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Some structural graph properties of the non-commuting graph of a class of finite Moufang loops

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Abstract

For any non-abelian group G, the non-commuting graph of G, $\Gamma = \Gamma_G$, is graph with vertex set $G\backslash Z(G)$, where $Z(G)$ is the set of elements of G that commute with every element of G and distinct non-central elements x and y of G are joined by an edge if and only if $xy \neq yx$. The non–commuting graph of a finite Moufang loop has been defined by Ahmadidelir. In this paper, we show that the multiple complete split-like graphs and the non-commuting graph of Chein loops of the form $M(D_{2n}, 2)$ are perfect (but not chordal). Then, we show that the non-commuting graph of a non-abelian group G is split if and only if the non-commuting graph of the Moufang loop $M(G, 2)$ is $3-$split. Precisely, we show that the non-commuting graph of the Moufang loop $M(G, 2)$, is $3-$split if and only if G is isomorphic to a Frobenius group of order $2n$, n is odd, whose Frobenius kernel is abelian of order n. Finally, we calculate the energy of generalized and multiple splite-like graphs, and discuss about the energy and also the number of spanning trees in the case of the non-commuting graph of Chein loops of the form $M(D_{2n}, 2)$.

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1. Introduction

Let Q be a set with one binary operation. Then it is a quasigroup if the equation $xy = z$ has a unique solution in Q whenever two of the three elements $x, y, z \in Q$ are specified. A quasigroup Q is a loop if Q possesses a neutral element e, i.e., if $ex = xe = x$ holds for every $x \in Q$. Moufang loops are loops in which any of the (equivalent) Moufang identities,

$$\begin{array}{ll} ((xy)x)z = x(y(xz)), & (M1) \\ x(y(zy)) = ((xy)z)y, & (M2) \\ (xy)(zx) = x((yz)x), & (M3) \\ (xy)(zx) = (x(yz))x. & (M4) \end{array}$$

holds for every $x, y, z \in Q$. Commutator of x, y and the associator of x, y and z are defined by $[x, y] = x^{-1}y^{-1}xy$ and $[x, y, z] = ((xy)z)^{-1}(x(yz))$, respectively. We define the commutant (or Moufang center) $C(Q)$ of Q as $\{x \in Q \mid xy = yx, \quad \forall y \in Q\}$. The center $Z(Q)$ of a Moufang loop Q is the set of all elements of Q which commute and associate with all other elements of Q. A non-empty subset P of Q is called a subloop of Q if P is itself a loop under the binary operation of Q, in particular, if this operation is associative on P, then it is a subgroup of Q. A subloop N of a loop Q is said to be normal in Q if $xN = Nx$; $x(yN) = (xy)N$; $N(xy) = (Nx)y$; for every $x, y \in Q$. In Moufang loop Q, the subloops $Z(Q)$ and $C(Q)$ are normal subloops. For more details about the Moufang loops one may see [8, 16, 13]. In 1974, Chein introduced a class of non-associative Moufang loops $M(G, 2)$, so called Chein loops. For a group G and a new element $u, (u \notin G)$, $M(G, 2) = G \cup Gu$ such that the multiplication with the new binary operation $\circ$ is defined as follows:

$$\begin{cases} g \circ h = gh, & g, h \in G, \\ g \circ (hu) = (hg)u, & g \in G,\ hu \in Gu, \\ (gu) \circ h = (gh^{-1})u, & gu \in Gu,\ h \in G, \\ (gu) \circ (hu) = h^{-1}g, & gu, hu \in Gu. \end{cases}$$

Clearly, the Moufang loop $M(G, 2)$ is non-associative if and only if G is non-abelian, see [8]. In [2], Ahmadidelir has investigated some probabilistic properties of $M(G, 2)$, such as its *commutativity degree*.

There are many papers on assigning a graph to a ring or a group in order to investigation of their algebraic properties. For any non-abelian group G the non-commuting graph of G, $\Gamma = \Gamma_G$ is a graph with vertex set $G \backslash Z(G)$, where distinct non-central elements x and y of G are joined by an edge if and only if $xy \neq yx$. This graph is connected with diameter 2 and girth 3 for a non-abelian finite group and has received some attention in existing literature. For instance, one may see [1, 10, 15, 17]. Similarly, the non-commuting graph of a finite Moufang loop has been defined by Ahmadidelir in [3]. He has defined this graph as follows: *Let M be a Moufang loop, then the vertex set is $M \backslash C(M)$ and two vertices x and y joined by an edge whenever $[x, y] \neq 1$.* He has shown that this graph is connected (as for groups) and obtained some results related to the non–commuting graph of a finite non-commutative Moufang loop.

We will denote a complete graph with n vertices by K_n. All graphs considered in this paper are finite and simple. For a graph Γ, we denote its vertex and edge sets by $V(\Gamma)$ and $E(\Gamma)$, respectively.

The complement of Γ is denoted by $\bar{\Gamma}$. A graph $\Gamma = (V, E)$, is called $k-$partite where $k > 1$, if it is possible to partition V into k subsets $V_1, V_2, \ldots, V_k$, such that every edge of E joins a vertex of V_i to a vertex of V_j, $i \neq j$. A clique in a graph Γ is an induced subgraph whose all vertices are pairwise adjacent. The maximum size of a clique in a graph Γ is called the clique number of Γ and denoted by $\omega(\Gamma)$. A subset X of the vertices of Γ is called an independent set (or stable) if the induced subgraph on X has no edges. The maximum size of an independent set in a graph Γ is called the independence number of Γ and denoted by $\alpha(\Gamma)$. The vertex chromatic number of a graph Γ is denoted by $\chi(\Gamma)$, and it is the minimum k for which $k-$vertex coloring of a graph Γ such that no two adjacent vertices have the same color. For a subset S of $V(\Gamma)$, $N_\Gamma[S]$ is the set of vertices in Γ which are in S or adjacent to a vertex in S. If $N_\Gamma[S] = V(\Gamma)$ then S is said to be a dominating set of the vertices in Γ. The minimum size of a dominating set of the vertices in Γ is dominating number of Γ and denoted by $\gamma(\Gamma)$. A vertex cover of a graph Γ is a set $Q \subseteq V(\Gamma)$ such that contains at least one endpoint of every edge. The minimum size of a vertex cover is denoted by $\beta(\Gamma)$. Our other used notations about graphs are standard and for more details one may see [6, 7, 11].

There is a relation between $\alpha(\Gamma)$ and $\beta(\Gamma)$ as follows:

Lemma 1.1. *([7], p. 296) Let Γ be a graph. Then $\alpha(\Gamma) + \beta(\Gamma) = n(\Gamma)$, where $n(\Gamma)$ is the number of vertices of Γ.* □

A perfect graph Γ, is a graph in which for every induced subgraph its clique number is equal to its chromatic number. A graph Γ is called weakly perfect graph if $\omega(\Gamma) = \chi(\Gamma)$. So, all perfect graphs are weakly perfect. A chordal graph is one in which all cycles of order four or more have a chord, which is an edge that is not part of cycle but connects two vertices of the cycle. The class of Chordal graphs is a subset of the class of perfect graphs. For more information about these types of graphs, one may see [12, 14]. We have the following Theorem about perfect graphs, called strongly perfect graph theorem, or Berg Theorem.

