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On inclusive distance vertex irregular labelings

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Abstract

For a simple graph G, a vertex labeling $f : V(G) \to \{1, 2, \ldots, k\}$ is called a k-*labeling*. The weight of a vertex v, denoted by $wt_f(v)$ is the sum of all vertex labels of vertices in the closed neighborhood of the vertex v. A vertex k-labeling is defined to be an *inclusive distance vertex irregular distance k-labeling* of G if for every two different vertices u and v there is $wt_f(u) \neq wt_f(v)$. The minimum k for which the graph G has a vertex irregular distance k-labeling is called the *inclusive distance vertex irregularity strength* of G. In this paper we establish a lower bound of the inclusive distance vertex irregularity strength for any graph and determine the exact value of this parameter for several families of graphs.

1. Introduction

Inspired by Miller, Rodger and Simanjuntak [5] who introduced a distance magic labeling, Slamin [6] introduced the concept of a distance vertex irregular labeling of graphs. A *distance vertex irregular labeling* of a graph is a mapping $g : V(G) \to \{1, 2, \ldots, k\}$ such that the set of vertex weights consists of distinct numbers, where the weight of a vertex $v \in V(G)$ under the labeling g is defined as

$$wt_g(v) = \sum_{u \in N_G(v)} g(u),$$

where $N_G(v)$ is a set of all neighbors of a vertex v, that is a set of vertices whose distance from v is 1. In other words the weight of a vertex v is the sum of all vertex labels of vertices in the neighborhood of the vertex v. The minimum k for which a graph G has a distance vertex irregular labeling is called the distance vertex irregularity strength of G, denoted by $dis(G)$. In his paper, Slamin also proved that $\text{dis}(K_n) = n$, for $n \geq 3$, $\text{dis}(P_n) = \lceil n/2 \rceil$, for $n \geq 4$, $\text{dis}(C_n) = \lceil (n+1)/2 \rceil$, for $n \equiv 0, 1, 2, 3 \pmod 8$ and $\text{dis}(W_n) = \lceil (n+1)/2 \rceil$, for $n \equiv 0, 1, 2, 5 \pmod 8$.

Bong et al. [2] generalized the concept to inclusive and non-inclusive vertex irregular d-distance vertex labeling. The difference between inclusive and non-inclusive labeling depend on the way to calculate the vertex weight whether the vertex label of vertex which we calculate its weight is included or not. The symbol d represents on how far the neighborhood is consider. Thus the original concept of Slamin can be called a non-inclusive vertex irregular 1-distance vertex labeling. An inclusive vertex irregular d-distance vertex labeling f is an irregular labeling of vertices in a graph G where the weights of a vertex $v \in V(G)$ is the sum of the label of v and all labels of vertices up to distance d from v,

$$wt_f(v) = f(v) + \sum_{u \in V(G): \ 1 \leq d(u,v) \leq d} f(u),$$

where $d(u, v)$ is the distance between vertex u to vertex v. They determined the inclusive 1-distance irregularity strength for path P_n ($n \equiv 0 \pmod 3$), star, double star $S(m, n)$ with $m \leq n$, lower bound for caterpillar, cycles and wheels.

In this paper we consider an inclusive vertex irregular d-distance vertex labeling with $d = 1$. For simplicity in this paper, we call the labeling an inclusive distance vertex irregular labeling. Thus an inclusive distance vertex labeling f is an irregular labeling of vertices in a graph G where the weights of a vertex $v \in V(G)$ is the sum of the label of v and all vertex labels of vertices in the neighborhood of the vertex v, and thus

$$wt_f(v) = \sum_{u \in N_G[v]} f(u),$$

where $N_G[v]$ is a set of all neighbors of a vertex v including v, that is a set of vertices whose distance from v is maximum 1.

The minimum k for which there exists an inclusive distance vertex irregular labeling of a graph G is called the *inclusive distance vertex irregularity strength* of G and is denoted by $\widehat{\text{dis}}(G)$. If such k does not exist we say that $\widehat{\text{dis}}(G) = \infty$.

We establish a lower bound of the inclusive distance vertex irregularity strength and determine the exact value of this parameter for several families of graphs. We are dealing with the given graph invariant for complete graphs, complete bipartite graphs, paths, cycles, fans and wheels.

2. Lower and upper bound

The following lemma gives a lower bound on the inclusive distance vertex irregularity strength of a graph G.

Lemma 2.1. *[2] Let G be a graph with maximum degree $\Delta(G)$ and minimum degree $\delta(G)$. Then*

$$\widehat{\text{dis}}(G) \geq \left\lceil \frac{|V(G)| + \delta(G)}{\Delta(G) + 1} \right\rceil.$$

Now we will deal with an upper bound of $\widehat{\text{dis}}(G)$. The following theorem gives a sufficient and necessary condition for $\widehat{\text{dis}}(G) < \infty$. Note that the graph G is not necessarily connected.

Theorem 2.1. *For a graph G holds $\widehat{\text{dis}}(G) = \infty$ if and only if there exist two distinct vertices $u, v \in V(G)$ such that $N_G[u] = N_G[v]$.*

Proof. Let us consider that in G there exist two distinct vertices $u, v \in V(G)$ such that

$$N_G[u] = N_G[v]. \tag{1}$$

Let f be any vertex labeling of G, $f : V(G) \to \{1, 2, \ldots, k\}$. For the weights of vertices u and v under a labeling f we get

$$wt_f(u) = \sum_{x \in N_G[u]} f(x) = \sum_{x \in N_G[u]} f(x) = \sum_{x \in N_G[v]} f(x) = \sum_{x \in N_G[v]} f(x) = wt_f(v).$$

According to the condition (1) we get $wt_f(u) = wt_f(v)$ which evidently means that f can not be distance vertex irregular.

Now, let us consider that for all vertices $u, v \in V(G)$ it holds that $N_G[u] \neq N_G[v]$. Let us denote the vertices of G arbitrarily by the symbols $v_1, v_2, \ldots, v_{|V(G)|}$. We define a vertex $2^{|V(G)|-1}$-labeling f of G in the following way:

$$f(v_i) = 2^{i-1}, \qquad \text{for } i = 1, 2, \ldots, |V(G)|.$$

Let us define the labeling θ such that

$$\theta_{i,j} = \begin{cases} 1, & \text{if } v_i \in N_G[v_j], \\ 0, & \text{if } v_i \notin N_G[v_j], \end{cases}$$

where $i = 1, 2, \ldots, |V(G)|$, $j = 1, 2, \ldots, |V(G)|$.

The weight of a vertex v_j is the sum of all vertex labels of vertices in a closed neighborhood of v_j. Thus, for $j = 1, 2, \ldots, |V(G)|$ we have

$$wt_f(v_j) = \sum_{v_i \in N_G[v_j]} f(v_i) = \sum_{i: \, v_i \in N_G[v_j]} 2^{i-1} = \sum_{i=1}^{|V(G)|} \theta_{i,j} 2^{i-1}. \tag{2}$$

To prove that vertex weights are all distinct it is enough to show that the sums $\sum_{i=1}^{|V(G)|} \theta_{i,j} 2^{i-1}$ are distinct for $j = 1, 2, \ldots, |V(G)|$. However, this is evident if we consider that the ordered $|V(G)|$-tuple $(\theta_{|V(G)|,j} \theta_{|V(G)|-1,j} \ldots \theta_{2,j} \theta_{1,j})$ corresponds to the binary code representation of the sum (2). As different vertices have distinct closed neighborhoods we immediately get that the $|V(G)|$-tuples are different for different vertices. Which implies that $\widehat{\text{dis}}(G) < 2^{|V(G)|-1}$. $\qquad \square$

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3. Complete and complete bipartite graph

Immediately from Theorem 2.1 we obtain the result for the inclusive distance vertex irregularity strength of complete graphs.

Corollary 3.1. *Let n be a positive integer. Then*

$$\widehat{\mathrm{dis}}(K_n) = \begin{cases} 1, & \text{for } n = 1, \\ \infty, & \text{for } n \geq 2. \end{cases}$$

From Theorem 2.1 we also obtain the following result.

Lemma 3.1. *Let $\widehat{\mathrm{dis}}(G) < \infty$. If $u, v \in V(G)$ are two distinct vertices such that $N_G(u) = N_G(v)$ then $f(u) \neq f(v)$ in any inclusive distance vertex irregular labeling f of G.*

We will use this lemma for finding the inclusive distance vertex irregularity strength of complete bipartite graphs $K_{m,n}$, $m, n \geq 1$.

Theorem 3.1. *Let m, n be positive integers $m \geq n \geq 1$. Then*

$$\widehat{\mathrm{dis}}(K_{m,n}) = \begin{cases} \infty, & \text{for } m = n = 1, \\ n + 2, & \text{for } m = n \geq 2, \\ m, & \text{for } m > n. \end{cases}$$

Proof. Let m, n be positive integers $m \geq n \geq 1$. Let us denote the vertices and edges of $K_{m,n}$ such that

$$V(K_{m,n}) = \{u_i : i = 1, 2, \ldots, m\} \cup \{v_j : j = 1, 2, \ldots, n\},$$
$$E(K_{m,n}) = \{u_i v_j : i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n\}.$$

We will consider three cases.

Case 1: $m = n = 1$. In this case the graph $K_{m,n} = K_{1,1}$ is isomorphic to K_2 and according to Corollary 3.1 we get $\widehat{\mathrm{dis}}(K_{1,1}) = \infty$.

Case 2: $m = n \geq 2$. Let $f : V(G) \to \{1, 2, \ldots, k\}$ be an inclusive distance vertex irregular labeling of $K_{n,n}$. According to Lemma 3.1 we get $k \geq n$ and the set of labels of vertices u_i, $i = 1, 2, \ldots, n$ must consist of distinct numbers and also the set of labels of vertices v_j, $j = 1, 2, \ldots, n$ must consist of distinct numbers. It is easy to see that $k > n$.

Let us consider that $k = n + 1$. Without loss of generality we can assume that

$$\{f(u_i) : i = 1, 2, \ldots, n\} = \{1, 2, \ldots, n\}$$

and

$$\{f(v_j) : j = 1, 2, \ldots, n\} = \{1, 2, \ldots, n + 1\} \setminus \{p\},$$

where p is an integer $1 \leq p \leq n$.

