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On cycle-irregularity strength of ladders and fan graphs

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Abstract

A simple graph $G = (V(G), E(G))$ admits an H-covering if every edge in $E(G)$ belongs to at least one subgraph of G isomorphic to a given graph H. A total k-labeling $\varphi : V(G) \cup E(G) \to \{1, 2, \dots, k\}$ is called to be an *H-irregular total k-labeling* of the graph G admitting an H-covering if for every two different subgraphs H' and H'' isomorphic to H there is $wt_\varphi(H') \neq wt_\varphi(H'')$, where $wt_\varphi(H) = \sum_{v \in V(H)} \varphi(v) + \sum_{e \in E(H)} \varphi(e)$. The *total H-irregularity strength* of a graph G, denoted by $\mathrm{ths}(G, H)$, is the smallest integer k such that G has an H-irregular total k-labeling. In this paper we determine the exact value of the cycle-irregularity strength of ladders and fan graphs.

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1. Introduction

Let G be a connected, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. By a *labeling* we mean any mapping that maps a set of graph elements to a set of numbers (usually positive integers), called *labels*. If the domain is the vertex-set or the edge-set, the labelings are

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called respectively *vertex labelings* or *edge labelings*. If the domain is $V(G) \cup E(G)$ then we call the labeling *total labeling*. The most complete recent survey of graph labelings is [12].

Bača, Jendroľ, Miller and Ryan in [9] defined the total labeling $\varphi : V(G) \cup E(G) \to \{1, 2, \dots, k\}$ to be an *edge irregular total k-labeling* of the graph G if for every two different edges xy and $x'y'$ of G one has

$$wt(xy) = \varphi(x) + \varphi(xy) + \varphi(y) \neq wt(x'y') = \varphi(x') + \varphi(x'y') + \varphi(y').$$

The *total edge irregularity strength*, $\text{tes}(G)$, is defined as the minimum k for which G has an edge irregular total k-labeling.

Ivančo and Jendroľ [14] posed a conjecture that for arbitrary graph G different from K_5 and maximum degree $\Delta(G)$,

$$\text{tes}(G) = \max\left\{\left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil\right\}.$$

This conjecture has been verified for complete graphs and complete bipartite graphs in [15] and [16], for the Cartesian, categorical and strong products of two paths in [17, 3, 2], for the categorical product of two cycles in [4], for generalized Petersen graphs in [13], for generalized prisms in [10], for corona product of a path with certain graphs in [19] and for large dense graphs with $(|E(G)| + 2)/3 \leq (\Delta(G) + 1)/2$ in [11].

There are several modifications of irregularity strength, namely the *total vertex irregularity strength* introduced in [9] and the *edge irregularity strength* introduced in [1]. In [20] there is confirmed the conjecture proposed by Nurdin, Baskoro, Salman and Gaos [18] for all trees with maximum degree five. The edge irregularity strength of some chain graphs is determined in [5].

An *edge-covering* of G is a family of subgraphs $H_1, H_2, \dots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs H_i, $i = 1, 2, \dots, t$. Then it is said that G admits an $(H_1, H_2, \dots, H_t)$-*(edge) covering*. If every subgraph H_i is isomorphic to a given graph H, then the graph G admits an *H-covering*. Note, that in this case every subgraph isomorphic to H must be in the H-covering.

Let G be a graph admitting H-covering. For the subgraph $H \subseteq G$ under the total k-labeling φ, we define the associated H-weight as

$$wt_\varphi(H) = \sum_{v \in V(H)} \varphi(v) + \sum_{e \in E(H)} \varphi(e).$$

A total k-labeling φ is called an *H-irregular total k-labeling* of the graph G if for every two different subgraphs H' and H'' isomorphic to H there is $wt_\varphi(H') \neq wt_\varphi(H'')$. The *total H-irregularity strength* of a graph G, denoted $\text{ths}(G, H)$, is the smallest integer k such that G has an H-irregular total k-labeling. If H is isomorphic to K_2, then the K_2-irregular total k-labeling is isomorphic to the edge irregular total k-labeling and thus the total K_2-irregularity strength of a graph G is equivalent to the total edge irregularity strength, that is $\text{ths}(G, K_2) = \text{tes}(G)$.

Analogously we can define an H-irregular edge k-labeling and an H-irregular vertex k-labeling. For the subgraph $H \subseteq G$ under the vertex k-labeling α, $\alpha : V(G) \to \{1, 2, \dots, k\}$, the associated H-weight is defined as

$$wt_\alpha(H) = \sum_{v \in V(H)} \alpha(v)$$

and under the edge k-labeling β, $\beta : E(G) \to \{1, 2, \dots, k\}$, we define the associated H-weight

$$wt_\beta(H) = \sum_{e \in E(H)} \beta(e).$$

A vertex k-labeling α is called an *H-irregular vertex k-labeling* of the graph G if for every two different subgraphs H' and H'' isomorphic to H there is $wt_\alpha(H') \neq wt_\alpha(H'')$. The *vertex H-irregularity strength* of a graph G, denoted by $\mathrm{vhs}(G, H)$, is the smallest integer k such that G has an H-irregular vertex k-labeling. Note, that $\mathrm{vhs}(G, H) = \infty$ if there exist two subgraphs in G isomorphic to H that have the same vertex sets. An edge k-labeling β is called an *H-irregular edge k-labeling* of the graph G if for every two different subgraphs H' and H'' isomorphic to H there is $wt_\beta(H') \neq wt_\beta(H'')$. The *edge H-irregularity strength* of a graph G, denoted by $\mathrm{ehs}(G, H)$, is the smallest integer k such that G has an H-irregular edge k-labeling.

The notion of the vertex (edge) H-irregularity strength was introduced in [6]. The total H-irregularity strength was defined in [7] and its lower bound is given by the following theorem.

Theorem 1.1. [7] *Let G be a graph admitting an H-covering given by t subgraphs isomorphic to H. Then*

$$\mathrm{ths}(G, H) \geq \left\lceil 1 + \frac{t-1}{|V(H)| + |E(H)|} \right\rceil.$$

The precise value of the total H-irregularity strength of G-amalgamation of graphs is given in [8] and it proves that the lower bound in Theorem 1.1 is tight.

Let G be a graph admitting H-covering. By the symbol $\mathbb{H}_m^S = (H_1^S, H_2^S, \dots, H_m^S)$, we denote the set of all subgraphs of G isomorphic to H such that the graph S, $S \ncong H$, is their maximum common subgraph. Thus $V(S) \subset V(H_i^S)$ and $E(S) \subset E(H_i^S)$ for every $i = 1, 2, \dots, m$. The next theorem presented in [7] gives another lower bound of the total H-irregularity strength.

