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Complete bipartite graph is a totally irregular total graph

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Abstract

A graph G is called a totally irregular total k-graph if it has a totally irregular total k-labeling $\lambda : V \cup E \to \{1, 2, \cdots, k\}$, that is a total labeling such that for any pair of different vertices x and y of G, their weights $wt(x)$ and $wt(y)$ are distinct, and for any pair of different edges e and f of G, their weights $wt(e)$ and $wt(f)$ are distinct. The minimum value k under labeling λ is called the total irregularity strength of G, denoted by $ts(G)$. For special cases of a complete bipartite graph $K_{m,n}$, the $ts(K_{1,n})$ and the $ts(K_{n,n})$ are already determined for any positive integer n. Completing the results, this paper deals with the total irregularity strength of complete bipartite graph $K_{m,n}$ for any positive integer m and n.

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1. Introduction

Graph labeling is a mapping that assigns integer (usually positive integer) to a vertex set or an edge set of a graph. Since the first appearance, graph labeling has been studied and modified under many conditions that lead to an interesting problem to deal with. When the domain is the union of vertex set and edge set, the labeling is called total labeling. Let G be a finite, simple, and undirected graph with vertex set $V(G)$ and edge set $E(G)$. For any total labeling $\lambda : V(G) \cup E(G) \to \{1, 2, \cdots, k\}$, the weight of a vertex v and the weight of an edge $e = uv$ are defined by $w(v) = \lambda(v) + \sum_{uv \in E(G)} \lambda(uv)$ and $w(uv) = \lambda(u) + \lambda(v) + \lambda(uv)$, respectively. If the weights of any pair of distinct vertices under total labeling λ are distinct, then λ is called a vertex irregular total k-labeling, and if the weights of any pair of distinct edges under total labeling λ are distinct, then λ is called an edge irregular total k-labeling. The minimum value k for which G has a vertex (or an edge) irregular total labeling λ is called the total vertex (or edge, resp.) irregularity strength of G and is denoted by $tvs(G)($ or $tes(G),$ resp.) [1].

The boundary for the $tvs(G)$ that for every (p, q)-graph G with minimum degree $\delta(G)$ and maximum degree $\Delta(G)$ is given in [1] by Baca, Jendrol, Miller, & Ryan, as follows.

$$\left\lceil \frac{p + \delta(G)}{\Delta(G) + 1} \right\rceil \leq tvs(G) \leq p + \Delta(G) - 2\delta(G) + 1; \tag{1}$$

while for the $tes(G)$ as follows.

$$\left\lceil \frac{|E(G)| + 2}{3} \right\rceil \leq tes(G) \leq |E(G)|. \tag{2}$$

In [14], Wijaya, Slamin, Surahmat and Jendroľ proved the sharpness of the lower bound of $tvs(G)$ for several cases of complete bipartite graph and gave the lower bound for a complete bipartite graph $K_{m,n}$, where $m \leq n$, except for $K_{2,2}$, as follows.

$$tvs(K_{m,n}) \geq max \left\{ \left\lceil \frac{m+n}{m+1} \right\rceil, \left\lceil \frac{2m+n-1}{n} \right\rceil \right\}. \tag{3}$$

In [4], Ivančo and Jendroľ proved that any tree T has an edge irregular total labeling and the lower bound (2) is sharp, as follows.

$$tes(T) = max \left\{ \left\lceil \frac{\Delta(T) + 1}{2} \right\rceil, \left\lceil \frac{|E(T)| + 2}{3} \right\rceil \right\}; \tag{4}$$

The lower bound (2) is also sharp for a complete bipartite graph as given in [5] by Jendroľ, Miškuf, and Soták, that is for $m, n \geq 2$,

$$tes(K_{m,n}) = \left\lceil \frac{mn + 2}{3} \right\rceil. \tag{5}$$

For many results of $tvs(G)$ and $tes(G)$ of some certain graphs, and various kind of graph labeling, one can refer to [2].