Then for the vertex weights we get

$$wt_f(v_j) = f(v_j) + \sum_{i=1}^{n} f(u_i) = f(v_j) + \sum_{i=1}^{n} i = f(v_j) + \frac{n(n+1)}{2},$$

for $j = 1, 2, \ldots, n$. Thus

$$\{wt_f(v_j) : j = 1, 2, \ldots, n\} = \{\tfrac{n(n+1)}{2} + 1, \tfrac{n(n+1)}{2} + 2, \ldots, \tfrac{n(n+1)}{2} + n + 1\} \setminus \{\tfrac{n(n+1)}{2} + p\}.$$

And

$$wt_f(u_i) = f(u_i) + \sum_{j=1}^{n} f(v_j) = f(u_i) + \sum_{j=1}^{n+1} j - p = f(u_i) + \sum_{j=1}^{n} j + (n+1) - p$$

$$= f(u_i) + \frac{n(n+1)}{2} + (n+1) - p,$$

for $i = 1, 2, \ldots, n$. Thus

$$\{wt_f(u_i) : i = 1, 2, \ldots, n\} = \{\tfrac{n(n+1)}{2} + n + 2 - p, \tfrac{n(n+1)}{2} + n + 3 - p, \ldots, \tfrac{n(n+1)}{2} + 2n + 2 - p\}.$$

Which is a contradiction as the vertex weights are not distinct. Thus $k \geq n + 2$.

Let us define the vertex labeling g of $K_{n,n}$ such that

$$g(u_i) = i, \qquad \text{for } i = 1, 2, \ldots, n,$$
$$g(v_j) = j + 2, \quad \text{for } j = 1, 2, \ldots, n.$$

Thus $g(u) \leq n + 2$ for all vertices in $K_{n,n}$.

Now we compute the vertex weights. For $i = 1, 2, \ldots, n$,

$$wt_g(u_i) = g(u_i) + \sum_{j=1}^{n} g(v_j) = i + \sum_{j=1}^{n}(j + 2) = i + \frac{n(n+1)}{2} + 2n$$

and for $j = 1, 2, \ldots, n$

$$wt_g(v_j) = g(v_j) + \sum_{i=1}^{n} g(u_i) = (j + 2) + \sum_{i=1}^{n} i = j + 2 + \frac{n(n+1)}{2} \leq \frac{n(n+1)}{2} + n + 2.$$

As $n \geq 2$ we obtain that the vertex weights are distinct.

Case 3: $m > n$. Let $f : V(G) \to \{1, 2, \ldots, k\}$ be an inclusive distance vertex irregular labeling of $K_{m,n}$. According to Lemma 3.1 we get $k \geq m$. Let us define the vertex labeling f of $K_{m,n}$ such that

$$f(u_i) = i, \quad \text{for } i = 1, 2, \ldots, m,$$
$$f(v_j) = j, \quad \text{for } j = 1, 2, \ldots, n.$$

Evidently the vertex labels are not greater than m.

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For the weights of the vertices we get the following. For $i = 1, 2, \ldots, m$

$$wt_f(u_i) = f(u_i) + \sum_{j=1}^{n} f(v_j) = i + \sum_{j=1}^{n} j = i + \frac{n(n+1)}{2},$$

thus

$$\{wt_f(u_i) : i = 1, 2, \ldots, m\} = \{\frac{n(n+1)}{2} + 1, \frac{n(n+1)}{2} + 2, \ldots, \frac{n(n+1)}{2} + m\}.$$

For $j = 1, 2, \ldots, n$

$$wt_f(v_j) = f(v_j) + \sum_{i=1}^{m} f(u_i) = j + \sum_{i=1}^{m} i = j + \sum_{i=1}^{n} i + \sum_{i=n+1}^{m} i = j + \frac{n(n+1)}{2} + \sum_{i=n+1}^{m} i$$

$$\geq j + \frac{n(n+1)}{2} + m,$$

which implies that that the weight sets do not overlap. □

Immediately from Theorem 3.1 we obtain the result for the inclusive distance vertex irregularity strength of stars $S_n = K_{1,n}$, which is also proved by Bong *et al.* [2].

Corollary 3.2. *Let n be a positive integer. Then*

$$\widehat{\mathrm{dis}}(S_n) = \begin{cases} \infty, & \text{for } n = 1, \\ n, & \text{for } n \geq 2. \end{cases}$$

The join $G \oplus H$ of two disjoint graphs G and H is the graph $G \cup H$ together with all the edges joining vertices of $V(G)$ and vertices of $V(H)$. In the next theorem we will deal with the inclusive distance vertex irregularity strength of joins $G \oplus K_1$, where G is an arbitrary graph, possibly also disconnected.

Theorem 3.2. *Let G be a graph with maximum degree $\Delta(G)$. Then*

$$\widehat{\mathrm{dis}}(G \oplus K_1) = \begin{cases} \infty, & \text{if } \Delta(G) = |V(G)| - 1, \\ \widehat{\mathrm{dis}}(G), & \text{if } \Delta(G) < |V(G)| - 1. \end{cases}$$

Proof. Let G be a graph with maximum degree $\Delta(G)$. Let us denote the vertices and edges of a graph $G \oplus K_1$ such that

$$V(G \oplus K_1) = V(G) \cup \{v\},$$
$$E(G \oplus K_1) = E(G) \cup \{vu : u \in V(G)\}.$$

Let $w \in V(G)$ be a vertex such that $\deg_G(w) = \Delta(G)$. If $\Delta(G) = |V(G)| - 1$ then in the graph $G \oplus K_1$ the degree of the vertex w will be $|V(G)|$, that is

$$N_{G \oplus K_1}[w] = V(G \oplus K_1).$$

But also for the vertex v we get $\deg_{G \oplus K_1}(v) = |V(G)|$ which means

$$N_{G \oplus K_1}[v] = V(G \oplus K_1).$$

According to Theorem 2.1 we get $\widehat{\mathrm{dis}}(G \oplus K_1) = \infty$.

Now let us consider that $\Delta(G) < |V(G)| - 1$. We distinguish two cases.

Case 1: $\widehat{\mathrm{dis}}(G) = \infty$. By Theorem 2.1 we get that there must exist in G two distinct vertices, say x, y, such that $N_G[x] = N_G[y]$. But this implies that

$$N_{G \oplus K_1}[x] = N_G[x] \cup \{v\} = N_G[y] \cup \{v\} = N_{G \oplus K_1}[y].$$

But this means that $\widehat{\mathrm{dis}}(G \oplus K_1) = \infty$.

Case 2: $\widehat{\mathrm{dis}}(G) < \infty$. Let $f : V(G) \to \{1, 2, \ldots, k\}$ be an inclusive distance vertex irregular labeling of a graph G such that $k = \widehat{\mathrm{dis}}(G)$. Let us define a vertex labeling $g : V(G \oplus K_1) \to \{1, 2, \ldots, k\}$ such that

$$\begin{aligned} g(u) &= f(u), \quad \text{for } u \in V(G), \\ g(v) &= 1. \end{aligned}$$

Evidently g is a k-labeling. For the vertex weights under the labeling g we get the following. If $u \in V(G)$ then

$$wt_g(u) = \sum_{x \in N_{G \oplus K_1}[u]} g(x) = \sum_{x \in N_G[u]} g(x) + g(v) = \sum_{x \in N_G[u]} f(x) + 1 = wt_f(u) + 1.$$

As f is an inclusive distance vertex irregular labeling of a graph G then the vertices $u \in V(G \oplus K_1) \setminus \{v\}$ have distinct weights under the labeling g.

The weight of a vertex v is

$$wt_g(v) = \sum_{x \in N_{G \oplus K_1}[v]} g(x) = \sum_{x \in N_G[u]} g(x) = 1 + \sum_{x \in V(G)} f(x).$$

As $\Delta(G) < |V(G)| - 1$ we get that for every vertex $u \in V(G)$

$$\sum_{x \in V(G)} f(x) > wt_f(u).$$

This implies that for every vertex $u \in V(G)$

$$wt_g(v) = 1 + \sum_{x \in V(G)} f(x) > 1 + wt_f(u) = wt_g(u).$$

Thus $\widehat{\mathrm{dis}}(G \oplus K_1) \le \widehat{\mathrm{dis}}(G)$.

To prove the equality it is sufficient to show that there does not exist an inclusive distance vertex irregular K-labeling of a graph $G \oplus K_1$ such that $K < \widehat{\mathrm{dis}}(G)$. On contrary let us consider

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that such labeling h exists. Thus $h : V(G \oplus K_1) \to \{1, 2, \ldots, K\}$ is an inclusive distance vertex irregular labeling of a graph $G \oplus K_1$. Under the labeling h the vertex weights of all vertices must be distinct, thus also for every two vertices $x, y \in V(G)$ we have

$$wt_h(x) \neq wt_h(y).$$

Now we subtract from both sides the label of the vertex v and we obtain

$$wt_h(x) - h(v) \neq wt_h(y) - h(v).$$

But this means that a restriction of the labeling h on the graph G is an inclusive distance vertex irregular K-labeling of G. And this is a contradiction as $K < \widehat{\mathrm{dis}}(G)$. This concludes the proof. $\square$

Immediately from Theorem 3.2 we get another proof of Corollary 3.1.

Let $K_{n_1,n_2,\ldots,n_p}$ denote the complete p-partite graph with partite sets of cardinalities n_i, $i = 1, 2, \ldots, p$. Using Theorems 3.2 and 3.1 we obtain a result for complete multipartite graphs of a special type.

Theorem 3.3. *Let m, n be positive integers $m \geq n \geq 1$. Then*

$$\widehat{\mathrm{dis}}(K_{m,n,1}) = \begin{cases} \infty, & \text{for } m = n = 1, \\ n + 2, & \text{for } m = n \geq 2, \\ m, & \text{for } m > n. \end{cases}$$

4. Path

Let P_n, $n \geq 2$ be a path on n vertices. We denote the vertices and edges of P_n such that

$$\begin{aligned} V(P_n) &= \{v_i : i = 1, 2, \ldots, n\}, \\ E(P_n) &= \{v_i v_{i+1} : i = 1, 2, \ldots, n-1\}. \end{aligned}$$

Bong *et al.* [2] proved that $\widehat{\mathrm{dis}}(P_n) = n/3 + 1$ for $n \equiv 0 \pmod 3$. As P_2 is isomorphic to a complete graph K_2 using Corollary 3.1 we get $\widehat{\mathrm{dis}}(P_2) = \infty$. For the inclusive distance vertex irregularity strength of a path we prove the following.

Theorem 4.1. *Let n be a positive integer $n \geq 2$. Then*

$$\widehat{\mathrm{dis}}(P_n) = \begin{cases} \infty, & \text{for } n = 2, \\ 3, & \text{for } n = 5, \\ \left\lceil \frac{n+1}{3} \right\rceil, & \text{for } n \not\equiv 2 \pmod 9, \ n \neq 5 \end{cases}$$

and

$$\frac{n+1}{3} \leq \widehat{\mathrm{dis}}(P_n) \leq \frac{n+1}{3} + 1,$$

when $n \equiv 2 \pmod 9$, $n \geq 11$.