Theorem 1.2. [7] *Let G be a graph admitting an H-covering. Let S_i, $i = 1, 2, \dots, z$, be all subgraphs of G such that S_i is a maximum common subgraph of m_i, $m_i \geq 2$, subgraphs of G isomorphic to H. Then*

$$\mathrm{ths}(G, H) \geq \max\left\{ \left\lceil 1 + \frac{m_1 - 1}{|V(H/S_1)| + |E(H/S_1)|} \right\rceil, \dots, \left\lceil 1 + \frac{m_z - 1}{|V(H/S_z)| + |E(H/S_z)|} \right\rceil \right\}.$$

In this paper we determine the exact value of the cycle-irregularity strength of ladders and fan graphs.

2. Total cycle-irregular labelings of ladders

Let $L_n \cong P_n \Box P_2$, $n \geq 3$, be a ladder with the vertex set $V(L_n) = \{v_i, u_i : i = 1, 2, \dots, n\}$ and the edge set $E(L_n) = \{v_i v_{i+1}, u_i u_{i+1} : i = 1, 2, \dots, n-1\} \cup \{v_i u_i : i = 1, 2, \dots, n\}$.

In [7] is determined the exact value of the total cycle-irregularity strength of ladders when the cycle is either of length 4 or 6.

Theorem 2.1. [7] *Let $L_n \cong P_n \square P_2$, $n \geq 3$, be a ladder admitting a C_{2m}-covering, $m = 2, 3$. Then*

$$\text{ths}(L_n, C_{2m}) = \left\lceil \tfrac{3m+n}{4m} \right\rceil.$$

In this section we extend the previous result for all feasible cycle-coverings.

Theorem 2.2. *Let $L_n \cong P_n \square P_2$, $n \geq 3$, be a ladder admitting a C_{2m}-covering, $2 \leq m \leq \lceil (n+1)/2 \rceil$. Then*

$$\text{ths}(L_n, C_{2m}) = \left\lceil \tfrac{3m+n}{4m} \right\rceil.$$

Proof. It is easy to see that the ladder $L_n \cong P_n \square P_2$, $n \geq 3$, admits a C_{2m}-covering for $m = 2, 3, \dots, \lceil (n+1)/2 \rceil$. Put $k = \left\lceil \frac{3m+n}{4m} \right\rceil$. According to Theorem 1.1 k is the lower bound of $\text{ths}(L_n, C_{2m})$. In order to show the converse inequality, it only remains to describe a C_{2m}-irregular total k-labeling $\varphi_m : V(L_n) \cup E(L_n) \to \{1, 2, \dots, k\}$ as follows

$$\begin{aligned}
\varphi_m(v_i) &= \left\lceil \tfrac{i+3m}{4m} \right\rceil && \text{for } i = 1, 2, \dots, n,\\
\varphi_m(u_i) &= \begin{cases} \left\lceil \frac{i}{4m} \right\rceil & \text{for } i \equiv 0, 3m \pmod{4m},\ i = 3m, 4m, 7m, 8m, \dots, n,\\ \left\lceil \frac{i+2m-1}{4m} \right\rceil & \text{for } i \not\equiv 0, 3m \pmod{4m},\ i = 1, 2, \dots, n, \end{cases}\\
\varphi_m(v_i v_{i+1}) &= \left\lceil \tfrac{i+m}{4m} \right\rceil && \text{for } i = 1, 2, \dots, n-1,\\
\varphi_m(u_i u_{i+1}) &= \left\lceil \tfrac{i+1}{4m} \right\rceil && \text{for } i = 1, 2, \dots, n-1,\\
\varphi_m(v_i u_i) &= \left\lceil \tfrac{i+2m}{4m} \right\rceil && \text{for } i = 1, 2, \dots, n.
\end{aligned}$$

We can see that all edge labels and vertex labels are at most k.

Every cycle C_{2m} in L_n is of the form

$$C_{2m}^i = v_i v_{i+1} \dots v_{i+m-1} u_{i+m-1} u_{i+m-2} \dots u_i v_i,$$

where $i = 1, 2, \dots, n-m+1$. It is easy to see that every edge of L_n belongs to at least one cycle C_{2m}^i if $m = 2, 3, \dots, \lceil (n+1)/2 \rceil$.

For the C_{2m}-weight of the cycle C_{2m}^i, $i = 1, 2, \dots, n-m+1$, under the total labeling φ_m, we get

$$\begin{aligned}
wt_{\varphi_m}(C_{2m}^i) &= \sum_{v \in V(C_{2m}^i)} \varphi_m(v) + \sum_{e \in E(C_{2m}^i)} \varphi_m(e)\\
&= \sum_{j=0}^{m-1} \varphi_m(v_{i+j}) + \sum_{j=0}^{m-1} \varphi_m(u_{i+j}) + \sum_{j=0}^{m-2} \varphi_m(v_{i+j} v_{i+j+1}) + \sum_{j=0}^{m-2} \varphi_m(u_{i+j} u_{i+j+1})\\
&\quad + \varphi_m(v_i u_i) + \varphi_m(v_{i+m-1} u_{i+m-1}) \qquad (1)
\end{aligned}$$

and for the C_{2m}-weight of the cycle C_{2m}^{i+1}, $i = 1, 2, \dots, n-m$, we obtain

$$wt_{\varphi_m}(C_{2m}^{i+1}) = \sum_{v \in V(C_{2m}^{i+1})} \varphi_m(v) + \sum_{e \in E(C_{2m}^{i+1})} \varphi_m(e)$$

$$= \sum_{j=1}^{m} \varphi_m(v_{i+j}) + \sum_{j=1}^{m} \varphi_m(u_{i+j}) + \sum_{j=1}^{m-1} \varphi_m(v_{i+j}v_{i+j+1}) + \sum_{j=1}^{m-1} \varphi_m(u_{i+j}u_{i+j+1}) + \varphi_m(v_{i+1}u_{i+1}) + \varphi_m(v_{i+m}u_{i+m}). \quad (2)$$

Now we count the difference between the C_{2m}-weights of the cycle C_{2m}^{i+1} and C_{2m}^{i} for $i = 1, 2, \dots, n-m$. According to (1) and (2) we get

$$\begin{aligned} wt_{\varphi_m}(C_{2m}^{i+1}) - wt_{\varphi_m}(C_{2m}^{i}) =& \varphi_m(v_{i+1}u_{i+1}) + \varphi_m(v_{i+m-1}v_{i+m}) + \varphi_m(v_{i+m}) + \varphi_m(v_{i+m}u_{i+m}) \\ &+ \varphi_m(u_{i+m}) + \varphi_m(u_{i+m-1}u_{i+m}) - \varphi_m(v_i) - \varphi_m(v_iv_{i+1}) \\ &- \varphi_m(v_iu_i) - \varphi_m(u_i) - \varphi_m(u_iu_{i+1}) - \varphi_m(v_{i+m-1}u_{i+m-1}). \end{aligned}$$

Let us distinguish four cases.