A graph is called k-regular, if the vertices of the graph are of the same degree k and a strongly regular graph S with parameters (n, k, λ, μ) is a $k-$regular graph of order n such that each pair of adjacent vertices has λ common neighbors and each pair of non-adjacent vertices has in which μ common neighbors. Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be undirected simple graphs. The union $\Gamma_1 \cup \Gamma_2$ of graphs Γ_1 and Γ_2 is a graph $\Gamma = (V, E)$ for which $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. The notation $n\Gamma$ is short for $\underbrace{\Gamma \cup \cdots \cup \Gamma}_{n-times}$.

The complete product $\Gamma_1 \nabla \Gamma_2$ of graph Γ_1 and Γ_2 is a graph obtained from $\Gamma_1 \cup \Gamma_2$ by joining every vertex of Γ_1 to every vertex of Γ_2. For every $a, b, n \in N$, a complete split, or simply, a split graph, is the graph $\bar{K}_a \nabla K_b$ and denoted by CS^a_b. By a theorem of Földes and Hammer ([12], Theorem 6.3), a graph is (complete) split iff contains no induced subgraph isomorphic to $2K2, C_4$ or C_5. Also, an undirected graph is split if and only if its complement is split ([12], Theorem 6.1). Clearly, every split graph is chordal and so perfect, but the converses are not true. More generally, a multiple complete split-like graph is $\bar{K}_a \nabla (nK_b)$ and denoted by $MCS^a_{b,n}$. Specially, in this paper, for $n = 3$ we call $MCS^a_{b,3}$ as a $3-$split graph.

We generalize the above definitions as follows:

Definition 1.1. *The generalized complete split-like graph is $GCS_k^a = \bar{K}_a \nabla S$ such that S is a strongly regular graph with parameters (n, k, λ, μ). The multiple generalized complete split-like graph is $GMCS_{k,m}^a = \bar{K}_a \nabla(mS)$.*

The laplacian matrix of a simple graph Γ with n vertices, is defined as $L(\Gamma) = D(\Gamma) - A(\Gamma)$, where $A(\Gamma)$ is its adjacency matrix and $D(\Gamma) = (d_1, \ldots, d_n)$ is the diagonal matrix of the vertex degrees in Γ. For any graph Γ, the energy of Γ is defined as $\xi(\Gamma) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix of Γ. A spanning tree of a graph Γ is an induced subgraph of Γ, which is a tree and contains every vertex of Γ.

In this paper, we show that the multiple complete split-like graphs are perfect (but not chordal) and deduce that the non-commuting graph of Chein loops of the form $M(D_{2n}, 2)$ is perfect but not chordal. Then, we show that the non-commuting graph of a non-abelian group G is split if and only if the non-commuting graph of the Moufang loop $M(G, 2)$ is $3-$split and then classify all Chein loops that their non-commuting graphs are $3-$split. Precisely, we show that for a non-abelian group G, the non-commuting graph of the Moufang loop $M(G, 2)$, is $3-$split if and only if G is isomorphic to a Frobenius group of order $2n$, n is odd, whose Frobenius kernel is abelian of order n. Finally, we calculate the energy of generalized and multiple splite-like graphs, and discuss about the energy and also the number of spanning trees in the case of the non-commuting graph of Chein loops of the form $M(D_{2n}, 2)$. We recall the following Proposition and Theorems in order to provide some tools to these purposes.

Theorem 1.1. *([5], p. 3: Schur complement) Let A be a $n \times n$ matrix partitioned as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where A_{11} and A_{22} are non-singular square matrices. Then the inverse of A, A^{-1} can be calculated by the following formula:*

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1} A_{12} (A/A_{11})^{-1} A_{21} A_{11}^{-1} & -A_{11}^{-1} A_{12} (A/A_{11})^{-1} \\ -(A/A_{11})^{-1} A_{21} A_{11} & (A/A_{11})^{-1} \end{bmatrix},$$

where

$$A/A_{11} = A_{22} - A_{21} A_{11}^{-1} A_{12}$$

and

$$detA = detA_1 \times det(A_{22} - A_{21} A_{11}^{-1} A_{12}),$$

such that $detA$ is the determinant of A. □

Theorem 1.2. *([14], Theorem 1) For $i = 1, 2$, let Γ_i be r_i-regular graphs with n_i vertices. Then the characteristic polynomial of the complete product of these two graphs is as follows:*

$$P_{\Gamma_1 \nabla \Gamma_2}(\lambda) = \frac{P_{\Gamma_1}(\lambda) P_{\Gamma_2}(\lambda)}{(\lambda - r_1)(\lambda - r_2)} [(\lambda - r_1)(\lambda - r_2) - n_1 n_2].$$

□

2. Some basic graph properties of the Moufang loop $M(D_{2n}, 2)$

Let D_{2n} denote the dihedral group of order $2n$, which has the following presentation:

$$D_{2n} = \left\langle a, b | \quad a^n = b^2 = (ab)^2 = 1 \right\rangle .$$

In this section, we want to study the non-commuting graph of the Moufang loops $M(D_{2n}, 2)$, simply denoted by Γ. We will use the following Lemma in next sections.

The following Lemma determines the structure of the non-commuting graph of the Moufang loop $M = M(D_{2n}, 2)$.

Lemma 2.1. *Let $M = M(D_{2n}, 2)$ and $\Gamma = \Gamma_M$ be its non-commuting graph.*

(a) *If n is odd then $\Gamma_M \cong \bar{K}_{n-1}\nabla S$, such that S is a strongly regular graph with parameters $(3n, n-1, n-2, 0)$.*

(b) *If n is even then $\Gamma_M \cong \bar{K}_{n-2}\nabla 3S$, such that S is a strongly regular graph with parameters $(n, n-2, n-3, n-2)$.*

Proof. a) By Lemma ([3], Lemma 4.4) and the definition of the non-commuting graph, for every odd integer n, we can partition the vertices of Γ into four sets, as follows:

$$\begin{aligned} t_1 &= \{a, a^2, \ldots, a^{n-1}\}, & t_2 &= \{b, ab, \ldots, a^{n-1}b\}, \\ t_3 &= \{u, au, \ldots, a^{n-1}u\}, & t_4 &= \{bu, abu, \ldots, a^{n-1}bu\}. \end{aligned}$$

For every $0 \le i, j \le n-1$, since $a^i a^j = a^j a^i$, t_1 is an independent set and from the relations $a^i \circ (a^j b) \neq (a^j b) \circ a^i$, $a^i \circ (a^j u) \neq (a^j u) \circ a^i$ and $a^i \circ (a^j bu) \neq (a^j bu) \circ a^i$, we find that all vertices of t_1 are adjacent to all vertices of each of the sets t_2, t_3 and t_4. Also, by the relations $(a^i b) \circ (a^j b) \neq (a^j b) \circ (a^i b)$, the induced subgraph $[t_2]$ of Γ, is a clique. Similarly, we can show that the induced subgraph $[t_3]$ and $[t_4]$ of Γ, are cliques. Hence, $\Gamma \cong \bar{K}_{n-1}\nabla 3K_n$ and the graph Γ is $3-$split and $3K_n \cong S$, where S is a strongly regular graph with parameters $(3n, n-1, n-2, 0)$.