Let $n \geq 3$ be a positive integer. As $\Delta(P_n) = 2$ and $\delta(P_n) = 1$ from Lemma 2.1 we obtain a lower bound of the inclusive distance vertex irregularity strength of a path

$$\widehat{\mathrm{dis}}(P_n) \geq \left\lceil \frac{n+1}{3} \right\rceil. \tag{3}$$

To prove the equality let us consider Lemmas 4.1 throughout 4.5.

Lemma 4.1. *Let n be a positive integer, $n \equiv 1, 7 \pmod{9}$, $n \geq 7$. Then*

$$\widehat{\mathrm{dis}}(P_n) = \frac{n+2}{3}.$$

Proof. Let $n \equiv 1, 7 \pmod{9}$, $n \geq 7$. According to (3) it suffices to show that there exists an inclusive distance vertex irregular $\lceil (n+1)/3 \rceil$-labeling of P_n.

Let $f : V(P_n) \to \{1, 2, \ldots, \lceil (n+1)/3 \rceil\}$ be a vertex labeling of P_n defined such that

$$f(v_i) = \left\lceil \frac{i}{3} \right\rceil, \qquad \text{for } i = 1, 2, \ldots, \frac{2n+1}{3},$$
$$f(v_i) = \left\lceil \frac{i+2}{3} \right\rceil, \qquad \text{for } i = \frac{2n+4}{3}, \frac{2n+7}{3}, \ldots, n.$$

It is easy to see that every vertex label is not greater than $(n+2)/3 = \lceil (n+1)/3 \rceil$.

For the vertex weights we get the following.

$$wt_f(v_1) = f(v_1) + f(v_2) = \left\lceil \frac{1}{3} \right\rceil + \left\lceil \frac{2}{3} \right\rceil = 2,$$
$$wt_f(v_i) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left\lceil \frac{i-1}{3} \right\rceil + \left\lceil \frac{i}{3} \right\rceil + \left\lceil \frac{i+1}{3} \right\rceil = i + 1,$$
$$\text{for } i = 2, 3, \ldots, \frac{2n-2}{3},$$
$$\text{thus the corresponding weights are } 3, 4, \ldots, \frac{2n+1}{3},$$
$$wt_f(v_{\frac{2n+1}{3}}) = f(v_{\frac{2n-2}{3}}) + f(v_{\frac{2n+1}{3}}) + f(v_{\frac{2n+4}{3}}) = \left\lceil \frac{\frac{2n-2}{3}}{3} \right\rceil + \left\lceil \frac{\frac{2n+1}{3}}{3} \right\rceil + \left\lceil \frac{\frac{2n+4}{3}+2}{3} \right\rceil = \frac{2n+7}{3},$$
$$wt_f(v_{\frac{2n+4}{3}}) = f(v_{\frac{2n+1}{3}}) + f(v_{\frac{2n+4}{3}}) + f(v_{\frac{2n+7}{3}}) = \left\lceil \frac{\frac{2n+1}{3}}{3} \right\rceil + \left\lceil \frac{\frac{2n+4}{3}+2}{3} \right\rceil + \left\lceil \frac{\frac{2n+7}{3}+2}{3} \right\rceil$$
$$= \begin{cases} \frac{2n+13}{3}, & \text{if } n \equiv 1 \pmod{9}, \\ \frac{2n+10}{3}, & \text{if } n \equiv 7 \pmod{9}, \end{cases}$$
$$wt_f(v_i) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left\lceil \frac{(i-1)+2}{3} \right\rceil + \left\lceil \frac{i+2}{3} \right\rceil + \left\lceil \frac{(i+1)+2}{3} \right\rceil = i + 3,$$
$$\text{for } i = \frac{2n+7}{3}, \frac{2n+10}{3}, \ldots, n-1,$$
$$\text{thus the corresponding weights are } \frac{2n+16}{3}, \frac{2n+19}{3}, \ldots, n+2,$$
$$wt_f(v_n) = f(v_{n-1}) + f(v_n) = \left\lceil \frac{(n-1)+2}{3} \right\rceil + \left\lceil \frac{n+2}{3} \right\rceil = \frac{2n+4}{3}.$$

Thus the vertex weights are distinct. This concludes the proof. □

Lemma 4.2. *Let n be a positive integer, $n \equiv 4 \pmod{9}$, $n \geq 4$. Then*

$$\widehat{\mathrm{dis}}(P_n) = \frac{n+2}{3}.$$

Proof. Let $n \equiv 4 \pmod 9$, $n \geq 4$. A vertex labeling $f : V(P_n) \to \{1, 2, \ldots, (n+2)/3\}$ is defined in the following way

$$f(v_i) = \left\lceil \tfrac{i}{3} \right\rceil, \qquad \text{for } i = 1, 2, \ldots, \tfrac{2n-5}{3},$$
$$f(v_i) = \left\lceil \tfrac{i+2}{3} \right\rceil, \quad \text{for } i = \tfrac{2n-2}{3}, \tfrac{2n+1}{3}, \ldots, n.$$

It is easy to see that every vertex label is not greater than $(n+2)/3 = \lceil (n+1)/3 \rceil$. Thus, according to (3) we only need to show that the corresponding vertex weights are distinct. In particular:

$$wt_f(v_1) = f(v_1) + f(v_2) = \left\lceil \tfrac{1}{3} \right\rceil + \left\lceil \tfrac{2}{3} \right\rceil = 2,$$
$$wt_f(v_i) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left\lceil \tfrac{i-1}{3} \right\rceil + \left\lceil \tfrac{i}{3} \right\rceil + \left\lceil \tfrac{i+1}{3} \right\rceil = i + 1,$$
$$\text{for } i = 2, 3, \ldots, \tfrac{2n-8}{3},$$
$$\text{thus the corresponding weights are } 3, 4, \ldots, \tfrac{2n-5}{3},$$
$$wt_f(v_{\frac{2n-5}{3}}) = f(v_{\frac{2n-8}{3}}) + f(v_{\frac{2n-5}{3}}) + f(v_{\frac{2n-2}{3}}) = \left\lceil \tfrac{\frac{2n-8}{3}}{3} \right\rceil + \left\lceil \tfrac{\frac{2n-5}{3}}{3} \right\rceil + \left\lceil \tfrac{\frac{2n-2}{3}+2}{3} \right\rceil = \tfrac{2n+1}{3},$$
$$wt_f(v_{\frac{2n-2}{3}}) = f(v_{\frac{2n-5}{3}}) + f(v_{\frac{2n-2}{3}}) + f(v_{\frac{2n+1}{3}}) = \left\lceil \tfrac{\frac{2n-5}{3}}{3} \right\rceil + \left\lceil \tfrac{\frac{2n-2}{3}+2}{3} \right\rceil + \left\lceil \tfrac{\frac{2n+1}{3}+2}{3} \right\rceil = \tfrac{2n+7}{3},$$
$$wt_f(v_i) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left\lceil \tfrac{(i-1)+2}{3} \right\rceil + \left\lceil \tfrac{i+2}{3} \right\rceil + \left\lceil \tfrac{(i+1)+2}{3} \right\rceil = i + 3,$$
$$\text{for } i = \tfrac{2n+1}{3}, \tfrac{2n+4}{3}, \ldots, n-1,$$
$$\text{thus the corresponding weights are } \tfrac{2n+10}{3}, \tfrac{2n+13}{3}, \ldots, n+2,$$
$$wt_f(v_n) = f(v_{n-1}) + f(v_n) = \left\lceil \tfrac{(n-1)+2}{3} \right\rceil + \left\lceil \tfrac{n+2}{3} \right\rceil = \tfrac{2n+4}{3}.$$

$\square$

Lemma 4.3. *Let n be a positive integer, $n \equiv 5 \pmod 9$, $n \geq 5$. Then*

$$\widehat{\mathrm{dis}}(P_n) = \begin{cases} 3, & \text{for } n = 5, \\ \tfrac{n+1}{3}, & \text{for } n \equiv 5 \pmod 9. \end{cases}$$

Proof. Let $n \equiv 5 \pmod 9$, $n \geq 5$. The lower bound for $\widehat{\mathrm{dis}}(P_n)$ is given by (3), thus

$$\widehat{\mathrm{dis}}(P_n) \geq \left\lceil \tfrac{n+1}{3} \right\rceil = \tfrac{n+1}{3}.$$

First, let $n = 5$. Thus $\widehat{\mathrm{dis}}(P_5) \geq 2$. We prove that $\widehat{\mathrm{dis}}(P_n) > 2$. Consider on contrary that there exists an inclusive distance vertex irregular 2-labeling of P_5. This means that all numbers $2, 3, 4, 5$ and 6 must be realizable as vertex weights. Note, that the weight 2 we can get only as $1 + 1$ and the weight 6 we can only get as $2 + 2 + 2$, this implies that two incident vertices, say v_1 and v_2, must be labeled by 1 and three incident vertices, say v_3, v_4 and v_5, must be labeled by label 2. But, in this case the weights of vertices v_2 and v_5 are the same (equal to 4). A contradiction.

An inclusive distance vertex irregular 3-labeling f of P_5 is

$$f(v_1) = f(v_2) = 1, \quad f(v_3) = f(v_4) = f(v_5) = 3.$$

For $n \geq 14$ let us consider a labeling $f : V(P_n) \rightarrow \{1, 2, \ldots, (n+1)/3\}$ defined in the following way

$$
\begin{aligned}
f(v_i) &= \tfrac{i+2}{3}, &&\text{for } i = 1, 4, \ldots, n-1, \\
f(v_i) &= \tfrac{i+1}{3}, &&\text{for } i = 2, 5, \ldots, \tfrac{2n+2}{3} - 5, \\
f(v_i) &= \tfrac{i+4}{3}, &&\text{for } i = \tfrac{2n+2}{3} - 2, \tfrac{2n+2}{3} + 1, \ldots, n-3, \\
f(v_n) &= \tfrac{n-5}{3}, && \\
f(v_i) &= \tfrac{i}{3}, &&\text{for } i = 3, 6, \ldots, n-5, \\
f(v_{n-2}) &= \tfrac{n+1}{3}. &&
\end{aligned}
$$