Finding both exact values is challenging even for some certain class of graphs. While many researchers work on evaluating each of the parameters of some certain graphs, Marzuki, Salman, and Miller [7] considering the vertex (and the edge) irregular total labeling of a graph at the same time. A graph G is called a *totally irregular total k-graph* if it has a totally irregular total k-labeling, that is an edge irregular total k-labeling and a vertex irregular total k-labeling at the same time. The minimum value k for which a graph G be a totally irregular total graph, is called the total irregularity strength of G, denoted by $ts(G)$. They [7] gave the lower bound for every graph G,

$$ts(G) \geq \max\{tes(G), tvs(G)\}; \tag{6}$$

and showed that the lower bound (6) is sharp for paths P_n, $n \neq 5$, and cycles C_n. For any path P_n of n vertices,

$$ts(P_n) = \begin{cases} \left\lceil \frac{n+2}{3} \right\rceil, & \text{for } n \in \{2, 5\}; \\ \left\lceil \frac{n+1}{3} \right\rceil, & \text{otherwise.} \end{cases} \tag{7}$$

For several cartesian product graphs, Ramdani and Salman in [8], showed that the lower bound (6) is sharp. Later, Ramdani, Salman, and Assiyatun [9] gave an upper bound of ts for some regular graphs. Then, in [10], Ramdani, Salman, Assiyatun, Semaničová-Feňovčíková, and Bača, proved that gear graphs, fungus graphs, $ts(Fg_n)$, for n even, $n \geq 6$; and disjoint union of stars are totally irregular total graphs with the ts equal to their tes.

In [12], Tilukay, Salman, and Persulessy proved that fan, wheel, triangular book, and friendship graphs are totally irregular total graphs with the ts equal to the lower bound (6) as well as double fans $DF_n, (n \geq 3)$, double triangular snakes $DT_p, (p \geq 3)$, joint-wheel graphs $WH_n, (n \geq 3)$, and $P_m + K_m (m \geq 3)$, that is given by Jeyanthi and Sudha in [6], and star graph $K_{1,n}$, double-stars, and caterpillar, that is given by Indriati, Widodo, Wijayanti, and Sugeng in [3]. They [3] obtained that for any positive integer $n \geq 3$,

$$ts(K_{1,n}) = \left\lceil \frac{n+1}{2} \right\rceil. \tag{8}$$

Next, Tilukay, Tomasouw, Rumlawang, and Salman in [13] found that complete graph K_n and complete bipartite graph $K_{n,n}$ are both totally irregular total graphs with their ts equal to the tes. They [9] obtained that for any positive integer $n \geq 2$,

$$ts(K_{n,n}) = \left\lceil \frac{n^2+2}{3} \right\rceil. \tag{9}$$

Taihuttu, Tilukay, Rumlawang, and Leleury [11] also provided the ts of a small result for a complete bipartite graph $K_{m,n}$, where $2 \leq m \leq 4$.

Completing the results of complete bipartite graphs above, in this paper, we proved that complete bipartite graph $K_{m,n}$ for any positive integer m and n is a totally irregular total graph by determining its total irregularity strength.

2. Complete Bipartite Graphs

Let $K_{m,n}$, where $m, n \geq 1$, be a complete bipartite graph with two vertex partition sets of cardinalities m and n. For simplifying the drawing of $K_{m,n}$ together with labels, let the labeling

$\lambda : V(K_{m,n}) \cup E(K_{m,n}) \to \{1, 2, \cdots, k\}$ represented by an $(m+1) \times (n+1)$ modification matrix $M_\lambda(K_{m,n}) = (a_{ij})$, where $a_{11} = 0$; first column $a_{i1}, i \neq 1$, consists of labels of m vertices in second partition set; first row $a_{1i}, i \neq 1$ consists of labels of n vertices in first partition set; and the remain entries consist of labels of edges joining these vertices.

Theorem 2.1. *Let $K_{m,n}$ be a complete bipartite graph with $2 \leq m < n$. Then*

$$ts(K_{m,n}) = \left\lceil \frac{mn+2}{3} \right\rceil.$$

Proof. Since $|V(K_{m,n})| = m + n$, $|E(K_{m,n})| = mn$, $\delta(G) = m$, $\Delta(G) = n$ with $m \leq n$ by equations (1), (2),(3), (5) and (6), for $2 \leq m < n$, we have

$$ts(K_{m,n}) \geq \left\lceil \frac{mn+2}{3} \right\rceil. \tag{10}$$

Next, we construct an irregular total labeling $\lambda : V(K_{m,n}) \cup E(K_{m,n}) \to \{1, 2, \cdots, k\}$ which is divided into two cases as follows. Let $k = \left\lceil \frac{mn+2}{3} \right\rceil$.

