Electronic Journal of Graph Theory and Applications **6** (1) (2018), 178–189

Electronic Journal of Graph Theory and Applications

On the intersection power graph of a finite group

Sudip Bera

Department of Mathematics
Visva-Bharati, Santiniketan-731235, India

sudipbera517@gmail.com

Abstract

Given a group G, the intersection power graph of G, denoted by $\mathcal{G}_I(G)$, is the graph with vertex set G and two distinct vertices x and y are adjacent in $\mathcal{G}_I(G)$ if there exists a non-identity element $z \in G$ such that $x^m = z = y^n$, for some $m, n \in \mathbb{N}$, i.e. $x \sim y$ in $\mathcal{G}_I(G)$ if $\langle x \rangle \cap \langle y \rangle \neq \{e\}$ and e is adjacent to all other vertices, where e is the identity element of the group G. Here we show that the graph $\mathcal{G}_I(G)$ is complete if and only if either G is cyclic p-group or G is a generalized quaternion group. Furthermore, $\mathcal{G}_I(G)$ is Eulerian if and only if $|G|$ is odd. We characterize all abelian groups and also all non-abelian p-groups G, for which $\mathcal{G}_I(G)$ is dominatable. Beside, we determine the automorphism group of the graph $\mathcal{G}_I(\mathbb{Z}_n)$, when $n \neq p^m$.

Keywords: automorphism group, intersection power graph, planar, p-groups.
Mathematics Subject Classification : 05C25
DOI: 10.5614/ejgta.2018.6.1.13

1. Introduction

Given an algebraic structure S, we can associate S to a directed or undirected graph in different ways. To study different algebraic structures using graph theory, different graphs have been formulated namely, commuting graph associate to a group [8], [20], power graph of a semigroup [11], strong power graph of a group [6], [18], normal subgroup based power graphs of a group [7], zero divisor graph of a rings [3] etc. Kelarev and Quinn introduced the directed power graph of a group [17]. Then Chakraborty et.al [11] introduced the undirected power graph $\mathcal{G}(G)$ of a

Received: 11 September 2017, Revised: 24 December 2017, Accepted: 10 January 2018.

semigroup G, where the vertex set of the graph is G and two distinct vertices x, y are adjacent if either $x = y^m$ or $y = x^n$ for some $m, n \in \mathbb{N}$. Again the commuting graph $\mathcal{C}(G)$ of a group G is the graph whose vertex set is G and two distinct vertices x, y are adjacent if $xy = yx$. Clearly for every group G, the power graph $\mathcal{G}(G)$ is a subgraph of the commuting graph $\mathcal{C}(G)$. In [1], Aalipour et.al characterized the finite groups G for which the power graph $\mathcal{G}(G)$ is same as the commuting graph. But when they are not equal, they measured how close the power graph is to the commuting graph by introducing a new graph, called enhanced power graph. The enhanced power graph $\mathcal{G}_e(G)$ of a group G is the graph whose vertex set is the group G and two distinct vertices x, y are adjacent if there exists $z \in G$ such that $x = z^m$ and $y = z^n$ for some $m, n \in \mathbb{N}$.

Here we define a graph of a finite group G namely intersection power graph, denoted by $\mathcal{G}_I(G)$. The vertex set of the graph is G and two distinct vertices x, y are adjacent in $\mathcal{G}_I(G)$ if $\langle x \rangle \cap \langle y \rangle \neq \{e\}$ and e is adjacent to all other vertices in $\mathcal{G}_I(G)$. Clearly the graph $\mathcal{G}_I(G)$ is connected.

Before proceeding further, let us talk about the motivation for defining this new graph. Let us closely examine the definitions of power graph, enhanced power graph and the intersection power graph. Frankly speaking all these three graphs are a bit misnomer, as we see that the term"power" in their names as nothing special to do with. In all these cases, we are considering the poset of cyclic subgroups of the finite group G. For example, in power graph x, y are adjacent if and only if either $\langle x \rangle \subset \langle y \rangle$ or $\langle y \rangle \subset \langle x \rangle$, i.e. the cyclic subgroups generated by x and y are comparable in the poset of cyclic subgroups of G. Now take the enhanced power graph of G, in this case two vertices x and y are adjacent if and only if there exists $z \in G$ such that $\langle x \rangle, \langle y \rangle \in \langle z \rangle$, i.e. the cyclic subgroups generated by x and y have a upper bound in the poset of cyclic subgroups of G. So the next natural task is to define a new graph Γ on G, where two vertices x and y are adjacent if and only if the cyclic subgroups generated by x and y have a lower bound in the poset of cyclic subgroups of G. Note that our definition of the intersection power graph is a slight modification of Γ. This observation also suggests that one can defined new graphs on algebraic structures by studying the poset of some suitable substructures and their Hasse diagram to visualize their algebraic properties through graphs.

In this article, some basic structures of intersection power graph have been studied. Throughout this article G stand for a finite group. We denote $o(x)$ to be the order of an element x in G. $|S|$ is the number of elements present in the set S. $\pi_e(G) = \{o(x) : x \in G\}$, $\pi(G) = \{p \in \mathbb{N} : p$ divides $|G|$ and p is a prime$\}$, For $a \in G, \pi(a) = \{p \in \mathbb{N} : p | o(a)$ and p is a prime$\}$. For any vertex v, $deg(v)$ is the number of vertices adjacent to v. For a graph Γ, $E(\Gamma)$ is the set of all edges in the graph Γ and $V(\Gamma)$ is the set of all vertices of the graph Γ and $e_1 = |E(\Gamma)|$. For a positive integer $r, [r] = \{1, 2, \cdots, r\}$. For a prime p, a group G is called a *p-group* if every element of G is of order p^m for some $m \in \{0\} \bigcup \mathbb{N}$. It follows from the Cauchy's Theorem that a finite group G is a *p-group* if and only if $|G| = p^t$ for some non negative integer t. We refer to [15], [19] for graph theory and to [14], [16] for group theoretic background.