b) Let n be an even integer. Again, we can partition the vertices of Γ into four sets, as follows:

$$\begin{aligned} t_1 &= \{a, a^2, \ldots, a^{\frac{n}{2}-1}, a^{\frac{n}{2}+1}, \ldots, a^{n-1}\}, & t_2 &= \{b, ab, \ldots, a^{n-1}b\}, \\ t_3 &= \{u, au, \ldots, a^{n-1}u\}, & t_4 &= \{bu, abu, \ldots, a^{n-1}bu\}. \end{aligned}$$

Since each pair of elements of t_1 commute, so the induced subgraph $[t_1]$ is an independent set, that means $[t_1] \cong \bar{K}_{n-2}$. Also, every element in M commutes with its inverse and since, $\forall x \in t_i, (i = 2, 3, 4)$, its inverse x^{-1} belongs to t_i. Therefore, every element of $t_i, (i = 2, 3, 4)$ is adjacent to each vertex in t_i, $i = 2, 3, 4$, except its inverse. Also any two elements x ,y in $t_i, (i = 2, 3, 4)$ commute if and only if $|i - j| = \frac{n}{2}$, where $x = a^i u$ or $a^i b$, $a^i bu$ and $y = a^j u$ or $a^j b$, $a^j bu$. Then $[t_i] \cong S$, where S is a strongly regular graph with parameters $(n, n-2, n-3, n-2)$. Finally, for every $2 \le i, j \le 4$ there is no edge of Γ such that joins a vertex of t_i to a vertex of t_j, $i \neq j$, but each vertex in t_1 joins to each vertex in $t_i, (i = 2, 3, 4)$. Therefore, $\Gamma_M \cong \bar{K}_{n-2}\nabla 3S$. □

In the following Theorem, we derive some important graph properties of $\Gamma_{M(D_{2n},2)}$.

Theorem 2.1. *Let $M = M(D_{2n}, 2)$ and $\Gamma = \Gamma_M$ be its non-commuting graph.*

(a) *If n is odd then:*

$$\omega(\Gamma) = n + 1, \qquad \chi(\Gamma) = n + 1,$$
$$\alpha(\Gamma) = n - 1, \qquad \beta(\Gamma) = 3n, \qquad \gamma(\Gamma) = 2.$$

(b) *If n is even then:*

$$\omega(\Gamma) = \frac{n}{2} + 1, \qquad \chi(\Gamma) = \frac{n}{2} + 1,$$
$$\alpha(\Gamma) = \begin{cases} 6, & (n = 6) \\ n - 2, & (n \geq 8) \end{cases}, \quad \beta(\Gamma) = \begin{cases} 16, & (n = 6) \\ 3n, & (n \geq 8) \end{cases}, \quad \gamma(\Gamma) = 2.$$

Proof. a) By Lemma 2.1, the non-commuting graph of $M(D_{2n}, 2)$ is a generalized complete split-like graph for any odd integer n. Then $\Gamma = \bar{K}_{n-1} \nabla S$ in which S is a strongly regular graph with parameters $(3n, n-1, n-2, 0)$, where $V(\bar{K}_{n-1}) = \{a, a^2, \ldots, a^{n-1}\}$ and $S \cong 3K_n$. So this graph is $3-$split. By the structure of Γ, since every vertex of each copy of K_n is joined to every vertex of $\bar{K}_{n-1}$, so we have the complete product $K_n \nabla [a^i]$, where $a^i \in \bar{K}_{n-1}$, $1 \leq i \leq n-1$. Also, $K_n \nabla [a^i]$ is the largest clique in Γ. So, $\omega(\Gamma) = n + 1$. We need n distinct colors for coloring any K_n and only one color for coloring $\bar{K}_{n-1}$ which is distinct with the previous ones. So, $\chi(\Gamma) = n + 1$. The set of vertices of $\bar{K}_{n-1}$ is the largest independent set, so $\alpha(\Gamma) = n - 1$. By Lemma 1.1, we have $\beta(\Gamma) = 4n - 1 - (n - 1) = 3n$. Clearly, the set of vertices of $3K_n$ has the minimum size of a vertex cover. Any vertex of $\bar{K}_{n-1}$ is dominating all vertices of S, and any vertex of S is dominating all vertices in $\bar{K}_{n-1}$. Thus $\gamma(\Gamma) = 2$.

b) By Lemma 2.1, the non-commuting graph of $M(D_{2n}, 2)$, for every even integer n, is a multiple generalized complete split-like graph as $\Gamma = \bar{K}_{n-2} \nabla 3S$, where S is a strongly regular graph with parameters $(n, n-2, n-3, n-2)$ and the set of vertices of $\bar{K}_{n-2}$ is an independent set as follows:

$$V(\bar{K}_{n-2}) = \{a, a^2, \ldots, a^{\frac{n}{2}-1}, a^{\frac{n}{2}+1}, \ldots, a^n - 1\}.$$

In order to find the clique number, we may choose one vertex of $\bar{K}_{n-2}$ and the other vertices from only one copy of S's. By definition, every vertex is not joined to its inverse, so, we can choose $\frac{n}{2}$ vertices of S and hence, $\omega(\Gamma) = \frac{n}{2} + 1$. The color of every vertex in S is co-color with its inverse. Therefore, the chromatic number of S is equal to $\frac{n}{2}$, and so the maximum color number for all the vertices of $3S$ is equal to $\frac{n}{2}$. By only one color distinct from $\frac{n}{2}-$color in $3S$, we can color $\bar{K}_{n-2}$. So, $\chi(\Gamma) = \frac{n}{2} + 1$. For $n = 6$, $\bar{K}_{n-2}$ have four independent vertices, but with two non-adjacent vertices chosen from any of the copies of S, we get 6 independent vertices. Therefore, in this case $\alpha(\Gamma) = 6$. Now, for $n \geq 8$, the set $\bar{K}_{n-2}$ is the largest independent set and so, $\alpha(\Gamma) = n - 2$. By using Lemma 1.1, we have $\beta(\Gamma) = n(\Gamma) - \alpha(\Gamma)$. Hence, if $n = 6$ then $\beta(\Gamma) = 16$, else if $n \geq 8$ then $\beta(\Gamma) = 4n - 2 - (n - 2) = 3n$. By choosing any vertex in $\bar{K}_{n-2}$ and the other in one of the copies of S, the domination set of Γ will be determined. Hence, $\gamma(\Gamma) = 2$. □

3. About perfectness and splitness of the non-commuting graph of a Moufang loop

In this section, first we show that the multiple complete split-like graphs are perfect and then characterize all Chein loops that their non-commuting graphs are $3-$split-like.