Case 1. $i \equiv 0 \pmod{4m}$

For the difference of weights of cycles we get

$$\begin{aligned} wt_{\varphi_m}(C_{2m}^{i+1}) - wt_{\varphi_m}(C_{2m}^{i}) =& \left\lceil \tfrac{i+1+2m}{4m} \right\rceil + \left\lceil \tfrac{i+2m-1}{4m} \right\rceil + \left\lceil \tfrac{i+4m}{4m} \right\rceil + \left\lceil \tfrac{i+3m}{4m} \right\rceil + \left\lceil \tfrac{i+3m-1}{4m} \right\rceil + \left\lceil \tfrac{i+m}{4m} \right\rceil \\ &- \left\lceil \tfrac{i+3m}{4m} \right\rceil - \left\lceil \tfrac{i+m}{4m} \right\rceil - \left\lceil \tfrac{i+2m}{4m} \right\rceil - \left\lceil \tfrac{i}{4m} \right\rceil - \left\lceil \tfrac{i+1}{4m} \right\rceil - \left\lceil \tfrac{i+3m-1}{4m} \right\rceil \\ =& \left\lceil \tfrac{i+2m+1}{4m} \right\rceil + \left\lceil \tfrac{i+2m-1}{4m} \right\rceil + 1 - \left\lceil \tfrac{i+2m}{4m} \right\rceil - \left\lceil \tfrac{i+1}{4m} \right\rceil. \end{aligned}$$

Since $i = 4mt$, $t = 1, 2, \dots$, thus

$$\begin{aligned} wt_{\varphi_m}(C_{2m}^{i+1}) - wt_{\varphi_m}(C_{2m}^{i}) =& \left\lceil \tfrac{4mt+2m+1}{4m} \right\rceil + \left\lceil \tfrac{4mt+2m-1}{4m} \right\rceil + 1 - \left\lceil \tfrac{4mt+2m}{4m} \right\rceil - \left\lceil \tfrac{4mt+1}{4m} \right\rceil \\ =& t + \left\lceil \tfrac{2m+1}{4m} \right\rceil + t + \left\lceil \tfrac{2m-1}{4m} \right\rceil + 1 - t - \left\lceil \tfrac{2m}{4m} \right\rceil - t - \left\lceil \tfrac{1}{4m} \right\rceil = 1. \end{aligned}$$

Case 2. $i \equiv 2m \pmod{4m}$

For the difference of weights of cycles we get

$$\begin{aligned} wt_{\varphi_m}(C_{2m}^{i+1}) - wt_{\varphi_m}(C_{2m}^{i}) =& \left\lceil \tfrac{i+1+2m}{4m} \right\rceil + \left\lceil \tfrac{i+2m-1}{4m} \right\rceil + \left\lceil \tfrac{i+4m}{4m} \right\rceil + \left\lceil \tfrac{i+3m}{4m} \right\rceil + \left\lceil \tfrac{i+m}{4m} \right\rceil + \left\lceil \tfrac{i+m}{4m} \right\rceil \\ &- \left\lceil \tfrac{i+3m}{4m} \right\rceil - \left\lceil \tfrac{i+m}{4m} \right\rceil - \left\lceil \tfrac{i+2m}{4m} \right\rceil - \left\lceil \tfrac{i+2m-1}{4m} \right\rceil - \left\lceil \tfrac{i+1}{4m} \right\rceil - \left\lceil \tfrac{i+3m-1}{4m} \right\rceil \\ =& \left\lceil \tfrac{i+2m+1}{4m} \right\rceil + 1 + \left\lceil \tfrac{i}{4m} \right\rceil + \left\lceil \tfrac{i+m}{4m} \right\rceil - \left\lceil \tfrac{i+2m}{4m} \right\rceil - \left\lceil \tfrac{i+1}{4m} \right\rceil - \left\lceil \tfrac{i+3m-1}{4m} \right\rceil. \end{aligned}$$

For $i = 4mt + 2m$, $t = 1, 2, \dots$, we get

$$\begin{aligned} wt_{\varphi_m}(C_{2m}^{i+1}) - wt_{\varphi_m}(C_{2m}^{i}) =& \left\lceil \tfrac{4mt+2m+2m+1}{4m} \right\rceil + 1 + \left\lceil \tfrac{4mt+2m}{4m} \right\rceil + \left\lceil \tfrac{4mt+2m+m}{4m} \right\rceil \\ &- \left\lceil \tfrac{4mt+2m+2m}{4m} \right\rceil - \left\lceil \tfrac{4mt+2m+1}{4m} \right\rceil - \left\lceil \tfrac{4mt+2m+3m-1}{4m} \right\rceil \\ =& t + 1 + \left\lceil \tfrac{1}{4m} \right\rceil + 1 + t + \left\lceil \tfrac{2m}{4m} \right\rceil + t + \left\lceil \tfrac{3m}{4m} \right\rceil - t - 1 - t - \left\lceil \tfrac{2m+1}{4m} \right\rceil \\ &- t - 1 - \left\lceil \tfrac{m-1}{4m} \right\rceil = 1. \end{aligned}$$

Case 3. $i \equiv 3m \pmod{4m}$

Now

$$wt_{\varphi_m}(C_{2m}^{i+1}) - wt_{\varphi_m}(C_{2m}^{i}) = \left\lceil \tfrac{i+1+2m}{4m} \right\rceil + \left\lceil \tfrac{i+2m-1}{4m} \right\rceil + \left\lceil \tfrac{i+4m}{4m} \right\rceil + \left\lceil \tfrac{i+3m}{4m} \right\rceil + \left\lceil \tfrac{i+m}{4m} \right\rceil + \left\lceil \tfrac{i+m}{4m} \right\rceil$$

$$- \left\lceil \tfrac{i+3m}{4m} \right\rceil - \left\lceil \tfrac{i+m}{4m} \right\rceil - \left\lceil \tfrac{i+2m}{4m} \right\rceil - \left\lceil \tfrac{i}{4m} \right\rceil - \left\lceil \tfrac{i+1}{4m} \right\rceil - \left\lceil \tfrac{i+3m-1}{4m} \right\rceil$$
$$= \left\lceil \tfrac{i+2m+1}{4m} \right\rceil + \left\lceil \tfrac{i+2m-1}{4m} \right\rceil + 1 + \left\lceil \tfrac{i+m}{4m} \right\rceil - \left\lceil \tfrac{i+2m}{4m} \right\rceil - \left\lceil \tfrac{i+1}{4m} \right\rceil - \left\lceil \tfrac{i+3m-1}{4m} \right\rceil.$$