Maximal number used as a vertex label is $(n+1)/3$. The vertex weights are

$$
\begin{aligned}
wt_f(v_1) =&\, f(v_1) + f(v_2) = \tfrac{1+2}{3} + \tfrac{2+1}{3} = 2, \\
wt_f(v_i) =&\, f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \tfrac{i-1}{3} + \tfrac{i+2}{3} + \tfrac{(i+1)+1}{3} = i+1, \\
&\text{for } i = 4, 7, \ldots, \tfrac{2n+2}{3} - 6, \\
&\text{thus the corresponding weights are } 5, 8, \ldots, \tfrac{2n+2}{3} - 5, \\
wt_f(v_i) =&\, f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \tfrac{i-1}{3} + \tfrac{i+2}{3} + \tfrac{(i+1)+4}{3} = i+2, \\
&\text{for } i = \tfrac{2n+2}{3} - 3, \tfrac{2n+2}{3}, \ldots, n-7, \\
&\text{thus the corresponding weights are } \tfrac{2n+2}{3} - 1, \tfrac{2n+2}{3} + 2, \ldots, n-5, \\
wt_f(v_{n-4}) =&\, f(v_{n-5}) + f(v_{n-4}) + f(v_{n-3}) = \tfrac{n-5}{3} + \tfrac{(n-4)+2}{3} + \tfrac{(n-3)+4}{3} = n-2, \\
wt_f(v_{n-1}) =&\, f(v_{n-2}) + f(v_{n-1}) + f(v_n) = \tfrac{n+1}{3} + \tfrac{(n-1)+2}{3} + \tfrac{n-5}{3} = n-1, \\
wt_f(v_i) =&\, f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \tfrac{(i-1)+2}{3} + \tfrac{i+1}{3} + \tfrac{i+1}{3} = i+1, \\
&\text{for } i = 2, 5, \ldots, \tfrac{2n+2}{3} - 5, \\
&\text{thus the corresponding weights are } 3, 6, \ldots, \tfrac{2n+2}{3} - 4, \\
wt_f(v_i) =&\, f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \tfrac{(i-1)+2}{3} + \tfrac{i+4}{3} + \tfrac{i+1}{3} = i+2, \\
&\text{for } i = \tfrac{2n+2}{3} - 2, \tfrac{2n+2}{3} + 1, \ldots, n-6, \\
&\text{thus the corresponding weights are } \tfrac{2n+2}{3}, \tfrac{2n+2}{3} + 3, \ldots, n-4, \\
wt_f(v_{n-3}) =&\, f(v_{n-4}) + f(v_{n-3}) + f(v_{n-2}) = \tfrac{(n-4)+2}{3} + \tfrac{(n-3)+4}{3} + \tfrac{n+1}{3} = n, \\
wt_f(v_n) =&\, f(v_{n-1}) + f(v_n) = \tfrac{(n-1)+2}{3} + \tfrac{n-5}{3} = \tfrac{2n-4}{3}, \\
wt_f(v_i) =&\, f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \tfrac{(i-1)+1}{3} + \tfrac{i}{3} + \tfrac{(i+1)+2}{3} = i+1, \\
&\text{for } i = 3, 6, \ldots, \tfrac{2n+2}{3} - 4, \\
&\text{thus the corresponding weights are } 4, 7, \ldots, \tfrac{2n+2}{3} - 3, \\
wt_f(v_i) =&\, f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \tfrac{(i-1)+4}{3} + \tfrac{i}{3} + \tfrac{(i+1)+2}{3} = i+2, \\
&\text{for } i = \tfrac{2n+2}{3} - 1, \tfrac{2n+2}{3} + 2, \ldots, n-5, \\
&\text{thus the corresponding weights are } \tfrac{2n+2}{3} + 1, \tfrac{2n+2}{3} + 4, \ldots, n-3, \\
wt_f(v_{n-2}) =&\, f(v_{n-3}) + f(v_{n-2}) + f(v_{n-1}) = \tfrac{(n-3)+4}{3} + \tfrac{n+1}{3} + \tfrac{(n-1)+2}{3} = n+1.
\end{aligned}
$$

Thus the set of vertex weights is $\{2, 3, \ldots, n+1\}$. ☐

Lemma 4.4. *Let n be a positive integer, $n \equiv 8 \pmod 9$, $n \geq 8$. Then*

$$\widehat{\mathrm{dis}}(P_n) = \tfrac{n+1}{3}.$$

Proof. Let $n \equiv 8 \pmod 9$, $n \geq 8$. We define a vertex labeling $f : V(P_n) \to \{1, 2, \ldots, (n+1)/3\}$ in the following way

$$
\begin{aligned}
f(v_i) &= \left\lceil \tfrac{i}{3} \right\rceil, &&\text{for } i = 1, 2, \ldots, \tfrac{2n-1}{3}, \\
f(v_i) &= \left\lfloor \tfrac{i}{3} \right\rfloor + 1, &&\text{for } i = \tfrac{2n+2}{3}, \tfrac{2n+5}{3}, \ldots, n.
\end{aligned}
$$

It is easy to see that every vertex label is at most $(n+1)/3 = \lceil (n+1)/3 \rceil$. Thus, according to (3) we only need to show that the corresponding vertex weights are distinct. In particular:

$$
\begin{aligned}
wt_f(v_1) &= f(v_1) + f(v_2) = \left\lceil \tfrac{1}{3} \right\rceil + \left\lceil \tfrac{2}{3} \right\rceil = 2, \\
wt_f(v_i) &= f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left\lceil \tfrac{i-1}{3} \right\rceil + \left\lceil \tfrac{i}{3} \right\rceil + \left\lceil \tfrac{i+1}{3} \right\rceil = i + 1, \\
&\qquad \text{for } i = 2, 3, \ldots, \tfrac{2n-4}{3}, \\
&\qquad \text{thus the corresponding weights are } 3, 4, \ldots, \tfrac{2n-1}{3}, \\
wt_f(v_{\frac{2n-1}{3}}) &= f(v_{\frac{2n-4}{3}}) + f(v_{\frac{2n-1}{3}}) + f(v_{\frac{2n+2}{3}}) = \left\lceil \tfrac{\frac{2n-4}{3}}{3} \right\rceil + \left\lceil \tfrac{\frac{2n-1}{3}}{3} \right\rceil + \left(\left\lfloor \tfrac{\frac{2n+2}{3}}{3} \right\rfloor + 1 \right) \\
&= \tfrac{2n+5}{3}, \\
wt_f(v_{\frac{2n+2}{3}}) &= f(v_{\frac{2n-1}{3}}) + f(v_{\frac{2n+2}{3}}) + f(v_{\frac{2n+5}{3}}) = \left\lceil \tfrac{\frac{2n-1}{3}}{3} \right\rceil + \left(\left\lfloor \tfrac{\frac{2n+2}{3}}{3} \right\rfloor + 1 \right) \\
&\quad + \left(\left\lfloor \tfrac{\frac{2n+5}{3}}{3} \right\rfloor + 1 \right) = \tfrac{2n+8}{3}, \\
wt_f(v_i) &= f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left(\left\lfloor \tfrac{i-1}{3} \right\rfloor + 1 \right) + \left(\left\lfloor \tfrac{i}{3} \right\rfloor + 1 \right) + \left(\left\lfloor \tfrac{i+1}{3} \right\rfloor + 1 \right) = i + 2, \\
&\qquad \text{for } i = \tfrac{2n+5}{3}, \tfrac{2n+8}{3}, \ldots, n-1, \\
&\qquad \text{thus the corresponding weights are } \tfrac{2n+11}{3}, \tfrac{2n+14}{3}, \ldots, n+1, \\
wt_f(v_n) &= f(v_{n-1}) + f(v_n) = \left(\left\lfloor \tfrac{n-1}{3} \right\rfloor + 1 \right) + \left(\left\lfloor \tfrac{n}{3} \right\rfloor + 1 \right) = \tfrac{2n+2}{3}.
\end{aligned}
$$

Thus the edge weights are distinct. ☐

Lemma 4.5. *Let n be a positive integer, $n \equiv 2 \pmod 9$, $n \geq 11$. Then*

$$\tfrac{n+1}{3} \leq \widehat{\mathrm{dis}}(P_n) \leq \tfrac{n+4}{3}.$$

Proof. Let $n \equiv 2 \pmod 9$, $n \geq 11$. We define a vertex labeling $f : V(P_n) \to \{1, 2, \ldots, k\}$ in the following way

$$
\begin{aligned}
f(v_i) &= \left\lceil \tfrac{i}{3} \right\rceil, &&\text{for } i = 1, 2, \ldots, \tfrac{2n+5}{3}, \\
f(v_i) &= \left\lceil \tfrac{i}{3} \right\rceil + 1, &&\text{for } i = \tfrac{2n+8}{3}, \tfrac{2n+11}{3}, \ldots, n.
\end{aligned}
$$

Evidently, the vertex labels are not greater that $\lceil n/3 \rceil + 1 = (n+4)/3$. For the vertex weights we get

$$
\begin{aligned}
wt_f(v_1) =& f(v_1) + f(v_2) = \left\lceil \tfrac{1}{3} \right\rceil + \left\lceil \tfrac{2}{3} \right\rceil = 2, \\
wt_f(v_i) =& f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left\lceil \tfrac{i-1}{3} \right\rceil + \left\lceil \tfrac{i}{3} \right\rceil + \left\lceil \tfrac{i+1}{3} \right\rceil = i + 1, \\
& \text{for } i = 2, 3, \ldots, \tfrac{2n+2}{3}, \\
& \text{thus the corresponding weights are } 3, 4, \ldots, \tfrac{2n+5}{3}, \\
wt_f(v_{\frac{2n+5}{3}}) =& f(v_{\frac{2n+2}{3}}) + f(v_{\frac{2n+5}{3}}) + f(v_{\frac{2n+8}{3}}) = \left\lceil \tfrac{\frac{2n+2}{3}}{3} \right\rceil + \left\lceil \tfrac{\frac{2n+5}{3}}{3} \right\rceil + \left(\left\lceil \tfrac{\frac{2n+8}{3}}{3} \right\rceil + 1 \right) \\
=& \tfrac{2n+11}{3}, \\
wt_f(v_{\frac{2n+8}{3}}) =& f(v_{\frac{2n+5}{3}}) + f(v_{\frac{2n+8}{3}}) + f(v_{\frac{2n+11}{3}}) = \left\lceil \tfrac{\frac{2n+5}{3}}{3} \right\rceil + \left(\left\lceil \tfrac{\frac{2n+8}{3}}{3} \right\rceil + 1 \right) \\
& + \left(\left\lceil \tfrac{\frac{2n+11}{3}}{3} \right\rceil + 1 \right) = \tfrac{2n+17}{3}, \\
wt_f(v_i) =& f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left(\left\lceil \tfrac{i-1}{3} \right\rceil + 1 \right) + \left(\left\lceil \tfrac{i}{3} \right\rceil + 1 \right) + \left(\left\lceil \tfrac{i+1}{3} \right\rceil + 1 \right) = i + 4, \\
& \text{for } i = \tfrac{2n+11}{3}, \tfrac{2n+14}{3}, \ldots, n - 1, \\
& \text{thus the corresponding weights are } \tfrac{2n+23}{3}, \tfrac{2n+26}{3}, \ldots, n + 3, \\
wt_f(v_n) =& f(v_{n-1}) + f(v_n) = \left(\left\lceil \tfrac{n-1}{3} \right\rceil + 1 \right) + \left(\left\lceil \tfrac{n}{3} \right\rceil + 1 \right) = \tfrac{2n+8}{3}.
\end{aligned}
$$

We proved that the vertex weights of distinct vertices are distinct. This means that f is an inclusive distance vertex irregular $((n+4)/3)$-labeling. Thus $\widehat{\mathrm{dis}}(P_n) \le (n+4)/3$. $\qquad\square$

The join of a path P_n, $n \ge 2$, and a complete graph K_1 is called a fan graph F_n. Combining Theorems 3.2 and 4.1 we obtain that for fans holds the following.