2. Definitions and some properties

Given a group G, the intersection power graph of G, denoted by $\mathcal{G}_I(G)$, is the graph with vertex set G and two distinct vertices x and y are adjacent in $\mathcal{G}_I(G)$ if there exists non-identity element $z \in G$ such that $x^m = z = y^n$, for some $m, n \in \mathbb{N}$. i.e. $x \sim y$ in $\mathcal{G}_I(G)$ if $\langle x \rangle \cap \langle y \rangle \neq \{e\}$, the identity element e of the group G is adjacent to all other vertices. From the definition the intersection power graph $\mathcal{G}_I(G)$ is connected. Let $a \in G$. We denote G_a to be the set of all generators of the cyclic subgroup $\langle a \rangle$ of G. Then $G = \cup_{a \in G} G_a$. Clearly $G_e = \{e\}$ and G_e is a clique in $\mathcal{G}_I(G)$. Now we show some basic properties of the intersection power graph $\mathcal{G}_I(G)$.

Proposition 2.1. *Let G be a group. Then for each non-identity element $a \in G, G_a$ forms a clique in $\mathcal{G}_I(G)$.*

Proof. Let x, y be any two vertices of $\mathcal{G}_I(G)$ in G_a. Since $\langle x \rangle = \langle y \rangle = \langle a \rangle$ we have $e \neq a \in \langle x \rangle \cap \langle y \rangle$ and hence $x \sim y$. Hence for each $a \in G, G_a$ is a clique in $\mathcal{G}_I(G)$. □

Corollary 2.1. *Let G be a group and $m \in \mathbb{N}$ for which there is an element $a \in G$ such that $o(a) = m$. Then $\mathcal{G}_I(G)$ has a complete subgraph isomorphic to $K_{\phi(m)+1}$.*

Proposition 2.2. *Let G be a group and $G_a \neq G_b$ for two distinct elements $a, b \in G$. If an element of G_a is adjacent to an element of G_b, then each element of G_a is adjacent to every elements of G_b.*

Proof. Suppose that $G_a \neq G_b$. Let $x \in G_a$, $y \in G_b$ with $x \sim y$. Then there exists $z (\neq e) \in G$ such that $z \in \langle x \rangle \cap \langle y \rangle$. Now for any $x_1 \in G_a$ and any $y_1 \in G_b$ we have $\langle x \rangle = \langle x_1 \rangle$ and $\langle y \rangle = \langle y_1 \rangle$. So $\langle x \rangle \cap \langle y \rangle \neq \{e\}$ implies that $\langle x_1 \rangle \cap \langle y_1 \rangle \neq \{e\}$. Hence all the vertices in G_a are adjacent to all the vertices in G_b. □

Corollary 2.2. *Let G be a cyclic group. Suppose that m_1, m_2 are two positive integers for which m_1, m_2 divide $|G|$ and gcd$(m_1, m_2) \neq 1$. Then $\mathcal{G}_I(G)$ has a complete subgraph isomorphic to $K_{\phi(m_1)+\phi(m_2)+1}$.*

Theorem 2.1. *Let G be a group. Then the intersection power graph $\mathcal{G}_I(G)$ of the group G contains a cycle if and only if $o(a) \geq 3$, for some $a \in G$.*

Proof. First suppose that $\pi_e(G) \subset \{1, 2\}$. Then for every $a \in G \setminus \{e\}$, G_a contains exactly one element. Therefore, $a, b \in G \setminus \{e\}$, $G_a \cap G_b = \{e\}$ implying a is not adjacent to b. Hence the intersection graph $\mathcal{G}_I(G)$ has no cycle.

Conversely, suppose that $a \in G$ such that $o(a) \geq 3$. Then $|G_a| \geq 2$. So the vertices in G_a with the identity form a cycle. Hence the result holds. □

If G is a finite group such that $o(a) = 2$ for every non-identity element a of G, then G is abelian and $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. Also a connected graph Γ is *tree* if and only if it has no cycle. Now we have the following corollary.

Corollary 2.3. *Let G be a group. Then the following conditions are equivalent.*

1. *$\mathcal{G}_I(G)$ is bipartite;*
2. *$\mathcal{G}_I(G)$ is tree;*
3. *$G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$;*
4. *$\mathcal{G}_I(G)$ is a star graph.*

3. Complete intersection power graph

In this section we characterize all groups G for which the intersection power graph $\mathcal{G}_I(G)$ is complete or Cayley graph of some group. A graph Γ is *complete* if any two vertices of the graph Γ are adjacent.

Theorem 3.1. *Let G be a group. Then the intersection power graph $\mathcal{G}_I(G)$ of the group G is complete if and only if either G is a cyclic p-group or G is a generalized quaternion group.*

Proof. First suppose that G is a cyclic p-group. Then $a, b \in G \setminus \{e\}$, $o(a) = p^{k_1}$ and $o(b) = p^{k_2}, k_1, k_2 \in \mathbb{N}$. If $k_1 \geq k_2$ then $\langle b\rangle \subset \langle a\rangle$ implies that $\langle b\rangle \cap \langle a\rangle \neq \{e\}$. So $a \sim b$ in the intersection graph $\mathcal{G}_I(G)$. Now suppose that G is a generalized quaternion group. Then G is a 2-group with unique nontrivial minimal subgroup H say and $|H| = 2$. Now $a, b \in G \setminus \{e\}$, $\langle a\rangle$ and $\langle b\rangle$ are cyclic 2-groups implies that $H \subset \langle a\rangle \cap \langle b\rangle$. Hence $a \sim b$ in $\mathcal{G}_I(G)$.