Since $i = 4mt + 3m$, $t = 1, 2, \dots$, it follows

$$\begin{aligned} wt_{\varphi_m}(C_{2m}^{i+1}) - wt_{\varphi_m}(C_{2m}^{i}) &= \left\lceil \tfrac{4mt+3m+2m+1}{4m} \right\rceil + \left\lceil \tfrac{4mt+3m+2m-1}{4m} \right\rceil + 1 + \left\lceil \tfrac{4mt+3m+m}{4m} \right\rceil \\ &\quad - \left\lceil \tfrac{4mt+3m+2m}{4m} \right\rceil - \left\lceil \tfrac{4mt+3m+1}{4m} \right\rceil - \left\lceil \tfrac{4mt+3m+3m-1}{4m} \right\rceil \\ &= t + 1 + \left\lceil \tfrac{m+1}{4m} \right\rceil + t + 1 + \left\lceil \tfrac{m-1}{4m} \right\rceil + 1 + t + 1 - t - 1 - \left\lceil \tfrac{m}{4m} \right\rceil - t \\ &\quad - \left\lceil \tfrac{3m+1}{4m} \right\rceil - t - 1 - \left\lceil \tfrac{2m-1}{4m} \right\rceil = 1. \end{aligned}$$

Case 4. $i \not\equiv 0, 2m, 3m \pmod{4m}$

In this case for the difference of weights of cycles we obtain

$$\begin{aligned} wt_{\varphi_m}(C_{2m}^{i+1}) - wt_{\varphi_m}(C_{2m}^{i}) &= \left\lceil \tfrac{i+1+2m}{4m} \right\rceil + \left\lceil \tfrac{i+2m-1}{4m} \right\rceil + \left\lceil \tfrac{i+4m}{4m} \right\rceil + \left\lceil \tfrac{i+3m}{4m} \right\rceil + \left\lceil \tfrac{i+3m-1}{4m} \right\rceil + \left\lceil \tfrac{i+m}{4m} \right\rceil \\ &\quad - \left\lceil \tfrac{i+3m}{4m} \right\rceil - \left\lceil \tfrac{i+m}{4m} \right\rceil - \left\lceil \tfrac{i+2m}{4m} \right\rceil - \left\lceil \tfrac{i+2m-1}{4m} \right\rceil - \left\lceil \tfrac{i+1}{4m} \right\rceil - \left\lceil \tfrac{i+3m-1}{4m} \right\rceil \\ &= \left\lceil \tfrac{i+2m+1}{4m} \right\rceil + \left\lceil \tfrac{i+4m}{4m} \right\rceil - \left\lceil \tfrac{i+2m}{4m} \right\rceil - \left\lceil \tfrac{i+1}{4m} \right\rceil. \end{aligned}$$

Let $i = 4mt + s$, $t = 0, 1, 2, \dots$ and $1 \le s \le 4m - 1$, $s \ne 2m, 3m$. Then we have

$$\begin{aligned} wt_{\varphi_m}(C_{2m}^{i+1}) - wt_{\varphi_m}(C_{2m}^{i}) &= \left\lceil \tfrac{4mt+s+2m+1}{4m} \right\rceil + \left\lceil \tfrac{4mt+s+4m}{4m} \right\rceil - \left\lceil \tfrac{4mt+s+2m}{4m} \right\rceil - \left\lceil \tfrac{4mt+s+1}{4m} \right\rceil \\ &= t + \left\lceil \tfrac{s+2m+1}{4m} \right\rceil + t + 1 + \left\lceil \tfrac{s}{4m} \right\rceil - t - \left\lceil \tfrac{s+2m}{4m} \right\rceil - t - \left\lceil \tfrac{s+1}{4m} \right\rceil \\ &= \left\lceil \tfrac{s+2m+1}{4m} \right\rceil + \left\lceil \tfrac{s}{4m} \right\rceil - \left\lceil \tfrac{s+2m}{4m} \right\rceil - \left\lceil \tfrac{s+1}{4m} \right\rceil + 1. \end{aligned}$$

If $1 \le s \le 2m - 1$ then

$$\left\lceil \tfrac{s+2m+1}{4m} \right\rceil = 1, \quad \left\lceil \tfrac{s}{4m} \right\rceil = 1, \quad \left\lceil \tfrac{s+2m}{4m} \right\rceil = 1 \quad \text{and} \quad \left\lceil \tfrac{s+1}{4m} \right\rceil = 1.$$

If $2m + 1 \le s \le 3m - 1$ or $3m + 1 \le s \le 4m - 1$ then

$$\left\lceil \tfrac{s+2m+1}{4m} \right\rceil = 2, \quad \left\lceil \tfrac{s}{4m} \right\rceil = 1, \quad \left\lceil \tfrac{s+2m}{4m} \right\rceil = 2 \quad \text{and} \quad \left\lceil \tfrac{s+1}{4m} \right\rceil = 1.$$

We can see that for every value of parameter s

$$wt_{\varphi_m}(C_{2m}^{i+1}) - wt_{\varphi_m}(C_{2m}^{i}) = 1.$$

Previous cases prove that the labeling φ_m is the desired C_{2m}-irregular total k-labeling of L_n. This concludes the proof. □

3. Total cycle-irregular labelings of fan graphs

A *fan graph* F_n, $n \ge 2$, is a graph obtained by joining all vertices of a path P_n to a further vertex. Thus F_n contains $n+1$ vertices, say, $u, v_1, v_2, \dots, v_n$ and $2n-1$ edges uv_i, $i = 1, 2, \dots, n$, and v_iv_{i+1}, $i = 1, 2, \dots, n-1$.

In [7] was given the exact value of the total C_3-irregularity strength of the fan graph F_n.

Theorem 3.1. [7] *Let F_n, $n \geq 2$, be a fan graph on $n+1$ vertices. Then*

$$\text{ths}(F_n, C_3) = \left\lceil \frac{n+3}{5} \right\rceil.$$

The next theorem completes this result for arbitrary cycle-covering.

