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The Ramsey numbers of fans versus a complete graph of order five

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Abstract

For two given graphs F and H, the Ramsey number $R(F, H)$ is the smallest integer N such that for any graph G of order N, either G contains F or the complement of G contains H. Let F_l denote a fan of order $2l + 1$, which is l triangles sharing exactly one vertex, and K_n a complete graph of order n. Surahmat et al. conjectured that $R(F_l, K_n) = 2l(n - 1) + 1$ for $l \geq n \geq 5$. In this paper, we show that the conjecture is true for $n = 5$.

Keywords: Ramsey number, fan, complete graph
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1. Introduction

All graphs considered in this paper are finite simple graphs. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The complement of G is denoted by $\overline{G}$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S in G and $G - S = G[V(G) - S]$, $N_S(v)$ denotes the set of the neighbors of a vertex v contained in S and $d_S(v) = |N_S(v)|$. If $S = V(G)$, we write $N(v) = N_G(v)$, $N[v] = N(v) \cup \{v\}$ and $d(v) = d_G(v)$. Let K_n be a complete graph of order n and mK_n the union of m vertex-disjoint copies of K_n. A fan of order $2l + 1$, denoted by F_l, is the join of K_1 and lK_2, that is l triangles sharing exactly one vertex, where the K_1 is called the center of F_l. For notations not defined here, we follow [1]. Let F and H be two given graphs. The Ramsey number $R(F, H)$ is the smallest integer N such that for any graph G of order N, either G contains F or $\overline{G}$ contains H. For a connected graph F of order p, Burr [2] established a general

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lower bound for $R(F,H)$, that is, $R(F,H) \geq (p-1)(\chi(H)-1)+s(H)$, if $p \geq s(H)$, where $\chi(H)$ is the chromatic number of H and $s(H)$ the minimum number of vertices in some color class under all vertex colorings by $\chi(H)$ colors. For the pair F_l and K_n, noting that $\chi(K_n)=n$ and $s(K_n)=1$, we have $R(F_l,K_n) \geq 2l(n-1)+1$ by Burr's lower bound. Gupta et al. showed the equality holds for $n=3$ and established the following.

Theorem 1.1 (Gupta et al. [3])**.** $R(F_l,K_3)=4l+1$ *for* $l \geq 2$.

Surahmat et al. proved the equality also holds for $n=4$ and obtained the following.

Theorem 1.2 (Surahmat et al. [5])**.** $R(F_l,K_4)=6l+1$ *for* $l \geq 3$.

Maybe motivated by Theorems 1.1 and 1.2, Surahmat et al. conjectured that the equality holds in a more general case in the same paper, and posed the following.

Conjecture 1 (Surahmat et al. [5])**.** $R(F_l,K_n)=2l(n-1)+1$ *for* $l \geq n \geq 5$.

Other results on Ramsey numbers of fans versus complete graphs can be found in the dynamic survey [4]. In this paper, we will confirm Conjecture 1 for $n=5$. The main result of this paper is as below.

Theorem 1.3. $R(F_l,K_5)=8l+1$ *for* $l \geq 5$.

2. Proof of Theorem 1.3

Since $4K_{2l}$ contains no F_l and its complement contains no K_5, $R(F_l,K_5) \geq 8l+1$. In the following, we need only to show that $R(F_l,K_5) \leq 8l+1$.

Let G be a graph of order $8l+1$ with $l \geq 5$, we need to show that either G contains an F_l or $\overline{G}$ contains a K_5. Suppose to the contrary that neither G contains an F_l nor $\overline{G}$ contains a K_5.

Let $v \in V(G)$. If $d(v) \leq 2l-1$, then $G-N[v]$ is a graph of order at least $6l+1$. By Theorem 1.2, $\overline{G}-N[v]$ contains a K_4, which implies that $\overline{G}$ contains a K_5, a contradiction. If $d(v) \geq 2l+3$, then a maximum matching M of $G[N(v)]$ contains at least l edges for otherwise $\overline{G}[N(v)-V(M)]$ is a complete graph of order at least 5, which implies that G has an F_l, a contradiction. Therefore, $2l \leq d(v) \leq 2l+2$ for any $v \in V(G)$.