Case 1. For $K_{2,i}$, where $i \in 3, 4, 5, 6, 7$; $K_{3,4}$; $K_{3,5}$; and $K_{4,14}$.

The labeling λ of each complete bipartite graphs represented by the modification matrices as follows.

i. $M_\lambda(K_{2,3}) = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & 3 \end{pmatrix}$; ii. $M_\lambda(K_{2,4}) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 4 & 4 & 4 & 4 \end{pmatrix}$;

iii. $M_\lambda(K_{2,5}) = \begin{pmatrix} 0 & 1 & 2 & 3 & 3 & 4 \\ 1 & 1 & 1 & 1 & 2 & 2 \\ 4 & 3 & 3 & 3 & 4 & 4 \end{pmatrix}$; iv. $M_\lambda(K_{2,6}) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 5 & 3 & 3 & 3 & 3 & 4 & 4 \end{pmatrix}$;

v. $M_\lambda(K_{2,7}) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 6 & 3 & 3 & 3 & 3 & 3 & 4 & 4 \end{pmatrix}$;

vi. $M_\lambda(K_{3,5}) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 2 & 3 & 4 & 5 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{pmatrix}$; vii. $M_\lambda(K_{3,6}) = \begin{pmatrix} 0 & 1 & 2 & 4 & 5 & 6 & 7 \\ 1 & 1 & 2 & 2 & 3 & 4 & 5 \\ 2 & 1 & 2 & 2 & 3 & 4 & 5 \\ 7 & 7 & 7 & 6 & 6 & 6 & 6 \end{pmatrix}$; and

viii. $M_\lambda(K_{4,14}) = \begin{pmatrix} 0 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 10 & 12 & 14 & 16 & 18 & 20 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 8 & 8 & 8 & 8 & 8 & 8 \\ 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 8 & 8 & 8 & 8 & 8 & 8 \\ 19 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 18 & 18 & 18 & 18 & 18 & 18 \\ 20 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 18 & 18 & 18 & 18 & 18 & 18 \end{pmatrix}$.

Figure 1 shows a totally irregular total 20-labeling of $K_{4,14}$ represented by $M_\lambda(K_{4,14})$.

It is easy to check that λ is optimal such that all edge-weights form arithmetic progression with different 1, that is $3, 4, \cdots, |E(K_{m,n})$ and all vertex-weights are distinct can be seen by summing

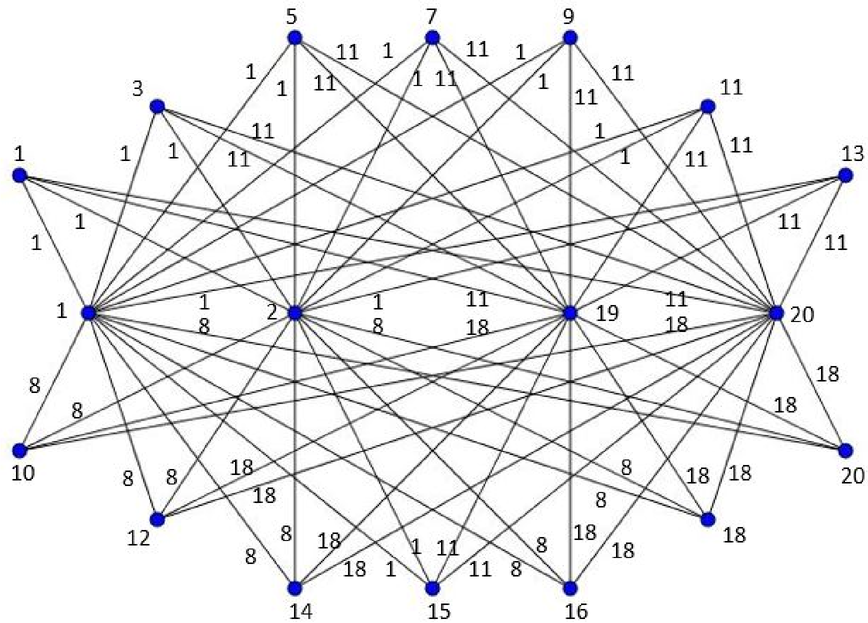

Figure 1. A totally irregular total 20-labeling of $K_{4,14}$.

all entries of each column or row, except for the first column and first row.

Case 2. For $K_{m,n}$, $2 \le m < n$, different from $K_{2,i}$, where $i \in 3, 4, 5, 6, 7$; $K_{3,4}$; $K_{3,5}$; and $K_{4,14}$.