Theorem 3.2. *Let F_n be a fan graph on $n+1$ vertices, $n \geq 2$ and $3 \leq m \leq \lceil (n+3)/2 \rceil$. Then*

$$\text{ths}(F_n, C_m) = \left\lceil \frac{n+m}{2m-1} \right\rceil.$$

Proof. Clearly, for every m, $3 \leq m \leq \lceil (n+3)/2 \rceil$, the fan graph F_n admits a C_m-covering with exactly $n-m+2$ cycles C_m. In view of the lower bound from Theorem 1.2 it suffices to prove the existence of a C_m-irregular total labeling $\varphi : V(F_n) \cup E(F_n) \to \{1, 2, \dots, \lceil (n+m)/(2m-1) \rceil\}$ such that $wt_\varphi(C_m^j) \neq wt_\varphi(C_m^i)$ for every $i, j = 1, 2, \dots, n-m+2$, $j \neq i$. We describe the irregular total labeling φ_m in the following way

$$\varphi_m(u) = 1,$$

$$\varphi_m(v_i) = \begin{cases} \lceil \frac{i+2}{2m-1} \rceil & \text{for } i \not\equiv m+1 \pmod{2m-1}, i = 1, 2, \dots, n, \\ \lceil \frac{i+2}{2m-1} \rceil + 1 & \text{for } i \equiv m+1 \pmod{2m-1}, i = m+1, 3m, \dots, n, \end{cases}$$

$$\varphi_m(v_i v_{i+1}) = \begin{cases} \lceil \frac{i+m}{2m-1} \rceil & \text{for } i \not\equiv m+1, 2m-3, 2m-1 \pmod{2m-1}, \\ & \quad i = 1, 2, \dots, n, \\ \lceil \frac{i+m}{2m-1} \rceil - 1 & \text{for } i \equiv m+1, 2m-3, 2m-1 \pmod{2m-1}, \\ & \quad i = m+1, 2m-3, 2m-1, 3m, 4m-4, 4m-2, \dots, n, \end{cases}$$

$$\varphi_m(v_i u) = \begin{cases} \lceil \frac{i+m}{2m-1} \rceil & \text{for } i \not\equiv m+1, 2m-2 \pmod{2m-1}, i = 1, 2, \dots, n, \\ \lceil \frac{i+m}{2m-1} \rceil - 1 & \text{for } i \equiv m+1, 2m-2 \pmod{2m-1}, \\ & \quad i = m+1, 2m-2, 3m, 4m-3, \dots, n. \end{cases}$$

Evidently, every edge label and vertex label is not greater than $\lceil (n+m)/(2m-1) \rceil$.

Every cycle C_m in F_n is of the form

$$C_m^i = v_i v_{i+1} \dots v_{i+m-2} u v_i,$$

where $i = 1, 2, \dots, n-m+2$.

For the C_m-weight of the cycle C_m^i, $i = 1, 2, \dots, n-m+2$, under the total labeling φ_m, we get

$$\begin{aligned} wt_{\varphi_m}(C_m^i) &= \sum_{v \in V(C_m^i)} \varphi_m(v) + \sum_{e \in E(C_m^i)} \varphi_m(e) \\ &= \sum_{j=0}^{m-2} \varphi_m(v_{i+j}) + \varphi_m(u) + \sum_{j=0}^{m-3} \varphi_m(v_{i+j} v_{i+j+1}) + \varphi_m(v_i u) + \varphi_m(v_{i+m-2} u) \qquad (3) \end{aligned}$$

and for the C_m-weight of the cycle C_m^{i+1}, $i = 1, 2, \dots, n-m+1$, we obtain

$$\begin{aligned} wt_{\varphi_m}(C_m^{i+1}) &= \sum_{v \in V(C_m^{i+1})} \varphi_m(v) + \sum_{e \in E(C_m^{i+1})} \varphi_m(e) \\ &= \sum_{j=1}^{m-1} \varphi_m(v_{i+j}) + \varphi_m(u) + \sum_{j=1}^{m-2} \varphi_m(v_{i+j}v_{i+j+1}) + \varphi_m(v_{i+1}u) + \varphi_m(v_{i+m-1}u). \quad (4) \end{aligned}$$

Now we count the difference between the C_m-weights of the cycle C_m^{i+1} and C_m^i for $i = 1, 2, \dots, n-m+1$. According to (3) and (4) we get

$$\begin{aligned} wt_{\varphi_m}(C_m^{i+1}) - wt_{\varphi_m}(C_m^i) =& \varphi_m(v_{i+m-1}) + \varphi_m(v_{i+m-2}v_{i+m-1}) + \varphi_m(v_{i+m-1}u) + \varphi_m(v_{i+1}u) \\ & - \varphi_m(v_i) - \varphi_m(v_iv_{i+1}) - \varphi_m(v_iu) - \varphi_m(v_{i+m-2}u). \end{aligned}$$

Let us distinguish nine cases.

Case 1. $i \equiv 2 \pmod{(2m-1)}$, i.e., $i = 2 + (2m-1)t$, $t = 0, 1, \dots$, then

$$\begin{aligned} wt_{\varphi_m}(C_m^{i+1}) - wt_{\varphi_m}(C_m^i) =& \varphi_m(v_{m+1+(2m-1)t}) + \varphi_m(v_{m+(2m-1)t}v_{m+1+(2m-1)t}) \\ & + \varphi_m(v_{m+1+(2m-1)t}u) + \varphi_m(v_{3+(2m-1)t}u) - \varphi_m(v_{2+(2m-1)t}) \\ & - \varphi_m(v_{2+(2m-1)t}v_{3+(2m-1)t}) - \varphi_m(v_{2+(2m-1)t}u) \\ & - \varphi_m(v_{m+(2m-1)t}u) \\ =& \left\lceil \tfrac{m+3+(2m-1)t}{2m-1} \right\rceil + 1 + \left\lceil \tfrac{2m+(2m-1)t}{2m-1} \right\rceil + \left\lceil \tfrac{2m+1+(2m-1)t}{2m-1} \right\rceil - 1 \\ & + \left\lceil \tfrac{m+3+(2m-1)t}{2m-1} \right\rceil - \left\lceil \tfrac{4+(2m-1)t}{2m-1} \right\rceil - \left\lceil \tfrac{m+2+(2m-1)t}{2m-1} \right\rceil \\ & - \left\lceil \tfrac{m+2+(2m-1)t}{2m-1} \right\rceil - \left\lceil \tfrac{2m+(2m-1)t}{2m-1} \right\rceil \\ =& (1+t) + 1 + (2+t) + (2+t) - 1 + (1+t) - (1+t) \\ & - (1+t) - (1+t) - (2+t) = 1. \end{aligned}$$