Theorem 3.1. *Every multiple complete split-like graph $MCS^a_{b,n} \cong \bar{K}_a \nabla (nK_b)$, $(n \geq 2)$ is perfect, but not chordal. Moreover, every complete split graph $CS^a_{b,n} \cong \bar{K}_a \nabla K_b$, is perfect and also chordal.*

Proof. Let $\Gamma \cong \bar{K}_a \nabla (nK_b)$ and C be an odd cycle. If all vertices of C lie in only one copy of K_b's, clearly this cycle has a chord. Also, if some vertices of C lie in more than one copy of K_b's, then since in this case C has some vertices of $\bar{K}_a$ and also these vertices in $\bar{K}_a$ are adjacent to each vertex of K_b, therefore, the cycle has a chord. In addition, the complement graph, $\bar{\Gamma}$, is a disconnected graph of the form $\bar{\Gamma} \cong K_a \cup S$ such that S is strongly regular graph with parameters $(nb, (n-1)b, (n-2)b, (n-1)b)$ or $S \cong T_{nb,b}$, which is a complete $n-$partite graph with nb vertices, and hence, each part has b vertices. Clearly, any cycle in K_a has a chord. If C be an odd cycle in S, then by structure of S, there is an intersection of C with more than three sections of S and these vertices are adjacent to any of the vertices in other sections and so, C has a chord. If C has an instruction with only two sections of S, then the induced subgraph of these sections will be a bipartite graph such that there is no any odd cycle in it. Now, by Berg Theorem ([9], Theorem 1.2) Γ is a perfect graph. Let $\Gamma \cong \bar{K}_a \nabla (nK_b)$ and $x_1, x_2 \in \bar{K}_a$, $x_1 \neq x_2$. Take x_3 and x_4 from two distinct copies of K_b's. Now the induced subgraph of Γ generated by x_1, x_2, x_3 and x_4 is a cycle of length four without a chord. So, by definition, Γ is not chordal.

Similar to the proof of the first part, $CS^a_{b,n} \cong \bar{K}_a \nabla K_b$ is perfect, but there is no cycle of length four or more without any chord and so this is a chordal graph. This completes the proof. □

Corollary 3.1. *The non-commuting graph of $M(D_{2n}, 2)$ is perfect but not chordal.*

Proof. Let $\Gamma = \Gamma(M(D_{2n}, 2))$, where n be an odd integer. Then by Lemma 2.1 *(a)*, $\Gamma \cong \bar{K}_{n-1} \nabla (3K_n)$ and by Theorem 3.1, Γ is perfect but not chordal.

If n be an even integer then by Lemma 2.1*(b)*, $\Gamma \cong \bar{K}_{n-2} \nabla 3S$ such that S is a strongly regular graph with parameters $(n, n-2, n-3, n-2)$. Assume that C is an odd cycle in Γ with length 5 or more, the length of the longest cycle without chord in each copy of S is equal to 4. Then there are some vertices of $\bar{K}_{n-2}$ in C, and these vertices are adjacent to each vertex in $3S$. Therefore, C have a chord. On the other hand, $\overline{\Gamma} \cong K_{n-2} \cup (\frac{n}{2}K_2 \nabla \frac{n}{2}K_2 \nabla \frac{n}{2}K_2)$. Let C be a cycle in $\overline{\Gamma}$. Clearly, every cycle in K_{n-2} have a chord and if C be an odd cycle in $\frac{n}{2}K_2 \nabla \frac{n}{2}K_2 \nabla \frac{n}{2}K_2$, then C have an intersection with more than two parts of $\frac{n}{2}K_2$, where one of them have more than one vertex in C, and these vertices adjacent to all vertices of C in other parts and so, C have a chord and by Theorem ([9], Theorem 1.2), Γ is perfect. The induced subgraph consist of any two vertices of $\bar{K}_{n-2}$ and two non-adjacent vertices of S is a cycle with length 4 without chord then Γ is not chordal. □

Remark 3.1. The generalized multiple complete split-like graph $GMCS^a_k$ is not perfect. As a counterexample, let we have a generalized complete split-like graph $\Gamma \cong \bar{K}_a \nabla (nS)$ in which S is a Peterson graph. This graph is not perfect, since it has a cycle of length 5 without any chord. Recall that a Peterson graph is a strongly regular graph with parameters $(10, 3, 0, 1)$.

Theorem 3.2. *Let G be a non-abelian group. Then its non-commuting graph Γ_G, is split if and only if the non-commuting graph of the Moufang loop $M(G,2)$, Γ_M, is $3-$split.*

Proof. Let Γ_M be $3-$split of the form $\Gamma_M = I\nabla 3C$, where I is an independent set and C is a complete graph. First we show that $Z(G) = C(M)$. By Lemma([3], Lemma 3.10), $C(M) \subseteq Z(G)$. Let $Z(G) \nsubseteq C(M)$. Then there exists $x \in Z(G)$ such that $x \notin C(M)$. Also, there exists $yu \in Gu$, where $x \circ (yu) \neq (yu) \circ x$, which yields $(yx)u \neq (yx^{-1})u$. Therefore, $x \neq x^{-1}$ and $x \in I$. So, every vertex y in each copy of C is adjacent to x and so $xy \neq yx$. But $x \in Z(G)$ then for every $g \in G$, we have $xg = gx$. Hence $G \subseteq I$. Now, let $g \in G \setminus Z(G)$. So, there exist $t \in G$ such that $tg \neq gt$ but in this case $t, g \in I$ and this is a contradiction, since I is an independent set. So, $G = Z(G)$ and this contradicts with non-abelianity of G. Thus $Z(G) = C(M)$. Obviously, every element of $3C$ is an involution. Let $x \in 3C$ and $x \neq x^{-1}$. So, since each element of Gu has order 2 then $x \in G$. Put $3C = C_1 \cup C_2 \cup C_3$, where each C_i is equal to a copy of C, $(1 \leq i \leq 3)$. Without loss of generality, let $x \in C_1$ and $x^{-1} \in C_2$ (note that $xx^{-1} = x^{-1}x$). Let $y \in G\setminus Z(G)$ and $y \notin \langle x\rangle$. Then since every element of G which commutes with x, also commutes with x^{-1}, so if $y \in C_1$ then $xy \neq yx$, and therefore $x^{-1}y \neq yx^{-1}$, but $x^{-1} \in C_2$ and this is a contradiction. Similarly, the case $y \in C_2$ will reach to a contradiction. So, $y \in I$ or $y \in C_3$. Now, consider the element xy. By the same reason as above, we have $xy \in I$ or $xy \in C_3$. Trivially, $xy \neq x, x^{-1}$. We have four cases as below:

Case 1. Let $y, xy \in I$. Then $y(xy) = (xy)y \Rightarrow yx = xy$. which is a contradiction, since y is adjacent to every element of C_1.

Case 2. Let $y \in I$ and $xy \in C_3$. Then $x \in C_1 \Rightarrow x(xy) = (xy)x$, $(x, y \in G) \Rightarrow xy = yx$ and we have the same contradiction as in case 1.

Case 3. Let $y \in C_3$ and $xy \in I$. Then $(xy)y \neq y(xy) \Rightarrow xy \neq yx$, which is also a contradiction since $y \in C_3$ and $x \in C_1$.

Case 4. Let $y, xy \in C_3$. Then we have $y(xy) \neq (xy)y \Rightarrow xy \neq yx$ and we obtain a similar contradiction as in case 3.