Conversely, suppose that the graph $\Gamma_I(G)$ is complete. First we show that G must be a p-group. If not, $|G|$ has at least two distinct prime factors, say p_1, p_2. Now there exists $x, y \in G$ such that $o(x) = p_1$ and $o(y) = p_2$ and $\langle x\rangle \cap \langle y\rangle = \{e\}$. This implies that x is not adjacent to y in $\mathcal{G}_I(G)$ a contradiction, so G must be a p-group. Now we show that G has a unique nontrivial minimal subgroup. If not, let $K_1 = \langle a_1\rangle$ and $K_2 = \langle a_2\rangle$ be two distinct nontrivial minimal subgroups of G. Since G is a p-group and $|K_1| = |K_2| = p$ we have $K_1 \cap K_2 = \{e\}$. So a_1 is not adjacent with a_2 in $\mathcal{G}_I(G)$. Which contradicts that $\mathcal{G}_I(G)$ is complete graph. So for a finite group G, the graph $\mathcal{G}_I(G)$ is complete implies that G is a p-group with a unique nontrivial minimal subgroup. So if G is abelian then it is a cyclic otherwise it is a generalized quaternion group [16]. □

Let G be a group and C be a subset of G that is closed under taking inverses and does not contain the identity. Then the *Cayley graph* $\Gamma(G, C)$ is the graph with the vertex set $V(\Gamma(G, C)) = G$ and two vertices a and b are adjacent if $ab^{-1} \in C$. Every complete graph with n-vertices is the Cayley graph $\Gamma(\mathbb{Z}_n, \mathbb{Z}_n \setminus \{0\})$. It is well known that every Cayley graph is regular.

Theorem 3.2. *Let G be a finite group. Then $\mathcal{G}_I(G)$ is a Cayley graph of some group if and only if either G is cyclic p-group or G is generalized quaternion group.*

Proof. Let G be a cyclic group of order p^m or it is generalized quaternion group. Then the intersection power graph $\mathcal{G}_I(G)$ is complete. Hence a Cayley graph.

Conversely, suppose that the graph $\mathcal{G}_I(G)$ is a Cayley graph of some group. Then $\mathcal{G}_I(G)$ is regular. Since the vertex e is adjacent to every other vertices, it follows that $\mathcal{G}_I(G)$ is complete. Hence the result. □

A graph Γ is said to be *planar* if it can be drawn in a plane so that no two edges intersect. A graph is planer if and only if it does not contain a graph which is isomorphic to either of the graphs $K_{3,3}$ or K_5.

Theorem 3.3. *Let G be a group. If there is a prime $p \geq 5$ such that p is a divisor of $|G|$. Then the intersection power graph $\mathcal{G}_I(G)$ is not planar.*

Proof. Suppose that, there is a prime $p \geq 5$ such that $p||G|$. Then there is an element a of order p. Now by Proposition 2.1, G_a forms a clique in $\mathcal{G}_I(G)$ and $|G_a| \geq 4$ as $\phi(p) \geq 4$. So the vertices in G_a with e forms a clique which is isomorphic to $K_{\phi(p)+1}$. So we get K_5 in $\mathcal{G}_I(G)$. Hence the intersection power graph is not planar. □

So for the intersection power graph $\mathcal{G}_I(G)$ to be planar, it is necessary that $|G| = 2^r 3^k, r, k \in \mathbb{N}$.

Theorem 3.4. *Let G be a group of order $2^r, r \in \mathbb{N}$. Then the intersection power graph $\mathcal{G}_I(G)$ is planar if and only if $o(a) \leq 4$, for all $a \in G$ and $\langle a_1 \rangle \cap \langle a_2 \rangle = \{e\}$, where $\langle a_1 \rangle$ and $\langle a_2 \rangle$ two distinct cyclic subgroups of G of order 4.*

Proof. Let $\mathcal{G}_I(G)$ be planar. Suppose there is an element $a \in G$ such that $o(a) = 2^m$ with $m \geq 3$. Now $\phi(2^m) = 2^{m-1} \geq 4$, as $m \geq 3$ and these 2^{m-1} vertices along with e form a clique containing a copy of K_5 in $\mathcal{G}_I(G)$, a contradiction. So order of each element of G is at most 4. Now let if possible there exists $a, b \in G$ such that $o(a) = o(b) = 4, \langle a \rangle \neq \langle b \rangle$ and $\langle a \rangle \cap \langle b \rangle \neq \{e\}$. Clearly $|G_a| = |G_b| = 2$. Now by Proposition 2.3 and $G_a \cup G_b \cup \{e\}$ forms a subgraph isomorphic to K_5.

Conversely, suppose that $o(a) \leq 4$ for all $a \in G$ and $\langle a_1 \rangle \cap \langle a_2 \rangle = \{e\}$, where $\langle a_1 \rangle$ and $\langle a_2 \rangle$ are two distinct cyclic subgroups of G of order 4 (for example, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ and D_4, the symmetric group of a square satisfies the above conditions). So for any two elements a_1, a_2 of order 4 with $\langle a_1 \rangle \neq \langle a_2 \rangle$, a_1 is not adjacent to a_2 in the graph $\mathcal{G}_I(G)$. Again any two vertices of order 2 of $\mathcal{G}_I(G)$ are not adjacent. Now for any element $x \in G$ with $o(x) = 4, \langle x \rangle$ has exactly one element y of order 2 and $x \sim y$. Hence it is clear that $\mathcal{G}_I(G)$ does not contain any copy of K_5. If possible there exists a copy of $K_{3,3}$ in $\mathcal{G}_I(G)$. That is we have two disjoint subsets of vertices namely, A and B such that $|A| = 3 = |B|$ and each vertex of A is adjacent to each vertex of B. Suppose that A contains a vertex v_1 of order 2. Note that the degree of the vertex v_1 is 3. Now by our assumptions v_1 belongs to exactly one cyclic subgroup H of order 4. Hence the elements of B are precisely identity and the two generators u_1, u_2 of H. Now $deg(u_1) = 3$ also and the adjacent vertices to u_1 are u_2, v_1 and e. So u_1 is not adjacent to each vertices in A, because u_2 and e does not belong to A, a contradiction. Now if we assume that A contains a vertex of order 4, then similarly as above we get a contradiction. Hence the intersection power graph is planar. □

Theorem 3.5. *Let G be a group of order $3^k, k \in \mathbb{N}$. Then the intersection graph $\mathcal{G}_I(G)$ is planar if and only if $o(a) = 3$, for all $a \in G \setminus \{e\}$.*

Proof. Let the graph $\mathcal{G}_I(G)$ be planar. Suppose that there is an element $a \in G$ such that $o(a) > 3$. So $o(a) = 3^r$, where $r \geq 2$. Then $\phi(3^r) = 23^{r-1} \geq 6$ and by Proposition 2.1, these 23^{r-1} vertices form a clique in the graph $\mathcal{G}_I(G)$. Since $23^{r-1} > 5$ we get a copy of K_5 in $\mathcal{G}_I(G)$, a contradiction.