Case 2. $i \equiv 3 \pmod{(2m-1)}$, i.e., $i = 3 + (2m-1)t$, $t = 0, 1, \dots$, then we get

$$\begin{aligned} wt_{\varphi_m}(C_m^{i+1}) - wt_{\varphi_m}(C_m^i) =& \varphi_m(v_{m+2+(2m-1)t}) + \varphi_m(v_{m+1+(2m-1)t}v_{m+2+(2m-1)t}) \\ & + \varphi_m(v_{m+2+(2m-1)t}u) + \varphi_m(v_{4+(2m-1)t}u) - \varphi_m(v_{3+(2m-1)t}) \\ & - \varphi_m(v_{3+(2m-1)t}v_{4+(2m-1)t}) - \varphi_m(v_{3+(2m-1)t}u) \\ & - \varphi_m(v_{m+1+(2m-1)t}u) \\ =& \left\lceil \tfrac{m+4+(2m-1)t}{2m-1} \right\rceil + \left\lceil \tfrac{2m+1+(2m-1)t}{2m-1} \right\rceil - 1 + \left\lceil \tfrac{2m+2+(2m-1)t}{2m-1} \right\rceil \\ & + \left\lceil \tfrac{m+4+(2m-1)t}{2m-1} \right\rceil - \left\lceil \tfrac{5+(2m-1)t}{2m-1} \right\rceil - \left\lceil \tfrac{m+3+(2m-1)t}{2m-1} \right\rceil \\ & - \left\lceil \tfrac{m+3+(2m-1)t}{2m-1} \right\rceil - \left\lceil \tfrac{2m+1+(2m-1)t}{2m-1} \right\rceil + 1 \\ =& (1+t) + (2+t) - 1 + (2+t) + (1+t) - (1+t) \end{aligned}$$

$$-(1+t)-(1+t)-(2+t)+1=1.$$

Case 3. $i \equiv m-1 \pmod{(2m-1)}$, i.e., $i = m-1+(2m-1)t$, $t = 0, 1, \ldots$, then we get

$$\begin{aligned} wt_{\varphi_m}(C_m^{i+1}) - wt_{\varphi_m}(C_m^i) =& \varphi_m(v_{2m-2+(2m-1)t}) + \varphi_m(v_{2m-3+(2m-1)t}v_{2m-2+(2m-1)t}) \\ &+ \varphi_m(v_{2m-2+(2m-1)t}u) + \varphi_m(v_{m+(2m-1)t}u) \\ &- \varphi_m(v_{m-1+(2m-1)t}) - \varphi_m(v_{m-1+(2m-1)t}v_{m+(2m-1)t}) \\ &- \varphi_m(v_{m-1+(2m-1)t}u) - \varphi_m(v_{2m-3+(2m-1)t}u) \\ =& \left\lceil \tfrac{2m+(2m-1)t}{2m-1} \right\rceil + \left\lceil \tfrac{3m-3+(2m-1)t}{2m-1} \right\rceil - 1 + \left\lceil \tfrac{3m-2+(2m-1)t}{2m-1} \right\rceil \\ &- 1 + \left\lceil \tfrac{2m+(2m-1)t}{2m-1} \right\rceil - \left\lceil \tfrac{m+1+(2m-1)t}{2m-1} \right\rceil - \left\lceil \tfrac{2m-1+(2m-1)t}{2m-1} \right\rceil \\ &- \left\lceil \tfrac{2m-1+(2m-1)t}{2m-1} \right\rceil - \left\lceil \tfrac{3m-3+(2m-1)t}{2m-1} \right\rceil \\ =& (2+t) + (2+t) - 1 + (2+t) - 1 + (2+t) - (1+t) \\ &- (1+t) - (1+t) - (2+t) = 1. \end{aligned}$$

Case 4. $i \equiv m \pmod{(2m-1)}$, i.e., $i = m+(2m-1)t$, $t = 0, 1, \ldots$ In this case holds

$$\begin{aligned} wt_{\varphi_m}(C_m^{i+1}) - wt_{\varphi_m}(C_m^i) =& \varphi_m(v_{2m-1+(2m-1)t}) + \varphi_m(v_{2m-2+(2m-1)t}v_{2m-1+(2m-1)t}) \\ &+ \varphi_m(v_{2m-1+(2m-1)t}u) + \varphi_m(v_{m+1+(2m-1)t}u) - \varphi_m(v_{m+(2m-1)t}) \\ &- \varphi_m(v_{m+(2m-1)t}v_{m+1+(2m-1)t}) - \varphi_m(v_{m+(2m-1)t}u) \\ &- \varphi_m(v_{2m-2+(2m-1)t}u) \\ =& \left\lceil \tfrac{2m+1+(2m-1)t}{2m-1} \right\rceil + \left\lceil \tfrac{3m-2+(2m-1)t}{2m-1} \right\rceil + \left\lceil \tfrac{3m-1+(2m-1)t}{2m-1} \right\rceil \\ &+ \left\lceil \tfrac{2m+1+(2m-1)t}{2m-1} \right\rceil - 1 - \left\lceil \tfrac{m+2+(2m-1)t}{2m-1} \right\rceil - \left\lceil \tfrac{2m+(2m-1)t}{2m-1} \right\rceil \\ &- \left\lceil \tfrac{2m+(2m-1)t}{2m-1} \right\rceil - \left\lceil \tfrac{3m-2+(2m-1)t}{2m-1} \right\rceil + 1 \\ =& (2+t) + (2+t) + (2+t) + (2+t) - 1 - (1+t) - (2+t) \\ &- (2+t) - (2+t) + 1 = 1. \end{aligned}$$

Case 5. $i \equiv m+1 \pmod{(2m-1)}$, i.e., $i = m+1+(2m-1)t$, $t = 0, 1, \ldots$, thus

$$\begin{aligned} wt_{\varphi_m}(C_m^{i+1}) - wt_{\varphi_m}(C_m^i) =& \varphi_m(v_{2m+(2m-1)t}) + \varphi_m(v_{2m-1+(2m-1)t}v_{2m+(2m-1)t}) \\ &+ \varphi_m(v_{2m+(2m-1)t}u) + \varphi_m(v_{m+2+(2m-1)t}u) - \varphi_m(v_{m+1+(2m-1)t}) \\ &- \varphi_m(v_{m+1+(2m-1)t}v_{m+2+(2m-1)t}) - \varphi_m(v_{m+1+(2m-1)t}u) \\ &- \varphi_m(v_{2m-1+(2m-1)t}u) \\ =& \left\lceil \tfrac{2m+2+(2m-1)t}{2m-1} \right\rceil + \left\lceil \tfrac{3m-1+(2m-1)t}{2m-1} \right\rceil - 1 + \left\lceil \tfrac{3m+(2m-1)t}{2m-1} \right\rceil \\ &+ \left\lceil \tfrac{2m+2+(2m-1)t}{2m-1} \right\rceil - \left\lceil \tfrac{m+3+(2m-1)t}{2m-1} \right\rceil - 1 - \left\lceil \tfrac{2m+1+(2m-1)t}{2m-1} \right\rceil + 1 \\ &- \left\lceil \tfrac{2m+1+(2m-1)t}{2m-1} \right\rceil + 1 - \left\lceil \tfrac{3m-1+(2m-1)t}{2m-1} \right\rceil \end{aligned}$$

$$\begin{aligned}=&(2+t)+(2+t)-1+(2+t)+(2+t)-(1+t)-1\\ &-(2+t)+1-(2+t)+1-(2+t)=1.\end{aligned}$$