Therefore, every element of $3C$ has order 2. On the other hand, Γ_G is always connected and it is the induced subgraph of Γ_M. Therefore, $\Gamma_G \cong K_m$, $(K_m \subseteq C)$ or $\Gamma_G \cong I'\nabla nC'$ such that $I' \subseteq I$, $C' \subseteq C$ and $nC' = \cup_{i=1}^{n} C_i$, where $1 \leq n \leq 3$, and each C_i is a subset of one copy of C's. If $\Gamma_G \cong K_m$, then the order of every element of G will be equal to 2, so G must be abelian, which is absurd. Therefore, we get, $\Gamma_G \cong I'\nabla nC'$. If $n = 1$ then Γ_G is split. Suppose that $1 \neq x, y \in G$, $x \in C_1$ and $y \in C_2$, then $xy = yx$ and there exists $z \in I'$ where $yz \neq zy$ and $xz \neq zx$. So, $xy \in G$. If $xy \in I'$, then $x(xy) \neq (xy)x$ and so, $x^2y \neq x(yx)$. Therefore, $x^2y \neq x(xy)$ and this is a contradiction. If $xy \in C_1$ then $x(yx) \neq (xy)x$ and $x^2y \neq x^2y$, and it is a contradiction, and if $xy \in C_2$ then $y(xy) \neq (xy)y$ and $y^2x \neq y^2x$, and it is also a contradiction. Finally, let $xy \in C_3$. Now, $xu \in M(G,2)$ then:

1) If $xu \in I$ or $xu \in C_1$, then $(xu) \circ x \neq x \circ (xu)$ and so $(xx^{-1})u \neq (xx)u$. Therefore, $u \neq x^2u$, this is a contradiction. So, every element of C in Γ_M is of order 2 therefore, $x^2 = 1$.

2) If $xu \in C_2$ then $(xu) \circ y \neq y \circ (xu)$ and so $(xy^{-1})u \neq (xy)u$. Thus $(xy)u \neq (xy)u$ and this is a contradiction.

3) If $xu \in C_3$ then $(xu) \circ (xy) \neq (xy) \circ (xu)$ and so $(x(xy)^{-1})u \neq (x(xy))u$ or $(x(y^{-1}x^{-1}))u \neq (x^2y)u$. So, $(x(yx))u \neq (x^2y)u$, or $(x(xy))u \neq yu$. Thus $(x^2y)u \neq yu$ and this is a contradiction.

Therefore, $\Gamma_G \cong I'\nabla C'$ and Γ_G is split.

Conversely, let Γ_G be split. Then $\Gamma_G \cong I\nabla C$. We show that Γ_M is $3-$split. By splitness of Γ_G and Lemmas ([4], Lemmas 2.4 and 2.5), we have, $Z(G) = 1$ and $C(M) \subseteq Z(G)$. So, $C(M) = 1$. Let $V(I) = \{a_1, a_2, \dots, a_k\}$ and $V(C) = \{b_1, b_2, \dots, b_t\}$. Then, $V(\Gamma_M)$ includes $V(I)$, $V(C)$ and the set of vertices of the form, $V(Iu) = \{a_1u, a_2u, \dots, a_ku\}$ and $V(Cu) = \{b_1u, b_2u, \dots, b_tu\}$. To prove $3-$splitness Γ_M, we consider and stablish the following claims.

Claim 1. *The induced subgraph containing the vertices in $V(Iu)$ forms a clique.*

Suppose that there exist two non-adjacent vertices a_iu and a_ju. So, $(a_iu)\circ(a_ju) = (a_ju)\circ(a_iu)$ and then $a_ia_j^{-1} = a_ja_i^{-1}$ or $(a_ia_j^{-1})^2 = 1$. Therefore, by Lemmas ([4], Lemmas 2.4 and 2.5), $I^* = I \cup \{1\}$ is a maximal subgroup of odd order and there is not any element of even order. So, $a_ia_j^{-1} \in C$, where in this case $(a_ia_j^{-1})a_j \neq a_j(a_ia_j^{-1})$. Then $a_i \neq a_j(a_ia_j^{-1})$ and $a_j^{-1}a_i \neq a_ia_j^{-1}$ and this is a contradiction.

Claim 2. *The induced subgraph containing the vertices in $V(Cu)$ is a clique.*

Suppose that there exist two vertices b_iu and b_ju such that are not adjacent. So, $(b_iu) \circ (b_ju) = (b_ju) \circ (b_iu)$. Therefore, $b_ib_j^{-1} = b_jb_i^{-1}$ and $b_ib_j = b_jb_i$, since, each element of C is an involution and which yields to a contradiction.

Claim 3. *There is no edge between $V(Iu)$ and $V(Cu)$.*

Suppose that there exist two vertices a_iu and b_ju such that $(a_iu) \circ (b_ju) \neq (b_ju) \circ (a_iu)$ then $b_j^{-1}a_i \neq a_i^{-1}b_j$ and $b_ja_i \neq a_i^{-1}b_j$, therefore $(b_ja_i)^2 \neq 1$. On the other hand $b_ja_i \in G$. So, $b_ja_i \in I$ or $b_ja_i \in C$.

1) If $b_ja_i \in I$ then $(b_ja_i)a_i = a_i(b_ja_i)$ and $b_ja_i = a_ib_j$, which yields to a contradiction.

2) If $b_ja_i \in C$ then $(b_ja_i)^2 = 1$ and this is a contradiction. Therefore, any two elements of $V(Iu)$ and $V(Cu)$ are non-adjacent.

Claim 4. *There is no edge between $V(C)$ and $V(Cu)$.*

Suppose that there exist two vertices b_i and b_ju such that $b_i \circ (b_ju) \neq (b_ju) \circ b_i$. Then $(b_jb_i)u \neq (b_jb_i^{-1})u$, so, $(b_jb_i)u \neq (b_jb_i)u$, and this is a contradiction. Therefore any two elements of $V(C)$ and $V(Cu)$ are non-adjacent.

Claim 5. *There is no edge between $V(C)$ and $V(Iu)$.*

Suppose that there exist two vertices b_i and a_ju such that $b_i \circ (a_ju) \neq (a_ju) \circ b_i$. Then $(a_jb_i)u \neq (a_jb_i^{-1})u$ and $a_jb_i \neq a_jb_i$. This is a contradiction. Therefore, any two vertices in $V(C)$ and $V(Iu)$ are non-adjacent.

Claim 6. *Every vertex in $V(Iu)$ is adjacent to every vertex in $V(I)$.*

Suppose that there exist two vertices a_i and a_ju such that $a_i \circ (a_ju) = (a_ju) \circ a_i$. Then $(a_ja_i)u = (a_ja_i^{-1})u$ and $a_ja_i = a_ja_i^{-1}$. So, $a_i = a_i^{-1}$. Therefore, $a_i^2 = 1$ and this is a contradiction.

Claim 7. *Every vertex in $V(Cu)$ is adjacent to every vertex in $V(I)$.*

Suppose that there exist two vertices $a_i \in I$ and $b_ju \in Cu$ such that $a_i \circ (b_ju) = (b_ju) \circ a_i$. Also, $(b_ja_i)u = (b_ja_i^{-1})u$ then $b_ja_i = b_ja_i^{-1}$ and $a_i = a_i^{-1}$, therefore $a_i^2 = 1$ and this is a contradiction.

Thus the non–commuting graph of $M(G, 2)$ is $3-$split, where the induced subgraphs containing the vertices of C and Cu and Iu are cliques and I is an independent set. □

Now, by using Theorems ([4], Theorem 2.3) and 3.2, we can classify all $3-$split Chein loops:

Corollary 3.2. *Let G be a non-abelian group. Then the non-commuting graph of the Moufang loop $M(G, 2)$, is $3-$split if and only if G is isomorphic to a Frobenius group of order $2n$, n is odd, whose Frobenius kernel is abelian of order n.* □

4. About the energy and the number of spanning trees of generalized and multiple splite-like graphs

In this section, we are going to calculate the energy of generalized complete and multiple splite-like graphs and derive the energy and also the number of spanning trees in the case of the non-commuting graph of Chein loops of the form $M(D_{2n}, 2)$.