Conversely suppose that $o(a) = 3$, for all $a \in G \setminus \{e\}$. Then $|G_a| = 2$ for any $a \in G \setminus \{e\}$. Let $a, b \in G$ such that $\langle a \rangle \neq \langle b \rangle$. Then for any $x \in G_a$ and $y \in G_b, \langle x \rangle \cap \langle y \rangle = \{e\}$. So x is not adjacent to b implies that $\mathcal{G}_I(G)$ planar. □

Combining the above two theorems we have the following theorem.

Theorem 3.6. *Let G be a group of order $2^m 3^n, m, n \in \mathbb{N}$. Then the intersection power graph is planar if and only if both of the following conditions hold:*

1. $o(a) = 2, 3$ *or* 4, *for all non-identity element* $a \in G$ *and.*
2. $\langle a_1 \rangle \cap \langle a_2 \rangle = \{e\}$, *where* $\langle a_1 \rangle$ *and* $\langle a_2 \rangle$ *two distinct cyclic subgroups of* G *of order* 4.

Proof. First suppose that G satisfies the above conditions. Then form Theorems 3.4 and 3.5 the intersection power graph is planar.

Conversely, let $\mathcal{G}_I(G)$ be planar. Now we show that G has no element of order $6k, k \in \mathbb{N}$. If possible there is an element $a \in G$ such that $o(a) = 6k$. Then $\langle a \rangle$ has $\phi(6k)$ generators and $\phi(6k) \geq 2$. Let $a_{16}, a_{26}, a_{12}, a_{13}, a_{23} \in \langle a \rangle$ with $o(a_{16}) = 6 = o(a_{26}), o(a_{12}) = 2$ and $o(a_{13}) = 3 = o(a_{23})$. Then take $A = \{a_{16}, a_{26}, e\}$ and $B = \{a_{12}, a_{13}, a_{23}\}$ as partition sets to form $K_{3,3}$ as subgraph of $\mathcal{G}_I(G)$. a contradiction. Hence the result holds. □

A graph Γ is called *Eulerian* if it has a closed trail containing all the vertices of Γ. An useful equivalent characterization of an Eulerian graph is that a graph Γ is Eulerian if and only if every vertex of Γ is of even degree.

Theorem 3.7. *Let G be a group of order n. Then the intersection power graph $\mathcal{G}_I(G)$ is Eulerian if and only if n is odd.*

Proof. The proof is similar to the proof of Theorem 2.5 in [4]. Suppose that the graph $\mathcal{G}_I(G)$ is Eulerian. Since the vertex e is edge connected with every other vertices of the graph $\mathcal{G}_I(G)$, it follows that the degree of e is $n-1$. Now $n-1$ is even implies that n is odd.

Conversely assume that n is odd. Then the degree of e in $\mathcal{G}_I(G)$ is $n-1$ and so even. Now we show that the degree of every non-identity element a is even. The vertex set of the intersection power graph can be written as $V(\mathcal{G}_I(G)) = \bigcup_{x \in G} G_x$. Now by Proposition 2.1, G_x form a clique for each $x \in G$. Again by Proposition 2.3, if $x \sim y$ then all the vertices in G_x are adjacent to all the vertices in G_y. Now G_y contains $\phi(o(y))$ vertices and every vertex is adjacent to e, so the degree of a vertex a in the graph $\mathcal{G}_I(G)$ is of the form $(\phi(o(a)) - 1) + \phi(o(x_1)) + \phi(o(x_2)) + \cdots + \phi(o(x_m)) + 1 = \phi(o(a)) + \phi(o(x_1)) + \phi(o(x_2)) + \cdots + \phi(o(x_m))$. Now n is odd implies that $o(x)$ is odd and so $\phi(o(x))$ is even for all $x \in G$. Thus the degree of every vertex of the graph $\mathcal{G}_I(G)$ is even. Hence the intersection power graph is Eulerian. □

4. Dominatability of intersection power graph

A vertex of a graph Γ is called a dominating vertex if it is adjacent to every other vertex. The identity element e is a dominating vertex of every intersection power graph $\mathcal{G}_I(G)$. We call an intersection power graph $\mathcal{G}_I(G)$ is dominatable if it has a dominating vertex other than e. In the context of power graphs, dominatability has been studied in [9], [10] and for enhanced power graph it was studied in [4]. Here we characterize all abelian groups and non-abelian p-groups G such that $\mathcal{G}_I(G)$ is dominatable. Throughout this section $p_1, p_2, \cdots, p_r$ are distinct primes and $\alpha_1, \alpha_2, \cdots, \alpha_r \in \mathbb{N} \cup \{0\}$. In the following theorem we characterize all finite abelian groups G for which the intersection power graph $\mathcal{G}_I(G)$ is dominatable.

Theorem 4.1. *Let G be a finite abelian group. Then the graph $\mathcal{G}_I(G)$ is dominatable if and only if G is a cyclic group.*

Proof. First suppose that the group G is cyclic. Then there exists $a \in G$ such that $G = \langle a \rangle$ and $\langle a \rangle \cap \langle x \rangle = \langle x \rangle$, for any $x \in G$. Hence a is a dominating vertex.