Suppose that G contains a subgraph $H=K_{2l-1}$. Choose $v_0 \in V(G)-V(H)$ such that $d_H(v_0)=\max\{d_H(v) \mid v \in V(G)-V(H)\}$. Obviously, $G-(V(H)\cup\{v_0\})$ is a graph of order $6l+1$. By Theorem 1.2, $G-(V(H)\cup\{v_0\})$ contains an independent set $\{u_1,u_2,u_3,u_4\}$. Since $\overline{G}$ has no K_5, we have $V(H)\cup\{v_0\} \subseteq \cup_{i=1}^{4}N(u_i)$. This implies that $\max\{d_H(u_i) \mid 1 \leq i \leq 4\} \geq \lceil (2l-1)/4 \rceil \geq 3$. By the choice of v_0, we have $d_H(v_0) \geq 3$. If $d_H(v_0) \geq 4$, then there is some u_i having at least two neighbors in $N_H(v_0)\cup\{v_0\}$; if $d_H(v_0)=3$, then $d_H(u_i) \leq d_H(v_0)=3$ for $1 \leq i \leq 4$, which implies that there exists some u_i such that $d_H(u_i) \geq 2$ and $N_H(u_i)\cap N_H(v_0) \neq \emptyset$. In both cases, $G[V(H)\cup\{v_0,u_i\}]$ contains an F_l, a contradiction. Hence, G contains no K_{2l-1}.

By Theorem 1.2, G has an independent set $U=\{u_1,u_2,u_3,u_4\}$. For $1 \leq i \leq 4$, set $X_i=\{v \mid d_U(v)=i,\ v \in V(G)\}$. Obviously,

$$\sum_{i=1}^{4} |X_i| = 8l-3, \tag{1}$$

$$\sum_{i=1}^{4} i|X_i| = \sum_{i=1}^{4} d(u_i). \tag{2}$$

Since $\sum_{i=1}^{4} d(u_i) \leq 8l+8$, by (1) and (2), we have

$$|X_1| \geq 8l-14+|X_3|+2|X_4| \geq 8l-14. \tag{3}$$

Let $X_{1i} = N_{X_1}(u_i)$ for $1 \leq i \leq 4$. Because $\overline{G}$ has no K_5, $G[X_{1i} \cup \{u_i\}]$ is a complete graph. Since G contains no K_{2l-1}, we have $|X_{1i} \cup \{u_i\}| \leq 2l-2$, which implies that $|X_{1i}| \leq 2l-3$ for $1 \leq i \leq 4$. Thus, $|X_1| = \sum_{i=1}^{4} |X_{1i}| \leq 8l-12$. By (3), we have $|X_3|+2|X_4| \leq 2$. By (1),

$$|X_2| \geq 7. \tag{4}$$

Assume without loss of generality that $|X_{11}| \geq |X_{12}| \geq |X_{13}| \geq |X_{14}|$. Then $|X_{11}| = |X_{12}| = 2l-3$, $|X_{13}|+|X_{14}| \geq 4l-8$ and $|X_{14}| \geq 2l-5$. Denote by U_i both the vertex set $X_{1i} \cup \{u_i\}$ and the graph $G[X_{1i} \cup \{u_i\}]$ for $1 \leq i \leq 4$, then U_1, U_2, U_3, U_4 are pairwise vertex-disjoint complete graphs with $|U_1| = |U_2| = 2l-2$, $|U_3|+|U_4| \geq 4l-6$ and $|U_4| \geq 2l-4$.

Let $Y_{ij} = N_{X_2}(u_i) \cap N_{X_2}(u_j)$ for $1 \leq i < j \leq 4$.

Claim 1. If $|U_i| = 2l-2$ for some i with $1 \leq i \leq 4$, then for any $y \in Y_{ij}$, $d_{U_j}(y) \geq 3$ and if $|U_i| = |U_j| = 2l-2$, then $Y_{ij} = \emptyset$.