For Case 2, we separate each of both partition sets of $V(K_{m,n})$ into 2 partition subsets as follows. For $a + b = n$ and $c + d = m$, let

$V(K_{m,n}) = \{u_i, v_j | 1 \le i \le a \text{ and } 1 \le j \le b\} \cup \{x_i, y_j | 1 \le i \le c \text{ and } 1 \le j \le d\}$.

$$\begin{aligned} E(K_{m,n}) = \ & \{u_i x_j | 1 \le i \le a, 1 \le j \le c\} \cup \{v_i x_j | 1 \le i \le b, 1 \le j \le c\} \cup \\ & \{u_i y_j | 1 \le i \le a, 1 \le j \le d\} \cup \{v_i y_j | 1 \le i \le b, 1 \le j \le d\}. \end{aligned}$$

Then, we define a, b, c, and d for any given m and n of $K_{m,n}$. It is shown in Table 1.

Next, by using the given condition in Table 1, we construct an irregular total labeling λ on $K_{m,n}$ in Table 2 - 5.

By verifying every label of all edges and vertices of $K_{m,n}$ for Case 2 given in Table 2 - 5, the maximum label is k, thus λ is a total k-labeling. More over, the construction of every label of all edges and vertices of $K_{m,n}$ for Case 2 given in Table 2 - 5 are optimal and resulting the edge weight-set $\{3, 4, \cdots, |E(K_{m,n})|\}$, while the vertex weight-sets of $K_{m,n}$ for Case 2 given in Table 2 - 5 are strictly increase.

Consider all the vertex-weights obtained in Table 2. It can be checked that for $8 \le n \le 10$, we have $b = 2$ which implies that $w(u_i) < w(x_1) < w(v_j) < w(y_1)$, while for $n \ge 11$, $w(u_i) < w(v_j) < w(x_1) < w(y_1)$, where $1 \le i \le a$ and $1 \le j \le b$. Thus, for $K_{2,n}$, where $n \ge 8$, a total k-labeling λ given in Table 2 is a totally irregular total k-labeling.

Table 1. Values of constrains a, b, c, and d for any given m and n defined in $V(K_{m,n})$

Subcase	m	n	a	b	c	d
2.1	2	$n \geq 8$	k	$n-k$	1	1
2.2	3	$n = 4$ or $n \geq 7$	$\lfloor \frac{n}{2} \rfloor$	$\lceil \frac{n}{2} \rceil$	2	1
2.3	even $m \geq 4$	$m+1 \leq n \leq \frac{3m+4}{2}$ or $3m+1 \leq n \leq 4m+2$ or $n \geq 5m+5$;	$\lceil \frac{2k}{m} \rceil$	$n - \lceil \frac{2k}{m} \rceil$	$\frac{m}{2}$	$\frac{m}{2}$
		$\frac{3m+6}{2} \leq n \leq 3m$ or $4m+3 \leq n \leq 5m+4$	$\lfloor \frac{n}{2} \rfloor$	$\lceil \frac{n}{2} \rceil$	$\frac{m}{2}$	$\frac{m}{2}$
2.4	odd $m \geq 5$	$m+1 \leq n \leq \frac{3m+21}{2}$ $\frac{3m+23}{2} \geq n \geq 2m+14$ $n \geq 2m+15$	$\lceil \frac{2k}{m-1} \rceil$ $n-m-7$ $n - \lceil \frac{2k}{m-1} \rceil$	$n - \lceil \frac{2k}{m-1} \rceil$ $m+7$ $\lceil \frac{2k}{m-1} \rceil$	$\lfloor \frac{m}{2} \rfloor$ $\lfloor \frac{m}{2} \rfloor$ $\lceil \frac{m}{2} \rceil$	$\lceil \frac{m}{2} \rceil$ $\lceil \frac{m}{2} \rceil$ $\lfloor \frac{m}{2} \rfloor$

Table 2. A construction of λ of $K_{m,n}$ in Subcase 2.1 and its associate weight

e	Label $\lambda(e)$	Weight $w(e)$	Constraint
u_i	i	$b+i+3$	$1 \leq i \leq a$
v_i	$a-b+i$	$n+b+i+3$	$1 \leq i \leq b$
x_1	1	$n+b^2+1$	
y_1	a	$n(b+2)+a+b^2$	
$u_i x_1$	1	$i+2$	$1 \leq i \leq a$
$v_i x_1$	$b+1$	$a+i+2$	$1 \leq i \leq b$
$u_i y_1$	$b+2$	$n+i+2$	$1 \leq i \leq a$
$v_i y_1$	$2b+2$	$n+a+i+2$	$1 \leq i \leq b$