Conversely, suppose that the graph $\mathcal{G}_I(G)$ is dominatable. We show that the group G is cyclic. Let $a \in G$ is dominating vertex. Suppose $\pi(G) = \{p_1, p_2 \cdots, p_r\}$. Now G has elements a_i with order p_i for all $i = 1, 2, \cdots, r$. Since a is a dominating vertex $a \sim a_i (i = i, 2, \cdots, r)$ in $\mathcal{G}_I(G)$. Now we show that G has a unique subgroup of order p_i for all $i = 1, 2, \cdots, r$. If possible there exists $x, y \in G$ such that $o(x) = p_i = o(y)$ and $\langle x \rangle \neq \langle y \rangle$, for some p_i. Then $a \sim x$ and $a \sim y$ implies that $\langle x \rangle$ and $\langle y \rangle$ are subgroups of order p_i of $\langle a \rangle$, a contradiction since $\langle a \rangle$ is a cyclic group. So G has a unique subgroup of order $p_i, i = 1, 2, \cdots, r$. Since G is abelian, $G \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p_r^{\alpha_r}}$. Hence G is a cyclic group. □

Theorem 4.2. *Let G be a non-abelian group of order $p_1 p_2 \cdots p_r$. Then the intersection power graph $\mathcal{G}_I(G)$ does not satisfy the dominatability property, where $r \geq 2$.*

Proof. Suppose that $a \in G, (a \neq e)$ is a dominating vertex. Now G has elements a_{p_i} of order $p_i, (i = 1, 2, \cdots, r)$. Since a is a dominating vertex, $a \sim a_{p_i}$ for all $i = 1, 2, \cdots, r$ and $o(a_{p_i}) = p_i$ is a prime implies that $a_{p_i} \in \langle a \rangle \cap \langle a_{p_i} \rangle$. So $\langle a_{p_i} \rangle$ is a subgroup of $\langle a \rangle$ and $p_i | o(a)$, for all $i = 1, 2, \cdots, r$ implies that $p_1 p_2 \cdots p_r$ is a divisor of $o(a)$. Now $|G| = p_1 p_2 \cdots p_r$ implies $o(a) = p_1 p_2 \cdots p_r$. Hence G is a cyclic group and $G = \langle a \rangle$ contradicts the group G is non-abelian. □

Now we turn our attention to the non-abelian p-groups.

Theorem 4.3. *Let G be a non-abelian p-group. Then the intersection power graph is dominatable if and only if G is generalized quaternion group.*

Proof. Suppose G is generalized quaternion group. Then the intersection power graph is complete implies that it is dominatable.

Conversely, Let $\mathcal{G}_I(G)$ be dominatable. Let $a \in G$ be a dominating vertex. Now we claim that G has unique nontrivial minimal subgroup. Let H_1 and H_2 be two nontrivial minimal subgroups of G. Clearly $|H_1| = p = |H_2|$, a prime implies that H_1 and H_2 are cyclic groups. Let $H_1 = \langle b \rangle$ and $H_2 = \langle d \rangle$. Now from the given condition $a \sim b$ and $a \sim d$. Again $o(b) = o(d) = p$, a prime and $\langle a \rangle \cap \langle b \rangle \neq \{e\}$ implies that $\langle b \rangle$ is a subgroup of $\langle a \rangle$. Similarly $\langle d \rangle$ is a subgroup of $\langle a \rangle$ that contradicts a cyclic group contains unique subgroup of each order. So our claim is true and G is generalized quaternion group [16]. □

5. The automorphism group of the graph $\mathcal{G}_I(\mathbb{Z}_n)$

In this section we determine the automorphism group of the intersection power graph of any finite cyclic group. Let G be a cyclic group of order n. Then $G \cong \mathbb{Z}_n$ implies that $\mathcal{G}_I(G) \cong \mathcal{G}_I(\mathbb{Z}_n)$. Denote the automorphism group of $\mathcal{G}_I(\mathbb{Z}_n)$ by Aut($\mathcal{G}_I(\mathbb{Z}_n)$). First note that, if $n = p^m$, p is a prime, then $\mathcal{G}_I(\mathbb{Z}_n)$ is complete implies that Aut$(\mathcal{G}_I(\mathbb{Z}_n)) = S_n$. Here we show that, if $n \neq p^m$ then Aut$(\mathcal{G}_I(\mathbb{Z}_n)) = \bigoplus_{\phi \neq I \subsetneq [r]} S_{P^I - 1} \bigoplus S_{(p_1^{\alpha_1} - 1)(p_2^{\alpha_2} - 1) \cdots (p_r^{\alpha_r} - 1) + 1}$, where $P^I - 1 = (p_{i_1}^{\alpha_{i_1}} - 1)(p_{i_2}^{\alpha_{i_2}} - 1) \cdots (p_{i_k}^{\alpha_{i_k}} - 1)$ for $I = \{i_1, i_2, \cdots, i_k\}$. Throughout this section G is a cyclic group of order n, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}, p_1, p_2, \cdots, p_r$ are distinct primes, $r \geq 2$ and $\alpha_1, \alpha_2, \cdots, \alpha_r \in \mathbb{N}$. Denote X_d, the set of all vertices of degree d of the graph $\mathcal{G}_I(G)$.

Lemma 5.1. *Let G be a group. Suppose that $a \in G$ be such that $o(a) = p_{i_1}^{x_{i_1}} p_{i_2}^{x_{i_2}} \cdots p_{i_k}^{x_{i_k}}$. Then $deg(a) = n - \frac{n}{p_{i_1}^{\alpha_{i_1}} p_{i_2}^{\alpha_{i_2}} \cdots p_{i_k}^{\alpha_{i_k}}}$, where $1 \leq x_{i_j} \leq \alpha_{i_j}$ for all $j = 1, 2, \cdots, k$ and $k \leq r$.*

Proof. Without loss of generality we assume that $o(a) = p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$. First we count the number of vertices v which are not adjacent with the vertex a. Now v is not adjacent with a implies that there is no common divisor (except 1) between $o(a)$ and $o(v)$. So for any vertex v which is not adjacent to a, $o(v)$ is of the form $p_{k+1}^{x_{k+1}} p_{k+2}^{x_{k+2}} \cdots p_r^{x_r}$, where $0 \leq x_i \leq \alpha_i$ for all i and at least one $x_i > 0$. So, $o(v)$ is a divisor of $p_{k+1}^{\alpha_{k+1}} p_{k+2}^{\alpha_{k+2}} \cdots p_r^{\alpha_r}$. Hence the number of vertices v which are not adjacent to a is

$$\begin{aligned} T = & \sum_{d \mid p_{k+1}^{\alpha_{k+1}} p_{k+2}^{\alpha_{k+2}} \cdots p_r^{\alpha_r},\ d \neq 1} \phi(d) \\ = & (p_{k+1}^{\alpha_{k+1}} p_{k+2}^{\alpha_{k+2}} \cdots p_r^{\alpha_r} - 1). \end{aligned}$$