Theorem 4.2. *Let n be a positive integer $n \ge 2$. Then*

$$
\widehat{\mathrm{dis}}(F_n) = \begin{cases}
\infty, & \text{for } n = 2, \\
3, & \text{for } n = 5, \\
\left\lceil \tfrac{n+1}{3} \right\rceil, & \text{for } n \not\equiv 2 \pmod 9, \ n \ne 5
\end{cases}
$$

and

$$
\tfrac{n+1}{3} \le \widehat{\mathrm{dis}}(F_n) \le \tfrac{n+1}{3} + 1,
$$

when $n \equiv 2 \pmod 9$, $n \ge 11$.

5. Cycle

We denote the vertices and edges of a cycle C_n on n, $n \ge 3$, vertices in the following way

$$
V(C_n) = \{v_i : i = 1, 2, \ldots, n\},
$$

$$E(C_n) = \{v_i v_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{v_1 v_n\}.$$

As $\Delta(C_n) = \delta(C_n) = 2$ using Lemma 2.1 we obtain a lower bound of the distance vertex irregularity strength of a cycle

$$\widehat{\mathrm{dis}}(C_n) \geq \left\lceil \tfrac{n+2}{3} \right\rceil . \tag{4}$$

As C_3 is isomorphic to a complete graph K_3 using Corollary 3.1 we get $\widehat{\mathrm{dis}}(C_3) = \infty$. From a lower bound we get $\widehat{\mathrm{dis}}(C_4) \geq 2$. But it is easy to prove that $\widehat{\mathrm{dis}}(C_4) \geq 4$. The corresponding inclusive distance vertex irregular 4-labeling f of C_4 is

$$f(v_i) = i, \quad \text{for } 1 \leq i \leq 4.$$

Some results for the inclusive distance vertex irregularity strength of cycles are obtained in [2]. Combining and extending these results we get the following.

Theorem 5.1. *Let n be a positive integer $n \geq 3$. Then*

$$\widehat{\mathrm{dis}}(C_n) = \begin{cases} \infty, & \text{for } n = 3, \\ 4, & \text{for } n = 4, \\ \left\lceil \tfrac{n+2}{3} \right\rceil, & \text{for } n \not\equiv 2, 3, 4 \pmod{18}, \, n \geq 5 \end{cases}$$

and

$$\left\lceil \tfrac{n+2}{3} \right\rceil \leq \widehat{\mathrm{dis}}(C_n) \leq \left\lceil \tfrac{n+2}{3} \right\rceil + 1,$$

when $n \equiv 2, 3, 4 \pmod{18}$, $n \geq 20$.

According to (4) if we want to prove the equality it is suffices to describe the corresponding inclusive distance vertex irregular labelings for cycles. These labelings are given in Lemmas 5.1, 5.2, 5.3 and 5.4.

Lemma 5.1. *Let n be a positive integer, $n \equiv 5, 0, 1 \pmod{6}$, $n \geq 5$. Then*

$$\widehat{\mathrm{dis}}(C_n) = \left\lceil \tfrac{n+2}{3} \right\rceil .$$

Proof. Let $n \equiv 1 \pmod{6}$, $n \geq 7$. We define a vertex labeling $f : V(C_n) \to \{1, 2, \ldots, (n+2)/3\}$ of C_n in the following way

$$\begin{aligned}
f(v_i) &= 2\left\lceil \tfrac{i}{3} \right\rceil - 1, & \text{for } i = 1, 2, \ldots, \tfrac{n+5}{2}, \\
f(v_i) &= \tfrac{2(n-i)}{3} + 2, & \text{for } i = \tfrac{n+7}{2}, \tfrac{n+13}{2}, \ldots, n, \\
f(v_i) &= \tfrac{2(n-i+1)}{3} + 1, & \text{for } i = \tfrac{n+9}{2}, \tfrac{n+15}{2}, \ldots, n-2, \\
f(v_i) &= \tfrac{2(n-i+2)}{3} + 1, & \text{for } i = \tfrac{n+11}{2}, \tfrac{n+17}{2}, \ldots, n-1.
\end{aligned}$$

Evidently, the vertex labels are not greater that $\lceil (n+2)/3 \rceil = (n+2)/3$. For the vertex weights we get

$$wt_f(v_1) = f(v_n) + f(v_1) + f(v_2) = \left(\tfrac{2(n-n)}{3} + 2 \right) + \left(2 \left\lceil \tfrac{1}{3} \right\rceil - 1 \right) + \left(2 \left\lceil \tfrac{2}{3} \right\rceil - 1 \right) = 4,$$

$$wt_f(v_i) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left(2 \left\lceil \tfrac{i-1}{3} \right\rceil - 1 \right) + \left(2 \left\lceil \tfrac{i}{3} \right\rceil - 1 \right) + \left(2 \left\lceil \tfrac{i+1}{3} \right\rceil - 1 \right)$$
$$= 2i - 1,$$
$$\text{for } i = 2, 3, \ldots, \tfrac{n+3}{2},$$
$$\text{thus the corresponding weights are } 3, 5, \ldots, n+2,$$

$$wt_f(v_{\frac{n+5}{2}}) = f(v_{\frac{n+3}{2}}) + f(v_{\frac{n+5}{2}}) + f(v_{\frac{n+7}{2}}) = \left(2 \left\lceil \tfrac{\frac{n+3}{2}}{3} \right\rceil - 1 \right) + \left(2 \left\lceil \tfrac{\frac{n+5}{2}}{3} \right\rceil - 1 \right)$$
$$+ \left(\tfrac{2\left(n - \frac{n+7}{2}\right)}{3} + 2 \right) = n+1,$$

$$wt_f(v_{\frac{n+7}{2}}) = f(v_{\frac{n+5}{2}}) + f(v_{\frac{n+7}{2}}) + f(v_{\frac{n+9}{2}}) = \left(2 \left\lceil \tfrac{\frac{n+5}{2}}{3} \right\rceil - 1 \right) + \left(\tfrac{2\left(n - \frac{n+7}{2}\right)}{3} + 2 \right)$$
$$+ \left(\tfrac{2\left(n - \frac{n+9}{2} + 1\right)}{3} + 1 \right) = n-1,$$

$$wt_f(v_i) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left(\tfrac{2(n-(i-1))}{3} + 2 \right) + \left(\tfrac{2(n-i+1)}{3} + 1 \right)$$
$$+ \left(\tfrac{2(n-(i+1)+2)}{3} + 1 \right) = 2n + 6 - 2i,$$
$$\text{for } i = \tfrac{n+9}{2}, \tfrac{n+15}{2}, \ldots, n-2,$$
$$\text{thus the corresponding weights are } 10, 16, \ldots, n-3,$$

$$wt_f(v_i) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left(\tfrac{2(n-(i-1)+1)}{3} + 1 \right) + \left(\tfrac{2(n-i+2)}{3} + 1 \right)$$
$$+ \left(\tfrac{2(n-(i+1))}{3} + 2 \right) = 2n + 6 - 2i,$$
$$\text{for } i = \tfrac{n+11}{2}, \tfrac{n+17}{2}, \ldots, n-1,$$
$$\text{thus the corresponding weights are } 8, 14, \ldots, n-5,$$

$$wt_f(v_i) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left(\tfrac{2(n-(i-1)+2)}{3} + 1 \right) + \left(\tfrac{2(n-i)}{3} + 2 \right)$$
$$+ \left(\tfrac{2(n-(i+1)+1)}{3} + 1 \right) = 2n + 6 - 2i,$$
$$\text{for } i = \tfrac{n+13}{2}, \tfrac{n+19}{2}, \ldots, n-3,$$
$$\text{thus the corresponding weights are } 12, 18, \ldots, n-7,$$

$$wt_f(v_n) = f(v_{n-1}) + f(v_n) + f(v_1) = \left(\tfrac{2(n-(n-1)+2)}{3} + 1 \right) + \left(\tfrac{2(n-n)}{3} + 2 \right) + \left(2 \left\lceil \tfrac{1}{3} \right\rceil - 1 \right)$$
$$= 6.$$

Thus the vertex weights are distinct numbers from the set $\{3, 4, \ldots, n+2\}$. This means that f is an inclusive distance vertex irregular $((n+2)/3)$-labeling. Combining this and (4), for $n \equiv 1$

$\pmod 6$, $n \geq 7$, we get

$$\widehat{\mathrm{dis}}(C_n) = \tfrac{n+2}{3}.$$

Let us consider the cycle C_{n-1} that we obtain from C_n such that

$$C_{n-1} = C_n - \{v_2\} \cup \{v_1 v_3\}.$$

It is easy to get that the restriction of the labeling f defined above on the graph $C_{n-1} = C_n - \{v_2\} \cup \{v_1 v_3\}$ is an inclusive distance vertex irregular $((n+2)/3)$-labeling of C_{n-1}. In particular, the vertex weights are $\{4, 5, \dots, n+2\}$. For $n - 1 \equiv 0 \pmod 6$, $n \geq 7$ according to (4) we get

$$\widehat{\mathrm{dis}}(C_{n-1}) \geq \left\lceil \tfrac{(n-1)+2}{3} \right\rceil = \tfrac{n+2}{3}.$$

This means that for $n \equiv 0 \pmod 6$, $n \geq 6$

$$\widehat{\mathrm{dis}}(C_n) = \left\lceil \tfrac{n+2}{3} \right\rceil.$$

Now we will deal with the case when the order of a cycle is congruent 5 modulo 6. We consider the cycle C_{n-2} obtained from C_n by deleting two vertices and adding corresponding edges

$$C_{n-2} = C_n - \{v_2, v_5\} \cup \{v_1 v_3, v_4 v_6\}.$$

Again, the restriction of the labeling f defined above on the graph $C_{n-2} = C_n - \{v_2, v_5\} \cup \{v_1 v_3, v_4 v_6\}$ is an inclusive distance vertex irregular $((n+2)/3)$-labeling of C_{n-2}. In particular, the vertex weights are $\{4, 5, \dots, 8, 10, 11, \dots, n+2\}$. As $(n-2) \equiv 5 \pmod 6$, $n \geq 5$ then according to (4)

$$\widehat{\mathrm{dis}}(C_{n-2}) \geq \left\lceil \tfrac{(n-2)+2}{3} \right\rceil = \tfrac{n+2}{3}.$$