For all the vertex-weights of $K_{3,n}$ obtained in Table 3, it is easy to checked that there is no two vertices and no two edges of $K_{3,4}$ or $K_{3,7}$ of the same weight. It can be checked also that for $K_{3,n}$, where $n \geq 8$ $w(u_i) < w(v_j) < w(x_r) < w(y_1)$, $1 \leq i \leq a$, $1 \leq j \leq b$, and $1 \leq r \leq 2$. Thus, for $K_{3,n}$, where $n = 4$ or $n \geq 7$, a total k-labeling λ given in Table 3 is a totally irregular total k-labeling.

Next, by evaluating all the vertex-weights of $K_{m,n}$, where m is even, $m \geq 4$ and $n \geq m+1$ obtained in Table 4, we have that for $1 \leq i \leq a$, $1 \leq j \leq b$, $1 \leq r \leq c$, $1 \leq s \leq d$,

i $w(x_r) < w(u_i) < w(v_j) < w(y_s)$, for $m+1 \leq n \leq \frac{3m+4}{2}$;

ii $w(u_i) < w(x_r) < w(v_j) < w(y_s)$, for $\frac{3m+4}{2} \leq n \leq 4m+2$;

iii $w(u_i) < w(v_j) < w(x_r) < w(y_s)$, for $n \geq 4m+3$.

Thus, for $K_{m,n}$, where m is even, $m \geq 4$ and $n \geq m+1$, a total k-labeling λ given in Table 4 is a totally irregular total k-labeling.

Table 3. A construction of f of $K_{m,n}$ in Subcase 2.2 and its associate weight

e	Label $\lambda(e)$	Weight $w(e)$	Constraint
u_i	i	$k+3i$	$1 \le i \le a$
v_i	$k-b+i$	$k+3a+3i-2$	$1 \le i \le b$
x_i	i	$\frac{kn}{2}-b+i$	$1 \le i \le c$
y_1	k	k^2-b	
u_ix_j	i	$2i+j$	$1 \le i \le a, 1 \le i \le c$
v_ix_j	$a+i-1$	$2a+2i+j$	$1 \le i \le b, 1 \le i \le c$
u_iy_1	k	$2k+i$	$1 \le i \le a$
v_iy_1	$k-1$	$2k+a+i$	$1 \le i \le b$

Table 4. A construction of λ of $K_{m,n}$ in Subcase 2.3 and its associate weight

e	Label $\lambda(e)$	Weight $w(e)$	Constraint
u_i	$d(i-1)+1$	$d(dn+d-k+i+1)+1$	$1 \le i \le a$
v_i	$k-bd$	$d(mn-3k-b-d+dn+i+4)+k$	$1 \le i \le b$
x_i	i	$a+b(dn-d-k+2)+i$	$1 \le i \le c$
y_i	$k-d+i$	$a(dn+d-k+1)+b(mn-2k+2)+$ $k-d+i$	$1 \le i \le d$
u_ix_j	1	$d(i-1)+j+2$	$1 \le i \le a, 1 \le j \le c$
v_ix_j	$d(n-1)-k+2$	$d(a+i-1)+j+2$	$1 \le i \le b, 1 \le j \le c$
u_iy_j	$d(n+1)-k+1$	$d(n+i-1)+j+2$	$1 \le i \le a, 1 \le j \le d$
v_iy_j	$mn-2k+2$	$mn-d(b-i+1)+j+2$	$1 \le i \le b, 1 \le j \le d$

Next, by evaluating all the vertex-weights of $K_{m,n}$, where m is odd, $m \ge 5$ and $n \ge m+1$ obtained in Table 5, we have that for $1 \le i \le a$, $1 \le j \le b$, $1 \le r \le c$, $1 \le s \le d$,

i $w(x_r) < w(u_i) < w(v_j) < w(y_s)$, for $m+1 \le n \le \frac{3m+21}{2}$;

ii $w(u_i) < w(x_r) < w(v_j) < w(y_s)$, for $\frac{3m+23}{2} \le n \le 2m+14$;

iii $w(u_i) < w(v_j) < w(x_r) < w(y_s)$, for $n \ge 2m+15$.

