2, \ldots, 2m) \colon c_j \in D$ or $u_j \in D\}$. Then since D is a double dominating set of G, we have

$$\bigcup_{i \in I} \{x_i, y_i\} \cup \bigcup_{j \in J} N_G[c_j] \cup \bigcup_{j \in J} \{u_j\} \supseteq \{x_1, x_2, \ldots, x_{3m}\}.$$

We conclude that $|I| + 3|J| \geq 9m$. Also $|I| + |J| \leq |D| - |L| \leq 3m$. Hence $|3I| + 3|J| \leq |I| + 3|J|$, thus $I = \emptyset$. We conclude that $x_i, y_i \notin D$ for $i = 1, 2, \ldots, 3m$. Since x_i $(i = 1, 2, \ldots, 3m)$ is dominated by D, we conclude that $|J| = 3m$ and $c' = \{c_j \colon j \in J\}$ is an exact cover for X. □

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[1] G. Domke, J. Hatting, S. Hedetniemi, R. Laskar, and L. Markus, Restrained domination in graphs, *Discrete Math.* **203** (2009), 61–69.

[2] M. Garey and D. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, Freeman, New York, 1979.

[3] F. Harary and T. Haynes, Double domination in graphs, *Ars Combin.* **55** (2000), 201–213.

[4] T. Haynes, S. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[5] T. Haynes, S. Hedetniemi, and P.J. Slater (eds.), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[6] S.M. Hosseini Moghaddam, D.A. Mojdeh, B. Samadib, and L. Volkmann, On the signed 2-independence number of graphs, *Electron. J. Graph Theory Appl.* **5** (1) (2017), 36–42.

[7] N. Jafari Rad, A note on the edge Roman domination in trees, *Electron. J. Graph Theory Appl.* **5** (1) (2017), 1–6.

[8] R. Kala and T. Vasantha, Restrained double domination number of a graph, *AKCE Int. J. Graphs Comb.* **5** (2008), 73–82.

[9] V. Kulli, B. Janakiram, and R. Iyer, The cototal domination number of a graph, *J. Discrete Math. Sci. Cryptogr.* **2** (1999), 179–184.

[10] S.J. Seo and P.J. Slater, Open-independent, open-locating-dominating sets, *Electron. J. Graph Theory Appl.* **5** (2) (2017), 179–193.

[11] V. Swaminathan and K. Dharmalingam, Degree equitable domination on graphs, *Kragujevac Journal of Mathematics* **35** (2011), 191–197.

[12] E. Vatandoost and F. Ramezani, On the domination and signed domination numbers of zero-divisor graph, *Electron. J. Graph Theory Appl.* **4** (2) (2016), 148–156.