Theorem 4.1. *Let Γ be a generalized complete split-like graph, $\Gamma \cong \bar{K}_a \nabla (nK_b)$. Then $\varepsilon(\Gamma) = 2n(b - 1)$.*

Proof. Let $P_{K_b}(\lambda)$ be the characteristic polynomial of K_b. Then,

$$P_{K_b}(\lambda) = (-1)^b(\lambda + 1)^{b-1}(\lambda - b + 1).$$

So,

$$P_{nK_b}(\lambda) = (-1)^{nb}(\lambda + 1)^{n(b-1)}(\lambda - b + 1)^n$$

and

$$P_{\bar{K}_a}(\lambda) = (-\lambda)^a.$$

By using Theorem 1.2, we have:

$$P_\Gamma(\lambda) = (-1)^{nb+a}(\lambda + 1)^{n(b-1)}(\lambda - b + 1)^{n-1}\lambda^{a-1}(\lambda^2 - (b - 1)\lambda - nab)$$

and by definition of the energy of a graph, we get:

$$\varepsilon(\Gamma) = n(b - 1) + (n - 1)(b - 1) + b - 1.$$

Hence, $\varepsilon(\Gamma) = 2n(b - 1)$. □

Corollary 4.1. *Let n be an odd integer. Let $G = D_{2n}$ and $M = M(G, 2)$. Then:*

(i) *if n is an odd integer, then*$\varepsilon(\Gamma_M) = 6(n-1)$*;*

(i) *if n is an even integer, then $\varepsilon(\Gamma_M) = 6(n-2)$.*

Moreover, in both cases, $\varepsilon(\Gamma_G)$ divides $\varepsilon(\Gamma_M)$.

Proof. Since, $\Gamma_M \cong \bar{K}_{n-1}\nabla 3K_n$, by Theorem 4.1, $\varepsilon(\Gamma_M) = 6(n-1)$. We know that $\Gamma_G \cong \bar{K}_{n-1}\nabla K_n$ and by Theorem 4.1, we have $\varepsilon(\Gamma_G) = 2(n-1)$. Thus $\varepsilon(\Gamma_G)$ divides $\varepsilon(\Gamma_M)$.
$ii)$ Now, let n be an even integer. Then, by Theorem 2.1, $\Gamma_M \cong \bar{K}_{n-2}\nabla 3S$, in which S is a strongly regular graph with parameters $(n, n-2, n-3, n-2)$. Thus, by Theorems ([5], Theorems 6.2 and 6.22), the adjacency matrix of S has exactly three distinct eigenvalues: $\lambda_1 = n-2$, whose multiplicity is 1, $\lambda_2 = 0$, whose multiplicity is 1 and $\lambda_3 = -1$, whose multiplicity is $n-2$. Therefore,

$$P_S(\lambda) = (\lambda - n + 2)(\lambda + 1)^{n-2}\lambda.$$

So,

$$P_{3S}(\lambda) = (\lambda - n + 2)^3(\lambda + 1)^{3n-6}\lambda^3$$

and

$$P_{\bar{K}_{n-2}}(\lambda) = \lambda^{n-2}.$$

By Theorem 1.2, we have:

$$P_{\Gamma_M}(\lambda) = (\lambda - n + 2)^2(\lambda + 1)^{3n-6}\lambda^{n-2}(\lambda^2 + (2-n)\lambda - 3n(n-2)).$$

Thus, $\varepsilon(\Gamma_M) = 6(n-2)$. We know that $\Gamma_G \cong \bar{K}_{n-2}\nabla S$, such that S is a strongly regular graph with parameters $(n, n-2, n-3, n-2)$. Therefore, by Theorems ([5], Theorems 6.2 and 6.22),

$$P_{\Gamma_G}(\lambda) = (\lambda + 1)^{n-2}\lambda^{n-2}(\lambda^2 + (2-n)\lambda - n(n-2)).$$

So, $\varepsilon(\Gamma_G) = 2(n-2)$. Thus $\varepsilon(\Gamma_G)$ divides $\varepsilon(\Gamma_M)$. □

Finally, in the following Theorems, we count the number of spanning trees of the non-commuting graph Γ_M, where $M = M(D_{2n}, 2)$, for odd and even n, separately, and they lead us to an important result.

Theorem 4.2. *The number of spanning trees of the non–commuting graph Γ_M, where $M = M(D_{2n}, 2)$ and n is odd, is equal to:*

$$\kappa(\Gamma_M) = (2n-1)^{3n-3}(n-1)^2(3n)^{n-2}.$$

Proof. There are $4n-1$ vertices in this graph, such that they are in t_1, t_2, t_3, t_4. Each of t_i, $2 \le i \le 4$, have n vertices of degree $2n-2$, and t_1 have $n-1$ vertices of degrees $3n$. By the structure of graph Γ_M in Lemma 2.1, the matrix of vertex degree, namely D of this graph is equal to:

$$D = \begin{bmatrix} (2n-2)I_{3n} & 0_{3n(n-1)} \\ 0_{(n-1)3n} & (3n)I_{n-1} \end{bmatrix}$$

and the adjacent matrix of graph has the form:

$$A = \begin{bmatrix} (J_n - I_n) \bigotimes I_3 & J_{3n(n-1)} \\ J_{(n-1)3n} & 0_{n-1} \end{bmatrix},$$

where, $\bigotimes$ denotes the tensor product of matrices. Thus,

$$L = D - A = \begin{bmatrix} ((2n-1)I_n - J_n) \bigotimes I_3 & -J_{3n(n-1)} \\ -J_{(n-1)3n} & (3n)I_{n-1} \end{bmatrix}.$$

Now, to calculate $\det(L+J)$, we have

$$L + J = \begin{bmatrix} (2n-1)I_n & J_n & J_n & 0 \\ J_n & (2n-1)I_n & J_n & 0 \\ J_n & J_n & (2n-1)I_n & 0 \\ 0 & 0 & 0 & (3n)I_{n-1} + J_{n-1} \end{bmatrix}.$$

Also, in this case we have

$$\det(L+J) = \det B \times \det C, \tag{1}$$

where,

$$B = \begin{bmatrix} (2n-1)I_n & J_n & J_n \\ J_n & (2n-1)I_n & J_n \\ J_n & J_n & (2n-1)I_n \end{bmatrix}$$

and $C = (3n)I_{n-1} + J_{n-1}$. So,

$$\det C = (3n)^{n-2}(4n-1) \tag{2}$$

and

$$B = \begin{bmatrix} E & J_{(2n)n} \\ J_{n(2n)} & F \end{bmatrix},$$

where,

$$E = \begin{bmatrix} (2n-1)I_n & J_n \\ J_n & (2n-1)I_n \end{bmatrix}$$

and $F = (2n-1)I_n$. By Theorem 1.1, we have

$$\det B = \det F \times \det(E - JF^{-1}J). \tag{3}$$

So, by using the following relations

$$\det F = (2n-1)^n, \qquad F^{-1} = \frac{1}{2n-1} I_n, \qquad JF^{-1}J = \frac{n}{2n-1} J_{2n}, \tag{4}$$

we have

$$E - JF^{-1}J = \frac{1}{2n-1} \begin{bmatrix} G & (n-1)J \\ (n-1)J & G \end{bmatrix},$$

where, $G = (2n-1)^2 I - nJ$ and

$$\det G = (2n-1)^{2n-2}(n-1)(3n-1), \quad G^{-1} = \frac{1}{(2n-1)^2}(I + \frac{n}{(n-1)(3n-1)}J). \tag{5}$$