This means that for $n \equiv 5 \pmod 6$, $n \geq 5$

$$\widehat{\mathrm{dis}}(C_n) = \left\lceil \tfrac{n+2}{3} \right\rceil.$$

$\square$

Lemma 5.2. *Let n be a positive integer, $n \equiv 8, 9, 10 \pmod{18}$, $n \geq 8$. Then*

$$\widehat{\mathrm{dis}}(C_n) = \left\lceil \tfrac{n+2}{3} \right\rceil.$$

Proof. Let $n \equiv 10 \pmod{18}$, $n \geq 10$. We define a vertex labeling $f : V(C_n) \to \{1, 2, \dots, (n+2)/3\}$ in the following way

$$
\begin{aligned}
f(v_i) &= \left\lceil \tfrac{i}{3} \right\rceil, & \text{for } i &= 1, 2, \dots, \tfrac{n+5}{3}, \\
f(v_i) &= \left\lceil \tfrac{i+1}{3} \right\rceil, & \text{for } i &= \tfrac{n+8}{3}, \tfrac{n+11}{3}, \dots, \tfrac{2n-5}{3}, \\
f(v_i) &= \left\lceil \tfrac{i+2}{3} \right\rceil, & \text{for } i &= \tfrac{2n-2}{3}, \tfrac{2n+1}{3}, \dots, n.
\end{aligned}
$$

Every vertex label is at most $\lceil (n+2)/3 \rceil = (n+2)/3$. The vertex weights are

$$
\begin{aligned}
wt_f(v_1) =& f(v_n) + f(v_1) + f(v_2) = \left\lceil \tfrac{n+2}{3} \right\rceil + \left\lceil \tfrac{1}{3} \right\rceil + \left\lceil \tfrac{2}{3} \right\rceil = \tfrac{n+8}{3}, \\
wt_f(v_i) =& f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left\lceil \tfrac{i-1}{3} \right\rceil + \left\lceil \tfrac{i}{3} \right\rceil + \left\lceil \tfrac{i+1}{3} \right\rceil = i + 1, \\
& \text{for } i = 2, 3, \ldots, \tfrac{n+2}{3}, \\
& \text{thus the corresponding weights are } 3, 4, \ldots, \tfrac{n+5}{3},
\end{aligned}
$$

$$
wt_f(v_{\frac{n+5}{3}}) = f(v_{\frac{n+2}{3}}) + f(v_{\frac{n+5}{3}}) + f(v_{\frac{n+8}{3}}) = \left\lceil \tfrac{\frac{n+2}{3}}{3} \right\rceil + \left\lceil \tfrac{\frac{n+5}{3}}{3} \right\rceil + \left\lceil \tfrac{\frac{n+8}{3}+1}{3} \right\rceil = \tfrac{n+11}{3},
$$

$$
wt_f(v_{\frac{n+8}{3}}) = f(v_{\frac{n+5}{3}}) + f(v_{\frac{n+8}{3}}) + f(v_{\frac{n+11}{3}}) = \left\lceil \tfrac{\frac{n+5}{3}}{3} \right\rceil + \left\lceil \tfrac{\frac{n+8}{3}+1}{3} \right\rceil + \left\lceil \tfrac{\frac{n+11}{3}+1}{3} \right\rceil = \tfrac{n+14}{3},
$$

$$
\begin{aligned}
wt_f(v_i) =& f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left\lceil \tfrac{(i-1)+1}{3} \right\rceil + \left\lceil \tfrac{i+1}{3} \right\rceil + \left\lceil \tfrac{(i+1)+1}{3} \right\rceil = i + 2, \\
& \text{for } i = \tfrac{n+11}{3}, \tfrac{n+14}{3}, \ldots, \tfrac{2n-8}{3}, \\
& \text{thus the corresponding weights are } \tfrac{n+17}{3}, \tfrac{n+20}{3}, \ldots, \tfrac{2n-2}{3},
\end{aligned}
$$

$$
wt_f(v_{\frac{2n-5}{3}}) = f(v_{\frac{2n-8}{3}}) + f(v_{\frac{2n-5}{3}}) + f(v_{\frac{2n-2}{3}}) = \left\lceil \tfrac{\frac{2n-8}{3}+1}{3} \right\rceil + \left\lceil \tfrac{\frac{2n-5}{3}+1}{3} \right\rceil + \left\lceil \tfrac{\frac{2n-2}{3}+2}{3} \right\rceil = \tfrac{2n+1}{3},
$$

$$
wt_f(v_{\frac{2n-2}{3}}) = f(v_{\frac{2n-5}{3}}) + f(v_{\frac{2n-2}{3}}) + f(v_{\frac{2n+1}{3}}) = \left\lceil \tfrac{\frac{2n-5}{3}+1}{3} \right\rceil + \left\lceil \tfrac{\frac{2n-2}{3}+2}{3} \right\rceil + \left\lceil \tfrac{\frac{2n+1}{3}+2}{3} \right\rceil = \tfrac{2n+4}{3},
$$

$$
\begin{aligned}
wt_f(v_i) =& f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left\lceil \tfrac{(i-1)+2}{3} \right\rceil + \left\lceil \tfrac{i+2}{3} \right\rceil + \left\lceil \tfrac{(i+1)+2}{3} \right\rceil = i + 3, \\
& \text{for } i = \tfrac{2n+1}{3}, \tfrac{2n+4}{3}, \ldots, n - 1, \\
& \text{thus the corresponding weights are } \tfrac{2n+10}{3}, \tfrac{2n+13}{3}, \ldots, n + 2,
\end{aligned}
$$

$$
wt_f(v_n) = f(v_{n-1}) + f(v_n) + f(v_1) = \left\lceil \tfrac{(n-1)+2}{3} \right\rceil + \left\lceil \tfrac{n+2}{3} \right\rceil + \left\lceil \tfrac{1}{3} \right\rceil = \tfrac{2n+7}{3}.
$$

The set of vertex weights consists of numbers $\{3, 4, \ldots, n + 2\}$. This means that f is an inclusive distance vertex irregular $((n + 2)/3)$-labeling. Thus, using (4), for $n \equiv 10 \pmod{18}$, $n \geq 10$, we have

$$
\widehat{\operatorname{dis}}(C_n) = \tfrac{n+2}{3}.
$$

As in the proof of the previous lemma we use the above described labeling to obtain $((n+2)/3)$-labelings of C_{n-1} and C_{n-2}.

The cycle C_{n-1} we obtain from C_n such that

$$
C_{n-1} = C_n - \{v_2\} \cup \{v_1 v_3\}.
$$

The restriction of the labeling f defined above on the graph $C_{n-1} = C_n - \{v_2\} \cup \{v_1 v_3\}$ is an inclusive distance vertex irregular $((n+2)/3)$-labeling of C_{n-1}. The vertex weights are $\{4, 5, \ldots, n+2\}$. As $(n - 1) \equiv 9 \pmod{18}$, $n \geq 10$ then according to (4)

$$
\widehat{\operatorname{dis}}(C_{n-1}) \geq \left\lceil \tfrac{(n-1)+2}{3} \right\rceil = \tfrac{n+2}{3}.
$$

This means that for $n \equiv 9 \pmod{18}$, $n \geq 9$

$$\widehat{\mathrm{dis}}(C_n) = \left\lceil \frac{n+2}{3} \right\rceil.$$

If $n \equiv 8 \pmod{18}$, $n \geq 8$ let us consider the cycle C_{n-2} obtained from C_n by deleting two vertices and adding corresponding edges

$$C_{n-2} = C_n - \{v_2, v_{n-1}\} \cup \{v_1 v_3, v_n v_{n-2}\}.$$

It is easy to see that the restriction of the labeling f defined above on the graph $C_{n-2} = C_n - \{v_2, v_{n-1}\} \cup \{v_1 v_3, v_n v_{n-2}\}$ is an inclusive distance vertex irregular $((n+2)/3)$-labeling of C_{n-2}. The set of vertex weights is $\{4, 5, \ldots, n+1\}$. As $(n-2) \equiv 8 \pmod{18}$, $n \geq 10$ then according to (4)

$$\widehat{\mathrm{dis}}(C_{n-2}) \geq \left\lceil \frac{(n-2)+2}{3} \right\rceil = \frac{n+2}{3}.$$

Thus for $n \equiv 8 \pmod{18}$, $n \geq 8$

$$\widehat{\mathrm{dis}}(C_n) = \left\lceil \frac{n+2}{3} \right\rceil.$$

$\square$

Lemma 5.3. *Let n be a positive integer, $n \equiv 14, 15, 16 \pmod{18}$, $n \geq 8$. Then*

$$\widehat{\mathrm{dis}}(C_n) = \left\lceil \frac{n+2}{3} \right\rceil.$$

Proof. Let $n \equiv 16 \pmod{18}$, $n \geq 16$. We consider a labeling $f : V(C_n) \to \{1, 2, \ldots, (n+2)/3\}$ defined such that

$$
\begin{aligned}
f(v_i) &= \left\lceil \tfrac{i}{3} \right\rceil, & &\text{for } i = 1, 2, \ldots, \tfrac{n+5}{3}, \\
f(v_i) &= \left\lceil \tfrac{i+2}{3} \right\rceil, & &\text{for } i = \tfrac{n+8}{3}, \tfrac{n+17}{3}, \ldots, \tfrac{2n-8}{3}, \\
f(v_i) &= \left\lceil \tfrac{i+2}{3} \right\rceil - 1, & &\text{for } i = \tfrac{n+11}{3}, \tfrac{n+20}{3}, \ldots, \tfrac{2n-5}{3}, \\
f(v_i) &= \left\lceil \tfrac{i+2}{3} \right\rceil, & &\text{for } i = \tfrac{n+14}{3}, \tfrac{n+23}{3}, \ldots, \tfrac{2n-2}{3}, \\
f(v_i) &= \left\lceil \tfrac{i+2}{3} \right\rceil, & &\text{for } i = \tfrac{2n+1}{3}, \tfrac{2n+4}{3}, \ldots, n.
\end{aligned}
$$