Thus, for $K_{m,n}$, where m is odd, $m \ge 5$ and $n \ge m+1$, a total k-labeling λ given in Table 5 is a totally irregular total k-labeling.

Based on the results of both cases, we have that λ is a totally irregular total $\lceil \frac{mn+2}{3} \rceil$-labeling. Thus, for $m < n, m \ge 2$, and $n \ge 3$, we obtain:

$$ts(K_{m,n}) \le \left\lceil \frac{mn+2}{3} \right\rceil. \tag{11}$$

By equation (10) and (11), we have $ts(K_{m,n}) = \lceil \frac{mn+2}{3} \rceil$, for $m < n, m \ge 2$, and $n \ge 3$.

□

Table 5. A construction of λ of $K_{m,n}$ in Subcase 2.4 and its associate weight

e	Label $\lambda(e)$	Weight $w(e)$	Constraint
u_i	$c(i-1)+1$; $d(i-1)+1$;	$d(cn+d-k)+mi+1$; $d^2(n+1)+d(n-k)+mi+1$;	$1 \leq i \leq a, n \leq 2m+14$; $1 \leq i \leq a, n \geq 2m+15$;
v_i	$k-c(b-i)$; $k-d(b-i)$;	$m(cn-k+i+2)+k+$ $c(dn-c-b)-d(k-a)$; $mi+k+c(dn-2d+c+a-k)+$ $d(2dn-2k-b+n+2)$;	$1 \leq i \leq b, n \leq 2m+14$; $1 \leq i \leq a, n \geq 2m+15$;
x_i	i;	$n+b(cn-c-k+1)+j$; $\frac{n}{2}(n+1)+b(dn-2d+c-k)+j$;	$1 \leq i \leq c, n \leq 2m+14$; $1 \leq i \leq c, n \geq 2m+15$;
y_i	$k-d+i$;	$\frac{n}{2}(n+1)+k(1-n)-d+cn^2+$ $ad+b(cn-k+2)+j$; $k-d+ad+n(dn+n-k+1)+$ $b(dn-k+1)+j$;	$1 \leq i \leq d, n \leq 2m+14$ $1 \leq i \leq d, n \geq 2m+15$;
u_ix_j	1; i;	$c(i-1)+j+2$; $d(i-1)+i+j+1$;	$1 \leq i \leq a, 1 \leq j \leq c,$ $m+1 \leq n \leq 2m+14$; $1 \leq i \leq a, 1 \leq j \leq c,$ $n \geq 2m+15$;
v_ix_j	$c(n-1)-k+2$; $d(n-2)+a+c-$ $k+i$;	$c(a+i-1)+j+2$ $d(a+i-2)+a+c+i+j$	$1 \leq i \leq b, 1 \leq j \leq c,$ $m+1 \leq n \leq 2m+14$; $1 \leq i \leq b, 1 \leq j \leq c,$ $n \geq 2m+15$;
u_iy_j	$cn+d-k+i$ $d(n+1)+n-k+1$	$c(n+i-1)+i+j+1$; $d(n+i-1)+n+j+2$;	$1 \leq i \leq a, 1 \leq j \leq d,$ $m+1 \leq n \leq 2m+14$; $1 \leq i \leq a, 1 \leq j \leq d,$ $n \geq 2m+15$;
v_iy_j	$2(cn-k+1)+a+i$; $2(dn-k+1)+n$	$c(n+a+i)+a-d+i+j+2$; $d(n+a+i-1)+n+j+2$;	$1 \leq i \leq b, 1 \leq j \leq d,$ $m+1 \leq n \leq 2m+14$; $1 \leq i \leq b, 1 \leq j \leq d,$ $n \geq 2m+15$;

3. Conclusion

By Equations (7), (8), (9), and Theorem 2.1, we can conclude that complete bipartite graph $K_{m,n}$ for any positive integer m and n is a totally irregular total graph with

$$ts(K_{m,n}) = \begin{cases} 2, & \text{for } m = n = 1; \\ \left\lceil \frac{n+1}{2} \right\rceil, & \text{for } m = 1, n \neq 1; \\ \left\lceil \frac{mn+2}{3} \right\rceil, & \text{otherwise.} \end{cases}$$

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