Now,

$$\det(E - JF^{-1}J) = (\frac{1}{2n-1})^{2n} \det(G) \times \det(G - (n-1)^2 JG^{-1}J), \tag{6}$$

where,

$$(n-1)^2 JG^{-1}J = \frac{n(n-1)}{3n-1}J$$

and

$$G - (n-1)^2 JG^{-1}J = \frac{1}{3n-1}((\alpha - \beta)I + \beta J),$$

such that, $\alpha = (n-1)(2n-1)(6n-1)$ and $\beta = -2n(2n-1)$. So,

$$\det(G - (n-1)^2 JG^{-1}J) = (2n-1)^{2(n-1)} \frac{8n^3 - 14n^2 + 7n - 1}{3n-1}. \tag{7}$$

By using the relations 5, 6 and 7, we have

$$\det(E - JF^{-1}J) = (2n-1)^{2(n-2)}(n-1)(8n^3 - 14n^2 + 7n - 1) \tag{8}$$

and by replacing relations 4 and 8 in 3 we get

$$\det B = (2n-1)^{3n-4}(n-1)(8n^3 - 14n^2 + 7n - 1). \tag{9}$$

Now, by replacing relations 2 and 9 in 1, we get

$$\det(L + J) = (2n-1)^{3(n-1)}(n-1)^2(4n-1)^2(3n)^{n-2}.$$

By Theorem ([5], Theorem 4.11), we have $\kappa = \frac{\det(L+J)}{(4n-1)^2}$. Therefore,

$$\kappa(\Gamma_M) = (2n-1)^{3(n-1)}(n-1)^2(3n)^{n-2}.$$

□

Theorem 4.3. *The number of spanning trees of the non-commuting graph Γ_M, where, $M = M(D_{2n}, 2)$ and n is even, is equal to:*

$$\kappa(\Gamma_M) = 2^{3n-3}(3n)^{n-3}(n-1)^{\frac{3n}{2}-3}(n-2)^{\frac{3n}{2}+2}.$$

Proof. There are $4n-2$ vertices in this graph and they are in t_1, t_2, t_3, t_4. Each of t_i, $2 \le i \le 4$, have n vertices of degree $2n-4$ and t_1 have $n-2$ vertices of degree $3n$. By the structure of the graph Γ in 2.1, the matrix of the vertex degree namely D, of this graph is:

$$D = \begin{bmatrix} 2(n-2)I_{3n} & 0 \\ 0 & 3nI_{n-2} \end{bmatrix}$$

and the adjacent matrix of the graph has the form:

$$A = \begin{bmatrix} X_n & 0 & 0 & J \\ 0 & X_n & 0 & J \\ 0 & 0 & X_n & J \\ J & J & J & 0 \end{bmatrix}.$$

By Lemma 2.1, each vertex in every $t_i \quad (2 \leq i \leq 4)$, is connected to the other vertices except its inverse element and itself, and so,

$$X = \begin{bmatrix} J-I & J-I \\ J-I & J-I \end{bmatrix},$$

such that I and J are square matrices of order $\frac{n}{2}$ in X. So,

$$L = D - A = \begin{bmatrix} Y_n & 0 & 0 & -J \\ 0 & Y_n & 0 & -J \\ 0 & 0 & Y_n & -J \\ -J & -J & -J & 3nI_{n-2} \end{bmatrix},$$

such that,

$$Y = \begin{bmatrix} (2n-3)I - J & I - J \\ I - J & (2n-3)I - J \end{bmatrix}.$$

Hence,

$$L + J = \begin{bmatrix} Z & J & J & 0 \\ J & Z & J & 0 \\ J & J & Z & 0 \\ 0 & 0 & 0 & 3nI + J \end{bmatrix}.$$

We have

$$Z = Y + J = \begin{bmatrix} (2n-3)I & I \\ I & (2n-3)I \end{bmatrix},$$

in which the order of I is equal to $\frac{n}{2}$. Now we obtain

$$\det(L + J) = \det B \times \det C, \tag{10}$$

where $C = 3nI_{n-2} + J_{n-2}$ and

$$B = \begin{bmatrix} Z & J & J \\ J & Z & J \\ J & J & Z \end{bmatrix}.$$

Therefore,

$$\det C = 2(3n)^{n-3}(2n-1) \tag{11}$$

and by using Theorem 1.1, we have

$$\det B = \det Z \times \det(D - JZ^{-1}J), \tag{12}$$

where,

$$D = \begin{bmatrix} Z & J \\ J & Z \end{bmatrix}$$

and

$$\det Z = (4(n-1)(n-2))^{\frac{n}{2}}. \tag{13}$$

Also,

$$Z^{-1} = \frac{1}{(2n-3)^2-1}\begin{bmatrix} (2n-3)I_{\frac{n}{2}} & -I_{\frac{n}{2}} \\ -I_{\frac{n}{2}} & (2n-3)I_{\frac{n}{2}} \end{bmatrix}$$

and so, $JZ^{-1} = \frac{1}{2(n-1)}J_{2n\times n}$ and $JZ^{-1}J = \frac{n}{2(n-1)}J_{2n\times 2n}$. So,

$$D - JZ^{-1}J = \begin{bmatrix} G & H \\ H & G \end{bmatrix}, \tag{14}$$

such that, $H = \frac{n-2}{2(n-1)}J$ and

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix},$$

where, $G_{11} = G_{22} = (2n-3)I - \frac{n}{2(n-1)}J$ and $G_{12} = G_{21} = I - \frac{n}{2(n-1)}J$.

By using elementary row or column operations in G we have

$$\begin{aligned} \det G &= \det(\tfrac{1}{2(n-1)}\begin{bmatrix} (n-1)(4n-6)I - nJ & 4(n-2)(1-n)I \\ 4(n-2)(1-n)I & 8(n-1)(n-2)I \end{bmatrix}) \\ &= (8(n-1)(n-2))^{\frac{n}{2}}\tfrac{1}{(2(n-1))^n}\det(2(n-1)^2I - nJ). \end{aligned}$$

Since,

$$\det(2(n-1)^2I - nJ) = 2^{\frac{n}{2}-2}(n-1)^{n-2}(n-2)(3n-2),$$

then

$$\det G = 2^{n-2}(n-1)^{\frac{n}{2}-2}(n-2)^{\frac{n}{2}+1}(3n-2). \tag{15}$$

By Theorem 1.1, G^{-1} is as follows:

$$G^{-1} = \begin{bmatrix} G_{11}^{-1} + (G_{11}^{-1}G_{12})(G/G_{11})^{-1}(G_{12}G_{11}^{-1}) & -G_{11}^{-1}G_{12}(G/G_{11})^{-1} \\ -(G/G_{11})^{-1}G_{12}G_{11}^{-1} & (G/G_{11})^{-1} \end{bmatrix},$$

such that, $G/G_{11} = G_{11} - G_{12}G_{11}^{-1}G_{12}$. Therefore,

$$G_{11}^{-1} = \frac{1}{(n-1)(2n-3)}(\frac{1}{2}I + \frac{n}{(n-2)(7n-6)}J)$$

and $G_{12} = I - \frac{n}{2(n-1)}J$. Then:

$$G_{12}G_{11}^{-1}G_{12} = \frac{1}{(2n-3)}(2(n-1)I + \frac{n(2n^2-15n+14)}{(7n-6)}J)$$

and

$$G_{11} - G_{12}G_{11}^{-1}G_{12} = \frac{8(n-1)(n-2)}{2n-3}((n-1)I - \frac{2n}{7n-6}J).$$