Case 6. $i \equiv 2m-3 \pmod{(2m-1)}$, i.e., $i = 2m-3+(2m-1)t$, $t = 0, 1, \dots$, thus

$$\begin{aligned}wt_{\varphi_m}(C_m^{i+1}) - wt_{\varphi_m}(C_m^i) =&\varphi_m(v_{3m-4+(2m-1)t}) + \varphi_m(v_{3m-5+(2m-1)t}v_{3m-4+(2m-1)t})\\ &+ \varphi_m(v_{3m-4+(2m-1)t}u) + \varphi_m(v_{2m-2+(2m-1)t}u)\\ &- \varphi_m(v_{2m-3+(2m-1)t}) - \varphi_m(v_{2m-3+(2m-1)t}v_{2m-2+(2m-1)t})\\ &- \varphi_m(v_{2m-3+(2m-1)t}u) - \varphi_m(v_{3m-5+(2m-1)t}u)\\ =&\left\lceil\tfrac{3m-2+(2m-1)t}{2m-1}\right\rceil + \left\lceil\tfrac{4m-5+(2m-1)t}{2m-1}\right\rceil + \left\lceil\tfrac{4m-4+(2m-1)t}{2m-1}\right\rceil\\ &+ \left\lceil\tfrac{3m-2+(2m-1)t}{2m-1}\right\rceil - 1 - \left\lceil\tfrac{2m-1+(2m-1)t}{2m-1}\right\rceil - \left\lceil\tfrac{3m-3+(2m-1)t}{2m-1}\right\rceil\\ &+ 1 - \left\lceil\tfrac{3m-3+(2m-1)t}{2m-1}\right\rceil - \left\lceil\tfrac{4m-5+(2m-1)t}{2m-1}\right\rceil\\ =&(2+t)+(2+t)+(2+t)+(2+t)-1-(1+t)\\ &-(2+t)+1-(2+t)-(2+t)=1.\end{aligned}$$

Case 7. $i \equiv 2m-2 \pmod{(2m-1)}$, i.e., $i = 2m-2+(2m-1)t$, $t = 0, 1, \dots$, thus

$$\begin{aligned}wt_{\varphi_m}(C_m^{i+1}) - wt_{\varphi_m}(C_m^i) =&\varphi_m(v_{3m-3+(2m-1)t}) + \varphi_m(v_{3m-4+(2m-1)t}v_{3m-3+(2m-1)t})\\ &+ \varphi_m(v_{3m-3+(2m-1)t}u) + \varphi_m(v_{2m-1+(2m-1)t}u)\\ &- \varphi_m(v_{2m-2+(2m-1)t}) - \varphi_m(v_{2m-2+(2m-1)t}v_{2m-1+(2m-1)t})\\ &- \varphi_m(v_{2m-2+(2m-1)t}u) - \varphi_m(v_{3m-4+(2m-1)t}u)\\ =&\left\lceil\tfrac{3m-1+(2m-1)t}{2m-1}\right\rceil + \left\lceil\tfrac{4m-4+(2m-1)t}{2m-1}\right\rceil + \left\lceil\tfrac{4m-3+(2m-1)t}{2m-1}\right\rceil\\ &+ \left\lceil\tfrac{3m-1+(2m-1)t}{2m-1}\right\rceil - \left\lceil\tfrac{2m+(2m-1)t}{2m-1}\right\rceil - \left\lceil\tfrac{3m-2+(2m-1)t}{2m-1}\right\rceil\\ &- \left\lceil\tfrac{3m-2+(2m-1)t}{2m-1}\right\rceil + 1 - \left\lceil\tfrac{4m-4+(2m-1)t}{2m-1}\right\rceil\\ =&(2+t)+(2+t)+(2+t)+(2+t)-(2+t)-(2+t)\\ &-(2+t)-(2+t)+1-(2+t)=1.\end{aligned}$$

Case 8. $i \equiv 2m-1 \pmod{(2m-1)}$, i.e., $i = 2m-1+(2m-1)t$, $t = 0, 1, \dots$, thus

$$\begin{aligned}wt_{\varphi_m}(C_m^{i+1}) - wt_{\varphi_m}(C_m^i) =&\varphi_m(v_{3m-2+(2m-1)t}) + \varphi_m(v_{3m-3+(2m-1)t}v_{3m-2+(2m-1)t})\\ &+ \varphi_m(v_{3m-2+(2m-1)t}u) + \varphi_m(v_{2m+(2m-1)t}u)\\ &- \varphi_m(v_{2m-1+(2m-1)t}) - \varphi_m(v_{2m-1+(2m-1)t}v_{2m+(2m-1)t})\\ &- \varphi_m(v_{2m-1+(2m-1)t}u) - \varphi_m(v_{3m-3+(2m-1)t}u)\\ =&\left\lceil\tfrac{3m+(2m-1)t}{2m-1}\right\rceil + \left\lceil\tfrac{4m-3+(2m-1)t}{2m-1}\right\rceil + \left\lceil\tfrac{4m-2+(2m-1)t}{2m-1}\right\rceil\\ &+ \left\lceil\tfrac{3m+(2m-1)t}{2m-1}\right\rceil - \left\lceil\tfrac{2m+1+(2m-1)t}{2m-1}\right\rceil - \left\lceil\tfrac{3m-1+(2m-1)t}{2m-1}\right\rceil + 1\end{aligned}$$

$$- \left\lceil \tfrac{3m-1+(2m-1)t}{2m-1} \right\rceil - \left\lceil \tfrac{4m-3+(2m-1)t}{2m-1} \right\rceil$$
$$=(2+t)+(2+t)+(2+t)+(2+t)-(2+t)$$
$$-(2+t)+1-(2+t)-(2+t)=1.$$