The numbers used as vertex labels are not greater than $\lceil (n+2)/3 \rceil = (n+2)/3$. For the vertex weights we have the following

$$
\begin{aligned}
wt_f(v_1) &= f(v_n) + f(v_1) + f(v_2) = \left\lceil \tfrac{n+2}{3} \right\rceil + \left\lceil \tfrac{1}{3} \right\rceil + \left\lceil \tfrac{2}{3} \right\rceil = \tfrac{n+8}{3}, \\
wt_f(v_i) &= f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left\lceil \tfrac{i-1}{3} \right\rceil + \left\lceil \tfrac{i}{3} \right\rceil + \left\lceil \tfrac{i+1}{3} \right\rceil = i + 1, \\
&\qquad \text{for } i = 2, 3, \ldots, \tfrac{n+2}{3}, \\
&\qquad \text{thus the corresponding weights are } 3, 4, \ldots, \tfrac{n+5}{3}, \\
wt_f(v_{\frac{n+5}{3}}) &= f(v_{\frac{n+2}{3}}) + f(v_{\frac{n+5}{3}}) + f(v_{\frac{n+8}{3}}) = \left\lceil \tfrac{\frac{n+2}{3}}{3} \right\rceil + \left\lceil \tfrac{\frac{n+5}{3}}{3} \right\rceil + \left\lceil \tfrac{\frac{n+8}{3}+2}{3} \right\rceil = \tfrac{n+11}{3},
\end{aligned}
$$

$$wt_f(v_{\frac{n+8}{3}}) = f(v_{\frac{n+5}{3}}) + f(v_{\frac{n+8}{3}}) + f(v_{\frac{n+11}{3}}) = \left\lceil \frac{\frac{n+5}{3}}{3} \right\rceil + \left\lceil \frac{\frac{n+8}{3}+2}{3} \right\rceil + \left(\left\lceil \frac{\frac{n+11}{3}+2}{3} \right\rceil - 1 \right)$$

$$= \frac{n+14}{3},$$

$$wt_f(v_i) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left\lceil \frac{(i-1)+2}{3} \right\rceil + \left(\left\lceil \frac{i+2}{3} \right\rceil - 1 \right) + \left\lceil \frac{(i+1)+2}{3} \right\rceil = i+2,$$

$$\text{for } i = \tfrac{n+11}{3}, \tfrac{n+20}{3}, \ldots, \tfrac{2n-5}{3},$$

thus the corresponding weights are $\frac{n+17}{3}, \frac{n+26}{3}, \ldots, \frac{2n+1}{3}$,

$$wt_f(v_i) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left(\left\lceil \frac{(i-1)+2}{3} \right\rceil - 1 \right) + \left\lceil \frac{i+2}{3} \right\rceil + \left\lceil \frac{(i+1)+2}{3} \right\rceil = i+2,$$

$$\text{for } i = \tfrac{n+14}{3}, \tfrac{n+23}{3}, \ldots, \tfrac{2n-11}{3},$$

thus the corresponding weights are $\frac{n+20}{3}, \frac{n+29}{3}, \ldots, \frac{2n-5}{3}$,

$$wt_f(v_i) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left\lceil \frac{(i-1)+2}{3} \right\rceil + \left\lceil \frac{i+2}{3} \right\rceil + \left(\left\lceil \frac{(i+1)+2}{3} \right\rceil - 1 \right) = i+2,$$

$$\text{for } i = \tfrac{n+17}{3}, \tfrac{n+26}{3}, \ldots, \tfrac{2n-8}{3},$$

thus the corresponding weights are $\frac{n+23}{3}, \frac{n+32}{3}, \ldots, \frac{2n-2}{3}$,

$$wt_f(v_{\frac{2n-2}{3}}) = f(v_{\frac{2n-5}{3}}) + f(v_{\frac{2n-2}{3}}) + f(v_{\frac{2n+1}{3}}) = \left(\left\lceil \frac{\frac{2n-5}{3}+2}{3} \right\rceil - 1 \right) + \left\lceil \frac{\frac{2n-2}{3}+2}{3} \right\rceil + \left\lceil \frac{\frac{2n+1}{3}+2}{3} \right\rceil$$

$$= \frac{2n+4}{3},$$

$$wt_f(v_{\frac{2n+1}{3}}) = f(v_{\frac{2n-2}{3}}) + f(v_{\frac{2n+1}{3}}) + f(v_{\frac{2n+4}{3}}) = \left\lceil \frac{\frac{2n-2}{3}+2}{3} \right\rceil + \left\lceil \frac{\frac{2n+1}{3}+2}{3} \right\rceil + \left\lceil \frac{\frac{2n+4}{3}+2}{3} \right\rceil$$

$$= \frac{2n+10}{3},$$

$$wt_f(v_i) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left\lceil \frac{(i-1)+2}{3} \right\rceil + \left\lceil \frac{i+2}{3} \right\rceil + \left\lceil \frac{(i+1)+2}{3} \right\rceil = i+3,$$

$$\text{for } i = \tfrac{2n+4}{3}, \tfrac{2n+7}{3}, \ldots, n-1,$$

thus the corresponding weights are $\frac{2n+13}{3}, \frac{2n+16}{3}, \ldots, n+2$,

$$wt_f(v_n) = f(v_{n-1}) + f(v_n) + f(v_1) = \left\lceil \frac{(n-1)+2}{3} \right\rceil + \left\lceil \frac{n+2}{3} \right\rceil + \left\lceil \frac{1}{3} \right\rceil = \frac{2n+7}{3}.$$

The set of vertex weights consists of numbers $\{3, 4, \ldots, n+2\}$. This means that f is an inclusive distance vertex irregular $((n+2)/3)$-labeling. Thus for $n \equiv 16 \pmod{18}$, $n \geq 16$, using (4), we obtain

$$\widehat{\mathrm{dis}}(C_n) = \tfrac{n+2}{3}.$$

Now we describe $((n+2)/3)$-labelings of cycles C_{n-1} and C_{n-2}. First we consider the cycle C_{n-1} obtained from the cycle C_n such that

$$C_{n-1} = C_n - \{v_2\} \cup \{v_1 v_3\}.$$

The restriction of the labeling f defined above on the graph $C_{n-1} = C_n - \{v_2\} \cup \{v_1 v_3\}$ is an inclusive distance vertex irregular $((n+2)/3)$-labeling of C_{n-1}. The vertex weights are $\{4, 5, \ldots, n+2\}$. According to (4) and as $(n-1) \equiv 15 \pmod{18}$, $n \geq 16$ we get

$$\widehat{\mathrm{dis}}(C_{n-1}) \geq \left\lceil \frac{(n-1)+2}{3} \right\rceil = \tfrac{n+2}{3}.$$

Thus for $n \equiv 15 \pmod{18}$, $n \geq 15$

$$\widehat{\mathrm{dis}}(C_n) = \left\lceil \tfrac{n+2}{3} \right\rceil.$$

Now we will deal with the cycle C_{n-2} obtained from C_n such that

$$C_{n-2} = C_n - \{v_2, v_{n-1}\} \cup \{v_1 v_3, v_n v_{n-2}\}.$$

Evidently, the restriction of the labeling f defined above on the graph $C_{n-2} = C_n - \{v_2, v_{n-1}\} \cup \{v_1 v_3, v_n v_{n-2}\}$ is an inclusive distance vertex irregular $((n+2)/3)$-labeling of C_{n-2}. The set of vertex weights is $\{4, 5, \ldots, n+1\}$. As $(n-2) \equiv 14 \pmod{18}$, $n \geq 16$ then according to (4)

$$\widehat{\mathrm{dis}}(C_{n-2}) \geq \left\lceil \tfrac{(n-2)+2}{3} \right\rceil = \tfrac{n+2}{3}.$$

Thus for $n \equiv 14 \pmod{18}$, $n \geq 14$

$$\widehat{\mathrm{dis}}(C_n) = \left\lceil \tfrac{n+2}{3} \right\rceil.$$

$\square$

Lemma 5.4. *Let n be a positive integer, $n \equiv 2, 3, 4 \pmod{18}$, $n \geq 20$. Then*

$$\left\lceil \tfrac{n+2}{3} \right\rceil \leq \widehat{\mathrm{dis}}(C_n) \leq \left\lceil \tfrac{n+2}{3} \right\rceil + 1.$$

Proof. The lower bound follows from (4). The upper bound we prove by describing the corresponding $((n+2)/3 + 1)$-labelings of cycles C_n.

Let $n \equiv 4 \pmod{18}$, $n \geq 22$. We consider a labeling $f : V(C_n) \to \{1, 2, \ldots, (n+2)/3 + 1\}$ of vertices of C_n defined such that

$$
\begin{aligned}
f(v_i) &= \left\lceil \tfrac{i}{3} \right\rceil, & \text{for } i &= 1, 2, \ldots, \tfrac{n+8}{3}, \\
f(v_i) &= \left\lceil \tfrac{i+2}{3} \right\rceil, & \text{for } i &= \tfrac{n+11}{3}, \tfrac{n+14}{3}, \ldots, \tfrac{n}{2}, \\
f(v_i) &= \left\lceil \tfrac{i}{3} \right\rceil + 1, & \text{for } i &= \tfrac{n}{2} + 1, \tfrac{n}{2} + 2, \ldots, \tfrac{2n-8}{3}, \\
f(v_i) &= \left\lceil \tfrac{i+2}{3} \right\rceil + 1, & \text{for } i &= \tfrac{2n-5}{3}, \tfrac{2n-2}{3}, \ldots, n.
\end{aligned}
$$