Hence the degree of the vertex a is

$$(n-1) - (p_{k+1}^{\alpha_{k+1}} p_{k+2}^{\alpha_{k+2}} \cdots p_r^{\alpha_r} - 1) = n - \frac{n}{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}}.$$

□

Now we determine the number of vertices v of the graph $\mathcal{G}_I(\mathbb{Z}_n)$ of same degree. Let v be any vertex of the graph $\mathcal{G}_I(\mathbb{Z}_n)$. If $o(v)$ contains prime factors $p_1, p_2, \cdots, p_k$, then from the above lemma $deg(v) = n - \frac{n}{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}}$. So the degree of any vertex of the graph $\mathcal{G}_I(\mathbb{Z}_n)$ depends only on the prime factors present in the order of that vertex not in the power of those primes.

Lemma 5.2. *Let G be a group. Let $a \in G$ be such that $o(a)$ contains the prime factors $p_{i_1} p_{i_2} \cdots p_{i_k}$ and $deg(a) = d$. Then the number of vertices of degree d in the graph $\mathcal{G}_I(\mathbb{Z}_n)$ is $(p_{i_1}^{\alpha_{i_1}} - 1)(p_{i_2}^{\alpha_{i_2}} - 1) \cdots (p_{i_k}^{\alpha_{i_k}} - 1)$, where $k \leq r$.*

Proof. Without loss of generality we assume that $o(a)$ contains the prime divisors $p_1, p_2, \cdots, p_k$. Let $Y = (y_1, y_2, \cdots, y_k)$ be a k-tuple of positive integers such that $1 \leq y_i \leq \alpha_i$ for all i. Clearly from Lemma 5.1 we get $d = n - \frac{n}{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}}$. Again remembering that $\pi(o(a)) = \pi(o(b))$ implies that $deg(a) = deg(b)$ in $\mathcal{G}_I(\mathbb{Z}_n)$. So the number of vertices of degree d is

$$\begin{aligned} S &= \sum_Y \phi(p_1^{y_1} p_2^{y_2} \cdots p_k^{y_k}) \\ &= \sum_Y \phi(p_1^{y_1}) \phi(p_2^{y_2}) \cdots \phi(p_k^{y_k}) \\ &= (\sum_{t=1}^{\alpha_1} \phi(p_1^t))(\sum_{t=1}^{\alpha_2} \phi(p_2^t)) \cdots (\sum_{t=1}^{\alpha_k} \phi(p_k^t)) \\ &= (\sum_{d_1 \mid p_1^{\alpha_1}} \phi(d_1) - 1)(\sum_{d_2 \mid p_2^{\alpha_2}} \phi(d_2) - 1) \cdots (\sum_{d_k \mid p_k^{\alpha_k}} \phi(d_k) - 1) \\ &= (p_1^{\alpha_1} - 1)(p_2^{\alpha_2} - 1) \cdots (p_k^{\alpha_k} - 1). \end{aligned}$$

□

Lemma 5.3. *Let G be a group. Then X_d forms a clique in the intersection power graph $\mathcal{G}_I(G)$.*

Proof. For any vertex $v \in G$ we denote D_v, to be the set of all prime divisor of $o(v)$. Then from the Lemma 5.1, $D_{v_1} = D_{v_2}$ for any two vertices in X_d. Now we show that any two vertices in X_d are adjacent in $\mathcal{G}_I(G)$. Suppose $v_1, v_2 \in X_d$. Then $D_{v_1} = D_{v_2}$ implies there exists a prime p such that p divides $o(v_1)$ and $o(v_2)$. Since G is cyclic, G has a unique subgroup H of order p. Now $|H| = p$ implies that $\langle v_1 \rangle \cap \langle v_2 \rangle \neq \{e\}$. Hence $v_1 \sim v_2$ in $\mathcal{G}_I(G)$. $\square$

Now combining Lemma 5.1, Lemma 5.2 and Lemma 5.3, we have our main theorem.

Theorem 5.1. *If $n \neq p^m$. Then Aut($\mathcal{G}_I(\mathbb{Z}_n)$) $= \bigoplus_{\phi \neq I \subsetneq [r]} S_{P^I - 1} \bigoplus S_{(p_1^{\alpha_1}-1)(p_2^{\alpha_2}-1)\cdots(p_r^{\alpha_r}-1)+1}$, where $P^I - 1 = (p_{i_1}^{\alpha_{i_1}} - 1)(p_{i_2}^{\alpha_{i_2}} - 1)\cdots(p_{i_k}^{\alpha_{i_k}} - 1)$ for $I = \{i_1, i_2, \cdots, i_k\}$.*

Now from Lemma 5.1 and Lemma 5.2 we determine the number of edges in the intersection power graph $\mathcal{G}_I(\mathbb{Z}_n)$. Already we have known that for the cyclic group of order p^m, where p is prime and $m \in \mathbb{N}$ the graph $\mathcal{G}_I(G)$ is complete. So in this case the edge number of the graph is $\frac{p^m(p^m-1)}{2}$. Now we determine the edge number of the graph $\mathcal{G}_I(\mathbb{Z}_n)$, where $n \neq p^s, p$ is prime and $s \in \mathbb{N}$.