Now, we have

$$G/G_{11} = \frac{8(n-1)(n-2)}{(2n-3)}((n-1)I - \frac{2n}{7n-6}J),$$

and

$$(G/G_{11})^{-1} = \frac{1}{8(n-1)^2(n-2)}((2n-3)I + \frac{2n}{3n-2}J).$$

Therefore,

$$G^{-1} = \frac{1}{4(n-1)(n-2)}(\begin{bmatrix}(2n-3)I & -I \\ -I & (2n-3)I\end{bmatrix} + \frac{2n}{3n-2}J).$$

Also, $HG^{-1}H = \frac{n(n-2)}{2(n-1)(3n-2)}J$ and

$$G - HG^{-1}H = \begin{bmatrix}(2n-3)I & I \\ I & (2n-3)I\end{bmatrix} - \frac{2n}{3n-2}J.$$

By using elementary row or column operations, we have

$$\det(G - HG^{-1}H) = \frac{2^n}{(3n-2)}(n-2)^{\frac{n}{2}+1}(n-1)^{\frac{n}{2}-1}(2n-1) \tag{16}$$

By relation 14, we get

$$\det(D - JZ^{-1}J) = \det G \times \det(G - HG^{-1}H).$$

Then, by relations 15 and 16, we have

$$\det(D - JZ^{-1}J) = 2^{2n-2}(n-1)^{n-3}(n-2)^{n+2}(2n-1). \tag{17}$$

Also, from relations 12, 13 and 17, we obtain

$$\det B = 2^{3n-2}(n-1)^{\frac{3n}{2}-3}(n-2)^{\frac{3n}{2}+2}(2n-1) \tag{18}$$

and by relations 10, 11 and 18, we have

$$\det(L+J) = 2^{3n-1}(3n)^{n-3}(n-1)^{\frac{3n}{2}-3}(n-2)^{\frac{3n}{2}+2}(2n-1)^2, \tag{19}$$

and from replacing 19 in $\kappa = \frac{\det(L+J)}{(4n-2)^2}$, we get

$$\kappa(\Gamma_M) = 2^{3n-3}(3n)^{n-3}(n-1)^{\frac{3n}{2}-3}(n-2)^{\frac{3n}{2}+2}.$$

□

Corollary 4.2. *Let $M = M(G,2)$, where $G = D_{2n}$. Then $\kappa(\Gamma_G)$ divides $\kappa(\Gamma_M)$.*

Proof. By Example 1 in [4], the non-commuting graph of $G = D_{2n}$, when in is odd, is a split graph and $\Gamma_G \cong I\nabla C$, where I is an independent set with $n-1$ vertices and $C \cong K_n$. So, the degree matrix of Γ_G has the form:

$$D = \begin{bmatrix} (2n-2)I_{n-1} & 0 \\ 0 & nI_n \end{bmatrix}$$

and the adjacency matrix of Γ_G is equal to:

$$A = \begin{bmatrix} J-I & J \\ J & 0 \end{bmatrix}.$$

So,

$$L = D - A = \begin{bmatrix} 2n-1)I - J & -J \\ -J & nI \end{bmatrix}$$

and

$$L + J = \begin{bmatrix} (2n-2)I & 0 \\ 0 & nI + J \end{bmatrix}.$$

Thus, $\det(L+J) = \det((2n-1)I) \times \det(nI+J)$ and this gives us:

$$\det(L+J) = (2n-1)^{n+1}n^{n-2}.$$

Therefore,

$$\kappa(\Gamma_G) = \frac{\det(L+J)}{(2n-1)^2} = (2n-1)^{n-1}n^{n-2}.$$

By Theorem 4.2, $\kappa(\Gamma_M) = (2n-1)^{3(n-1)}(n-1)^2(3n)^{n-2}$. Hence, the proof is complete and $\kappa(\Gamma_G)$ divides $\kappa(\Gamma_M)$, where n is an odd integer.

Now, let n be an even integer. Then $\Gamma_G \cong \overline{K}_{n-2}\nabla S$, where S is a strongly regular graph with parameters $(n, n-2, n-4, n-2)$. Also, the degree matrix, D, of Γ_G is equal to:

$$D = \begin{bmatrix} (2n-4)I & 0 \\ 0 & nI \end{bmatrix}$$

and the adjacency matrix of Γ_G, namely A, has the form:

$$A = \begin{bmatrix} X & J \\ J & 0 \end{bmatrix},$$

where,

$$X = \begin{bmatrix} J-I & J-I \\ J-I & J-I \end{bmatrix},$$

in which, I and J are of order $\frac{n}{2}$. So,

$$L = D - A = \begin{bmatrix} Y & -J \\ -J & nI \end{bmatrix},$$

where,

$$Y = \begin{bmatrix} (2n-3)I - J & I - J \\ I - J & (2n-3)I - J \end{bmatrix}.$$

Hence,

$$L + J = \begin{bmatrix} Z & 0 \\ 0 & nI + J \end{bmatrix},$$

where,

$$Z = \begin{bmatrix} (2n-3)I & I \\ I & (2n-3)I \end{bmatrix}.$$

Since, $\det(L+J) = \det Z \times \det(nI+J)$, $\det Z = (4(n-1)(n-2))^{\frac{n}{2}}$ and $\det(nI+J) = n^{n-3}(2n-2)$, then

$$\det(L+J) = 2^{n+1} n^{n-3} (n-1)^{\frac{n}{2}+1} (n-2)^{\frac{n}{2}}.$$

Therefore,

$$\kappa(\Gamma_G) = \frac{\det(L+J)}{(2n-2)^2} = 2^{n-1} n^{n-3} (n-1)^{\frac{n}{2}-1} (n-2)^{\frac{n}{2}}.$$

Also, by Theorem 4.3, we have

$$\kappa(\Gamma_M) = 2^{3n-3} (3n)^{n-3} (n-1)^{\frac{3n}{2}-3} (n-2)^{\frac{3n}{2}+2}.$$

This proves that $\kappa(\Gamma_G)$ divides $\kappa(\Gamma_M)$. □

5. Conclusion

In this research work, we studied some properties of the non–commuting graph of a class of finite Moufang loops. Also, we proved that the multiple complete-like graphs and the non-commuting graph of Chein loops of the form $M(D_{2n}, 2)$ are perfect, and both graphs are non chordal. Finally, we characterized when the non-commuting graph of Moufang loop $M(G, 2)$ is 3-splite and we give the energy of generalized and multiple splite-like graphs. In future, we will try to study the similar graph properties of the non–commuting graph for the simple Moufang loops and characterize relations between any group G with the non–commuting graph $M(G, 2)$.

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