Case 9. $i \not\equiv 2,3,m-1,m,m+1,2m-3,2m-2,2m-1 \pmod{(2m-1)}$. Then

$$\begin{aligned} wt_{\varphi_m}(C_m^{i+1}) - wt_{\varphi_m}(C_m^i) =& \varphi_m(v_{i+m-1}) + \varphi_m(v_{i+m-2}v_{i+m-1}) + \varphi_m(v_{i+m-1}u) + \varphi_m(v_{i+1}u) \\ & - \varphi_m(v_i) - \varphi_m(v_iv_{i+1}) - \varphi_m(v_iu) - \varphi_m(v_{i+m-2}u) \\ =& \left\lceil \tfrac{(i+m-1)+2}{2m-1} \right\rceil + \left\lceil \tfrac{(i+m-2)+m}{2m-1} \right\rceil + \left\lceil \tfrac{(i+m-1)+m}{2m-1} \right\rceil + \left\lceil \tfrac{(i+1)+m}{2m-1} \right\rceil \\ & - \left\lceil \tfrac{i+2}{2m-1} \right\rceil - \left\lceil \tfrac{i+m}{2m-1} \right\rceil - \left\lceil \tfrac{i+m}{2m-1} \right\rceil - \left\lceil \tfrac{(i+m-2)+m}{2m-1} \right\rceil \\ =& 2\left\lceil \tfrac{i+m+1}{2m-1} \right\rceil - 2\left\lceil \tfrac{i+m}{2m-1} \right\rceil + \left\lceil \tfrac{i}{2m-1} \right\rceil - \left\lceil \tfrac{i+2}{2m-1} \right\rceil + 1. \end{aligned}$$

Now we distinguish three subcases.

If $i = 1+(2m-1)t$, $t = 0,1,\dots$, then

$$\begin{aligned} wt_{\varphi_m}(C_m^{i+1}) - wt_{\varphi_m}(C_m^i) =& 2\left\lceil \tfrac{1+(2m-1)t+m+1}{2m-1} \right\rceil - 2\left\lceil \tfrac{1+(2m-1)t+m}{2m-1} \right\rceil + \left\lceil \tfrac{1+(2m-1)t}{2m-1} \right\rceil \\ & - \left\lceil \tfrac{1+(2m-1)t+2}{2m-1} \right\rceil + 1 = 2(1+t) - 2(1+t) + (1+t) - (1+t) + 1 \\ =& 1. \end{aligned}$$

If $i = s+(2m-1)t$, $t = 0,1,\dots$ and $4 \le s \le m-2$, then

$$\begin{aligned} wt_{\varphi_m}(C_m^{i+1}) - wt_{\varphi_m}(C_m^i) =& 2\left\lceil \tfrac{s+(2m-1)t+m+1}{2m-1} \right\rceil - 2\left\lceil \tfrac{s+(2m-1)t+m}{2m-1} \right\rceil + \left\lceil \tfrac{s+(2m-1)t}{2m-1} \right\rceil \\ & - \left\lceil \tfrac{s+(2m-1)t+2}{2m-1} \right\rceil + 1 = 2(1+t) - 2(1+t) + (1+t) - (1+t) + 1 \\ =& 1. \end{aligned}$$

If $i = s+(2m-1)t$, $t = 0,1,\dots$ and $m+2 \le s \le 2m-4$, in this case we get

$$\begin{aligned} wt_{\varphi_m}(C_m^{i+1}) - wt_{\varphi_m}(C_m^i) =& 2\left\lceil \tfrac{s+(2m-1)t+m+1}{2m-1} \right\rceil - 2\left\lceil \tfrac{s+(2m-1)t+m}{2m-1} \right\rceil + \left\lceil \tfrac{s+(2m-1)t}{2m-1} \right\rceil \\ & - \left\lceil \tfrac{s+(2m-1)t+2}{2m-1} \right\rceil + 1 = 2(2+t) - 2(2+t) + (1+t) - (1+t) + 1 \\ =& 1. \end{aligned}$$

Thus, according to all these cases we get that

$$wt_{\varphi_m}(C_m^{i+1}) - wt_{\varphi_m}(C_m^i) = 1$$

for every i, $i = 1,2,\dots,n-m+1$. This concludes the proof.

□

4. Conclusion

In this paper we determined the exact value of the cycle-irregularity strength of ladders and fan graphs. We proved that for the ladder $L_n \cong P_n \Box P_2$, $n \geq 3$, admitting a C_{2m}-covering, $2 \leq m \leq \lceil (n+1)/2 \rceil$, $\text{ths}(L_n, C_{2m}) = \left\lceil \frac{3m+n}{4m} \right\rceil$. Moreover, for the fan graph F_n on $n+1$ vertices, $n \geq 2$ and $3 \leq m \leq \lceil (n+3)/2 \rceil$, $\text{ths}(F_n, C_m) = \left\lceil \frac{n+m}{2m-1} \right\rceil$.

For the edge (vertex) cycle-irregularity strength of ladders was proved the following.

Theorem 4.1. [6] *Let $L_n \cong P_n \Box P_2$, $n \geq 2$, be a ladder. Then*

$$\text{ehs}(L_n, C_4) = \left\lceil \frac{n+2}{4} \right\rceil.$$

Theorem 4.2. [6] *Let $L_n \cong P_n \Box P_2$, $n \geq 3$, be a ladder. Let m be a positive integer, $m \leq \lceil (n+1)/2 \rceil$. Then*

$$\text{vhs}(L_n, C_{2m}) = \left\lceil \frac{m+n}{2m} \right\rceil.$$

In [6] is also given the exact value for the vertex cycle-irregularity strength for fan graphs.

Theorem 4.3. [6] *Let F_n be a fan graph on $n+1$ vertices, $n \geq 2$ and $3 \leq m \leq \lceil (n+3)/2 \rceil$. Then*

$$\text{vhs}(F_n, C_m) = \left\lceil \frac{n}{m-1} \right\rceil.$$

According to results proved in [6] it is needed to find the edge cycle-irregularity strength for ladders and fans for every feasible length of cycles. We suppose that these parameters equal to the lower bounds. We conclude the paper with the following conjectures.

Conjecture 1. *Let $L_n \cong P_n \Box P_2$, $n \geq 2$, be a ladder admitting a C_{2m}-covering, $3 \leq m \leq \lceil (n+1)/2 \rceil$. Then*

$$\text{ehs}(L_n, C_{2m}) = \left\lceil \frac{m+n}{2m} \right\rceil.$$

Conjecture 2. *Let F_n be a fan graph on $n+1$ vertices, $n \geq 2$ and $3 \leq m \leq \lceil (n+3)/2 \rceil$. Then*

$$\text{ehs}(F_n, C_m) = \left\lceil \frac{n+1}{m} \right\rceil.$$

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