Every vertex label is smaller or equal to $\lceil (n+2)/3 \rceil + 1 = (n+2)/3 + 1$. The vertex weights under the labeling f are

$$
\begin{aligned}
wt_f(v_1) &= f(v_n) + f(v_1) + f(v_2) = \left(\left\lceil \tfrac{n+2}{3} \right\rceil + 1 \right) + \left\lceil \tfrac{1}{3} \right\rceil + \left\lceil \tfrac{2}{3} \right\rceil = \tfrac{n+11}{3}, \\
wt_f(v_i) &= f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left\lceil \tfrac{i-1}{3} \right\rceil + \left\lceil \tfrac{i}{3} \right\rceil + \left\lceil \tfrac{i+1}{3} \right\rceil = i + 1, \\
&\qquad \text{for } i = 2, 3, \ldots, \tfrac{n+5}{3}, \\
&\qquad \text{thus the corresponding weights are } 3, 4, \ldots, \tfrac{n+8}{3}, \\
wt_f(v_{\frac{n+8}{3}}) &= f(v_{\frac{n+5}{3}}) + f(v_{\frac{n+8}{3}}) + f(v_{\frac{n+11}{3}}) = \left\lceil \tfrac{\frac{n+5}{3}}{3} \right\rceil + \left\lceil \tfrac{\frac{n+8}{3}}{3} \right\rceil + \left\lceil \tfrac{\frac{n+11}{3}+2}{3} \right\rceil = \tfrac{n+14}{3},
\end{aligned}
$$

$$wt_f(v_{\frac{n+11}{3}}) = f(v_{\frac{n+8}{3}}) + f(v_{\frac{n+11}{3}}) + f(v_{\frac{n+14}{3}}) = \left\lceil \frac{\frac{n+8}{3}}{3} \right\rceil + \left\lceil \frac{\frac{n+11}{3}+2}{3} \right\rceil + \left\lceil \frac{\frac{n+14}{3}+2}{3} \right\rceil = \frac{n+20}{3},$$

$$wt_f(v_i) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left\lceil \frac{(i-1)+2}{3} \right\rceil + \left\lceil \frac{i+2}{3} \right\rceil + \left\lceil \frac{(i+1)+2}{3} \right\rceil = i + 3,$$

$$\text{for } i = \tfrac{n+14}{3}, \tfrac{n+17}{3}, \ldots, \tfrac{n}{2} - 1,$$

$$\text{thus the corresponding weights are } \tfrac{n+23}{3}, \tfrac{n+26}{3}, \ldots, \tfrac{n}{2} + 2,$$

$$wt_f(v_{\frac{n}{2}}) = f(v_{\frac{n-2}{2}}) + f(v_{\frac{n}{2}}) + f(v_{\frac{n+2}{2}}) = \left\lceil \frac{\frac{n-2}{2}+2}{3} \right\rceil + \left\lceil \frac{\frac{n}{2}+2}{3} \right\rceil + \left(\left\lceil \frac{\frac{n+2}{2}}{3} \right\rceil + 1 \right) = \tfrac{n}{2} + 3,$$

$$wt_f(v_{\frac{n+2}{2}}) = f(v_{\frac{n}{2}}) + f(v_{\frac{n+2}{2}}) + f(v_{\frac{n+4}{2}}) = \left\lceil \frac{\frac{n}{2}+2}{3} \right\rceil + \left(\left\lceil \frac{\frac{n+2}{2}}{3} \right\rceil + 1 \right) + \left(\left\lceil \frac{\frac{n+4}{2}}{3} \right\rceil + 1 \right)$$

$$= \tfrac{n}{2} + 5,$$

$$wt_f(v_i) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left(\left\lceil \frac{i-1}{3} \right\rceil + 1 \right) + \left(\left\lceil \frac{i}{3} \right\rceil + 1 \right) + \left(\left\lceil \frac{i+1}{3} \right\rceil + 1 \right) = i + 4,$$

$$\text{for } i = \tfrac{n}{2} + 2, \tfrac{n}{2} + 3, \ldots, \tfrac{2n-11}{3},$$

$$\text{thus the corresponding weights are } \tfrac{n}{2} + 6, \tfrac{n}{2} + 7, \ldots, \tfrac{2n+1}{3},$$

$$wt_f(v_{\frac{2n-8}{3}}) = f(v_{\frac{2n-11}{3}}) + f(v_{\frac{2n-8}{3}}) + f(v_{\frac{2n-5}{3}}) = \left(\left\lceil \frac{\frac{2n-11}{3}}{3} \right\rceil + 1 \right) + \left(\left\lceil \frac{\frac{2n-8}{3}}{3} \right\rceil + 1 \right)$$

$$+ \left(\left\lceil \frac{\frac{2n-5}{3}+2}{3} \right\rceil + 1 \right) = \tfrac{2n+4}{3},$$

$$wt_f(v_{\frac{2n-5}{3}}) = f(v_{\frac{2n-8}{3}}) + f(v_{\frac{2n-5}{3}}) + f(v_{\frac{2n-2}{3}}) = \left(\left\lceil \frac{\frac{2n-8}{3}}{3} \right\rceil + 1 \right) + \left(\left\lceil \frac{\frac{2n-5}{3}+2}{3} \right\rceil + 1 \right)$$

$$+ \left(\left\lceil \frac{\frac{2n-2}{3}+2}{3} \right\rceil + 1 \right) = \tfrac{2n+10}{3},$$

$$wt_f(v_i) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = \left(\left\lceil \frac{(i-1)+2}{3} \right\rceil + 1 \right) + \left(\left\lceil \frac{i+2}{3} \right\rceil + 1 \right) + \left(\left\lceil \frac{(i+1)+2}{3} \right\rceil + 1 \right)$$

$$= i + 6,$$

$$\text{for } i = \tfrac{2n-2}{3}, \tfrac{2n+1}{3}, \ldots, n - 1,$$

$$\text{thus the corresponding weights are } \tfrac{2n+16}{3}, \tfrac{2n+19}{3}, \ldots, n + 5,$$

$$wt_f(v_n) = f(v_{n-1}) + f(v_n) + f(v_1) = \left(\left\lceil \frac{(n-1)+2}{3} \right\rceil + 1 \right) + \left(\left\lceil \frac{n+2}{3} \right\rceil + 1 \right) + \left\lceil \frac{1}{3} \right\rceil = \tfrac{2n+13}{3}.$$

The set of vertex weights consist of numbers $\{3, 4, \ldots, \frac{n+14}{3}, \frac{n+20}{3}, \frac{n+23}{3}, \ldots, \frac{n}{2} + 3, \frac{n}{2} + 5, \frac{n}{2} + 6,$ $\ldots, \frac{2n+4}{3}, \frac{2n+10}{3}, \frac{2n+13}{3}, \ldots, n + 5\}$. This means that f is an inclusive distance vertex irregular $((n + 2)/3 + 1)$-labeling. Thus for $n \equiv 4 \pmod{18}$, $n \geq 22$

$$\widehat{\text{dis}}(C_n) \leq \tfrac{n+2}{3} + 1.$$

First we consider the cycle C_{n-1} obtained from C_n in the following way

$$C_{n-1} = C_n - \{v_2\} \cup \{v_1 v_3\}.$$

The restriction of the labeling f defined above on the graph $C_{n-1} = C_n - \{v_2\} \cup \{v_1 v_3\}$ is an inclusive distance vertex irregular $((n + 2)/3 + 1)$-labeling of C_{n-1}. The vertex weights are

$\{4, 5, \ldots, \frac{n+14}{3}, \frac{n+20}{3}, \frac{n+23}{3}, \ldots, \frac{n}{2} + 3, \frac{n}{2} + 5, \frac{n}{2} + 6, \ldots, \frac{2n+4}{3}, \frac{2n+10}{3}, \frac{2n+13}{3}, \ldots, n + 5\}$. Thus for $n \equiv 3 \pmod{18}$, $n \geq 21$

$$\widehat{\text{dis}}(C_n) \leq \lceil \tfrac{n+2}{3} \rceil + 1.$$

Now let us consider the cycle C_{n-2} obtained from C_n such that

$$C_{n-2} = C_n - \{v_2, v_{n-1}\} \cup \{v_1 v_3, v_n v_{n-2}\}.$$

It is easy to see that the restriction of the labeling f defined above on the graph $C_{n-2} = C_n - \{v_2, v_{n-1}\} \cup \{v_1 v_3, v_n v_{n-2}\}$ is an inclusive distance vertex irregular $((n + 2)/3 + 1)$-labeling of C_{n-2}. The vertex weights are distinct numbers from the set $\{4, 5, \ldots, \frac{n+14}{3}, \frac{n+20}{3}, \frac{n+23}{3}, \ldots, \frac{n}{2} + 3, \frac{n}{2} + 5, \frac{n}{2} + 6, \ldots, \frac{2n+4}{3}, \frac{2n+10}{3}, \frac{2n+13}{3}, \ldots, n + 4\}$. However, this proves that for $n \equiv 2 \pmod{18}$, $n \geq 20$

$$\widehat{\text{dis}}(C_n) \leq \lceil \tfrac{n+2}{3} \rceil + 1.$$

$\square$

The join of a cycle C_n, $n \geq 3$, and a complete graph K_1 is a graph known as a wheel W_n. Thus from Theorems 5.1 and 3.2, we have the following.

Theorem 5.2. *Let n be a positive integer $n \geq 3$. Then*

$$\widehat{\text{dis}}(W_n) = \begin{cases} \infty, & \text{for } n = 3, \\ 4, & \text{for } n = 4, \\ \lceil \frac{n+2}{3} \rceil, & \text{for } n \not\equiv 2, 3, 4 \pmod{18}, n \geq 5 \end{cases}$$

and

$$\lceil \tfrac{n+2}{3} \rceil \leq \widehat{\text{dis}}(W_n) \leq \lceil \tfrac{n+2}{3} \rceil + 1,$$

when $n \equiv 2, 3, 4 \pmod{18}$, $n \geq 20$.

While Bong *et al.* [2] also proved that $\widehat{\text{dis}}(W_n) = \widehat{\text{dis}}(C_n)$.

6. Conclusion

In the foregoing sections we studied the existence of inclusive vertex irregular distance labelings of graphs. We established a lower bound of the distance vertex irregularity strength and determined the exact value of this parameter for complete and complete bipartite graphs and for join graphs $G \oplus K_1$.

For a path P_n we determined the exact value of the inclusive distance vertex irregularity strength for every $n \geq 2$ except for $n \equiv 2 \pmod{9}$, when $n \geq 11$. For $n \equiv 2 \pmod{9}$, $n \geq 11$, we found only an upper bound. Consequently, we propose the following open problem.

Problem 6.1. *For a path P_n, $n \equiv 2 \pmod{9}$, $n \geq 11$, determine the exact value of the inclusive distance vertex irregularity strength.*

For cycle C_n we determined the exact value of the inclusive distance vertex irregularity strength for every $n \geq 3$ except for $n \equiv 2, 3, 4 \pmod{18}$ when $n \geq 20$. For these values of n we described the inclusive diatance vertex irregular $(\lceil (n+2)/3 \rceil + 1)$-labeling which gives an upper bound of the inclusive distance vertex irregularity strength. So, we suggest the following open problem.

Problem 6.2. *For the cycle C_n, $n \equiv 2, 3, 4 \pmod{18}$, $n \geq 20$, determine the exact value of the inclusive distance vertex irregularity strength.*

For both cases mentioned in Problems 6.1 and 6.2 we suppose that the corresponding parameters reach the lower bounds.

According to obtained results for inclusive distance vertex irregularity strength of paths, cycles, fan graphs and wheels it seems that the subgraph relation posses the hereditary properties with respect to $\widehat{\text{dis}}(G)$. We state the following open problem.

Problem 6.3. *Determine if $H \subset G$ implies $\widehat{\text{dis}}(H) \leq \widehat{\text{dis}}(G)$.*

Another interesting problem is whether the corresponding hereditary properties are preserved also for other graph operations, for example Cartesian product or subdivision of edges.

Acknowledgement

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