Theorem 5.2. *Let G be a group. Then the number of edges in the intersection power graph $\mathcal{G}_I(\mathbb{Z}_n)$ is $\frac{n^2-n-1-(2p_1^{\alpha_1}-1)(2p_2^{\alpha_2}-1)\cdots(2p_r^{\alpha_r}-1)}{2}$.*

Proof. Let G be a cyclic group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$. Suppose $v \in G$ such that $o(v)$ contains the primes $p_{i_1}, p_{i_2}, \cdots, p_{i_m}$. Then from Lemma 4.1, the degree of v is $n - \frac{n}{p_{i_1}^{\alpha i_1} p_{i_2}^{\alpha i_2} \cdots p_{i_m}^{\alpha i_m}}$. Again from Lemma 4.2, the number of vertices of degree $n - \frac{n}{p_{i_1}^{\alpha i_1} p_{i_2}^{\alpha i_2} \cdots p_{i_m}^{\alpha i_m}}$ is $(p_{i_1}^{\alpha_{i_1}} - 1)(p_{i_2}^{\alpha_{i_2}} - 1)\cdots(p_{i_m}^{\alpha_{i_m}} - 1)$. Now $deg(e) = n - 1$. Let $S = \{i_1, i_2, \cdots, i_m\} \subset [r]$ with $|S| \geq 1$ and denote

$$P_S = (p_{i_1}^{\alpha_{i_1}} - 1)(p_{i_2}^{\alpha_{i_2}} - 1)\cdots(p_{i_m}^{\alpha_{i_m}} - 1)(n - \frac{n}{p_{i_1}^{\alpha i_1} p_{i_2}^{\alpha i_2} \cdots p_{i_m}^{\alpha i_m}})$$

Let Γ be any graph. Then Γ satisfies the relation $2e_1 = \sum_{v \in V(\Gamma)} deg(v)$. So for the intersection graph $\mathcal{G}_I(G)$,

$$
\begin{aligned}
2e_1 &= \textstyle\sum_{S\subset[r]} P_S + (n-1)\\
&= \sum_{S\subset[r]} (p_{i_1}^{\alpha_{i_1}}-1)(p_{i_2}^{\alpha_{i_2}}-1)\cdots(p_{i_m}^{\alpha_{i_m}}-1)(n-\frac{n}{p_{i_1}^{\alpha i_1}p_{i_2}^{\alpha i_2}\cdots p_{i_m}^{\alpha i_m}}) + (n-1)\\
&= n\sum_{S\subset[r]} (p_{i_1}^{\alpha_{i_1}}-1)(p_{i_2}^{\alpha_{i_2}}-1)\cdots(p_{i_m}^{\alpha_{i_m}}-1) - n\sum_{S\subset[r]} \frac{(p_{i_1}^{\alpha_{i_1}}-1)(p_{i_2}^{\alpha_{i_2}}-1)\cdots(p_{i_m}^{\alpha_{i_m}}-1)}{p_{i_1}^{\alpha i_1}p_{i_2}^{\alpha i_2}\cdots p_{i_m}^{\alpha i_m}} + (n-1)\\
&= n(1+(p_1^{\alpha_1}-1))(1+(p_2^{\alpha_2}-1))\cdots(1+(p_r^{\alpha_r}-1)) - n\\
&\quad - n\sum_{S\subset[r]}(1-\frac{1}{p_{i_1}^{\alpha_{i_1}}})(1-\frac{1}{p_{i_2}^{\alpha_{i_2}}})\cdots(1-\frac{1}{p_{i_m}^{\alpha_{i_m}}}) + (n-1)\\
&= n(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}) - n - n(1+(1-\frac{1}{p_1^{\alpha_1}}))(1+(1-\frac{1}{p_2^{\alpha_2}}))\cdots(1+(1-\frac{1}{p_r^{\alpha_r}})) - n + (n-1)\\
&= n^2 - n - n(2-\frac{1}{p_1^{\alpha_1}})(2-\frac{1}{p_1^{\alpha_1}})\cdots(2-\frac{1}{p_r^{\alpha_r}}) - 1\\
&= n^2 - n - n(2-\frac{1}{p_1^{\alpha_1}})(2-\frac{1}{p_1^{\alpha_1}})\cdots(2-\frac{1}{p_r^{\alpha_r}}) - 1\\
&= n^2 - n - 1 - (2p_1^{\alpha_1}-1)(2p_2^{\alpha_2}-1)\cdots(2p_r^{\alpha_r}-1)
\end{aligned}
$$

Hence the result holds. □

6. Vertex connectivity of $\mathcal{G}_I(\mathbb{Z}_n)$

The *vertex connectivity* of a graph Γ, denoted by $\kappa(\Gamma)$, is the minimum number of vertices whose deletion increases the number of connected components of the graph Γ or has only one vertex. In [5] Bera et al. proved that the vertex connectivity $\kappa(\mathcal{G}(\mathbb{Z}_n))$ of the power graph $\mathcal{G}(\mathbb{Z}_n)$ is $\kappa(\mathcal{G}(\mathbb{Z}_n)) = \phi(p_1p_2\cdots p_{r-1})(p_r-2)+p_1p_2\cdots p_{r-1}$, where $n = p_1p_2\cdots p_r$ and $p_i (i = 1, 2, \cdots, r)$ are primes such that $p_1 < p_2 < p_3 \cdots < p_r$. In this section we give an upper bound of the vertex connectivity of the intersection power graph of any finite cyclic group. Throughout this section we denote the group G is a cyclic group of order n, where $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}, p_1 < p_2 < \cdots < p_r$ are distinct primes, $r \geq 2$ and $\alpha_1, \alpha_2, \cdots, \alpha_r \in \mathbb{N}$.

Theorem 6.1. *Let G be a group. Then $\kappa(\mathcal{G}_I(\mathbb{Z}_n)) \leq 2 + (p_1^{\alpha_1}-1)p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_r^{\alpha_r} - p_1^{\alpha_1}$.*

Proof. Let $S = \{i_1, i_2, \cdots, i_k\} \subset [r]$, where $1 \leq k \leq (r-1)$. Now for $S = \{i_1, i_2, \cdots, i_k\} \subset [r]$, consider $V_S \subset V(\mathcal{G}_I(\mathbb{Z}_n))$ such that $v \in V_S$ if and only if $\pi(v) = \{p_{i_1}, p_{i_2}, \cdots, p_{i_k}\}$. Let S be any subset $[r]$ such that $|S| = r-1$. Then we have r such subsets of the set $[r]$. And by definition of V_S, for each such subset $S(|S| = r-1)$ there is a unique subset V_S of $V(\mathcal{G}_I(\mathbb{Z}_n))$. Now we prove that to disconnect the graph we have to delete all the vertices present in any $r-1$ subsets $V_S(|S| = r-1)$ of $V(\mathcal{G}_I(\mathbb{Z}_n))$ (i.e. to disconnect the graph we can keep all the vertices present in at most one such $V_S(|S| = r-1)$ in the graph). In fact if we keep vertices present in two such sets, namely V_{S_1}, V_{S_2} such that $|S_1| = r-1 = |S_2|$ in the intersection graph then we show that the graph is connected. Let a_1, a_2 be two vertices in $\mathcal{G}_I(\mathbb{Z}_n)$. Then there is a prime divisor

p_k of $o(a_1)$ such that either p_k is a divisor of order of each vertex in V_{S_1} or each vertex in V_{S_2}. So either a_1 is adjacent to each vertex of V_{S_1} or to each vertex of V_{S_2}. Similarly it is true for the case a_2. Again each vertex of V_{S_1} is adjacent to each vertex of V_{S_2}. So there is a path between the vertices a_1 and a_2. So we delete all the vertices in all $V_S(S \subset [r], |S| = r-1)$ except the vertices in $V_{\acute{S}}$, where $\acute{S} = \{2, 3, \cdots, r\}$. Also we delete all other vertices v such that order of v is of the form $p_1^{x_1} p_2^{x_2} \cdots p_r^{x_r}$, where $(x_1, x_2, \cdots, x_r)$ be such that $1 \leq x_1 \leq \alpha_1$ and at least one $x_k (k \neq 1) > 0$ and the identity e of the group. Now it is easy to see that the resulting graph is disconnected. In this case the total number of deleted vertices is $1 + \sum_{(x_1, x_2, \cdots, x_r)} \phi(p_1^{x_1} p_2^{x_2} \cdots p_r^{x_r})$, [here $(x_1, x_2, \cdots, x_r)$ be such that $1 \leq x_1 \leq \alpha_1$ and at least one $x_k (k \neq 1) > 0$].
$= 1 + \sum_{(x_1, x_2, \cdots, x_r)} \phi(p_1^{x_1} p_2^{x_2} \cdots p_r^{x_r}) - (p_1^{\alpha_1} - 1)$, [here $(x_1, x_2, \cdots, x_r)$ be such that $x_1 \geq 1$ and $x_i \geq 0$ for all $i \neq 1$]
$= 1 + (\phi(p_1) + \phi(p_1^2) + \cdots + \phi(p_1^{\alpha_1}))(1 + \phi(p_2) + \phi(p_2^2) + \cdots + \phi(p_2^{\alpha_2})) \cdots (1 + \phi(p_r) + \phi(p_r^2) + \cdots + \phi(p_r^{\alpha_r})) - (p_1^{\alpha_1} - 1)$
$= 2 + (p_1^{\alpha_1} - 1) p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r} - p_1^{\alpha_1}$.
Hence $\kappa(\mathcal{G}_I(\mathbb{Z}_n)) \leq 2 + (p_1^{\alpha_1} - 1) p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r} - p_1^{\alpha_1}$. □

Acknowledgement

The author is thankful to Dr. A.K. Bhuniya and Mr. S.K. Mukherjee for some fruitful suggestions on the paper. The research is partially supported by UGC-NET fellowship, India, award letter no-2061441005.

References

[1] G. Aalipour, S. Akbari, P.J. Cameron, R. Nikandish and F. Shaveisi, On the structure of the power graph and the enhanced power graph of a group, *Electron. J. Combin.* **24** (3) (2017).

[2] J. Abawajy, A. Kelarev and M. Chowdhury, Power graphs: a survey, *Electron. J. Graph Theory Appl.* **1** (2) (2013), 125–147.

[3] I. Beck, Coloring of commutative ring, *J. Algebra* **116** (1988), 208–226.

[4] S. Bera and A.K. Bhuniya, On enhanced power graphs of finite groups, *J. Algebra Appl.* (2017), 1850146.

[5] S. Bera and A.K. Bhuniya, On the vertex connectivity of the power graphs of finite cyclic groups, *Research gate* (2016).

[6] A.K. Bhuniya and S. Bera, On some characterizations of strong power graphs of finite groups, *Spec. Matrices* **4** (2016), 121–129.

[7] A.K. Bhuniya and S. Bera, Normal subgroup based power graphs of a finite group, *Comm. Algebra* **45** (2016), 3251–3259.

[8] R. Brauer and K.A. Fowler, On groups of even order,*The Annals of Mathematics* **62** (3) (1955), 567–583.

[9] P.J. Cameron and S. Ghosh, The power graph of a finite group, *Discrete Math.* **311** (2011), 1220–1222.

[10] P.J. Cameron, The power graph of a finite group, II, *J. Group Theory* **13** (6) (2010), 779–783.

[11] I. Chakrabarty, S. Ghosh and M.K. Sen, Undirected power graphs of semigroups, *Semigroup Forum* **78** (2009), 410–426.

[12] S. Chattopadhyay and P. Panigrahi, Power graphs of finite groups of even order, *Communications in Computer and Information Science* **283** (2012), 62–67.

[13] S. Chattopadhyay and P. Panigrahi, Connectivity and Planarity of Power Graphs of Finite Cyclic Dihedral and Dicyclic Groups, *Algebra and Discrete Mathematics*, accepted.

[14] J.A. Gallian, *Contemporary Abstract Algebra*, Narosa Publising House. 1999.

[15] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer-Verlag, New York Inc, 2001.

[16] D. Gorenstein, *Finite groups*, New York, Harper and Row, Publishers, 1968.

[17] A.V. Kelarev and S.J. Quinn, A combinatorial property and power graphs of groups, *Contrib. General Algebra* **12** (2000), 229–235.

[18] G. Singh and K. Manilal, Some Generalities on Power Graphs and Strong Power Graphs, *Int. J. Contemp. Math Sciences* **5** (55) (2010), 2723–2730.

[19] D.B. West, *Introduction to Graph Theory*, 2nd ed. Pearson Education, 2001.

[20] T.J. Woodcock, *Commuting Graphs of Finite Groups*. PhD thesis, University of Virginia, 2010.