Proof. Since G contains no K_{2l-1}, $U_i - N(y) \neq \emptyset$. In this case, $G[U_i \cup U_j - N(y)]$ is a complete graph for otherwise any two nonadjacent vertices in $G[U_i \cup U_j - N(y)]$ together with $U \cup \{y\} - \{u_i, u_j\}$ form a K_5 in $\overline{G}$, a contradiction. Since G has no F_l and both U_i and U_j are complete graphs, we have $d_{U_j}(u) \leq 3$ for any $u \in U_i$, which implies that $|U_j - N(y)| \leq 3$. Noting that $|U_j| \geq 2l-4$ and $l \geq 5$, we have $d_{U_j}(y) \geq |U_j| - |U_j - N(y)| \geq (2l-4)-3 \geq 3$.

If $|U_i| = |U_j| = 2l-2$ and $Y_{ij} \neq \emptyset$, then for any $y \in Y_{ij}$, $d_{U_i}(y) + d_{U_j}(y) \leq 2l$ since otherwise $G[N[y]]$ contains an F_l with y as center, a contradiction. Thus we have $|U_i \cup U_j - N(y)| \geq |U_i| + |U_j| - 2l \geq 6$ since $l \geq 5$. By the arguments in the first part, we have $|U_i - N(y)| = |U_j - N(y)| = 3$. Thus, $G[U_i \cup (U_j - N(y))]$ contains an F_l with u as center for any $u \in U_i - N(y)$, a contradiction. Hence $Y_{ij} = \emptyset$. □

If $|U_4| = 2l-2$, then by Claim 1, $X_2 = \cup_{1 \leq i < j \leq 4} Y_{ij} = \emptyset$ which contradicts (4). Hence we have $2l-4 \leq |U_4| \leq 2l-3$.

Assume $|U_3| = 2l-2$. By Claim 1, we have $X_2 = \cup_{1 \leq i < j \leq 4} Y_{ij} = Y_{14} \cup Y_{24} \cup Y_{34}$, that is, $X_2 \subseteq N(u_4)$. If $|U_4| = 2l-3$, then since $\sum_{i=1}^{4} d(u_i) \leq 8l+8$, by (1), (2) and (3), either $|X_2| = 10$, $|X_3| = |X_4| = 0$ and $\sum_{i=1}^{4} d(u_i) = 8l+7$ or $|X_2| = 9$, $|X_3| = 1$, $|X_4| = 0$ and $\sum_{i=1}^{4} d(u_i) = 8l+8$. If $|U_4| = 2l-4$, then for the same reason, we have $|X_2| = 11$, $|X_3| = |X_4| = 0$ and $\sum_{i=1}^{4} d(u_i) = 8l+8$. Thus we have $|X_2| \geq 9$ in both cases, which implies that $d(u_4) \geq |X_{14}| + |X_2| \geq 2l-5+9 = 2l+4$, a contradiction. Therefore, $|U_3| \leq 2l-3$.

Since $|U_3|+|U_4| \geq 4l-6$ and $|U_4| \leq 2l-3$, we are now left to consider the case when $|U_3| = |U_4| = 2l-3$. Since $\sum_{i=1}^{4} d(u_i) \leq 8l+8$, by (1), (2) and (3), we have $|X_2| = 11$, $|X_3| = |X_4| = 0$ and $\sum_{i=1}^{4} d(u_i) = 8l+8$, which implies $d_{X_2}(u_4) = 6$. Let $N_{X_2}(u_4) = \{y_i \mid 1 \leq i \leq 6\}$. Since $\overline{G}$ contains no K_5, $G[N_{X_2}(u_4)]$ contains at least one edge, say $y_1y_2 \in E(G)$. Since G has no F_l,

$G[\{y_3, y_4, y_5, y_6\}]$ contains no edge. Because $\overline{G}$ has no K_5, we have $|\{y_3, y_4, y_5, y_6\} \cap (N(u_1) \cup N(u_2))| \geq 2$. Assume that $\{y_3, y_4\} \subseteq N(u_1) \cup N(u_2)$. By Claim 1, $d_{U_4}(y_3) \geq 3$ and $d_{U_4}(y_4) \geq 3$, which implies that $d_{X_{14}}(y_3) \geq 2$ and $d_{X_{14}}(y_4) \geq 2$. In this case, there exist $u', u'' \in X_{14}$ such that $u'y_3, u''y_4 \in E(G)$, which implies that $G[U_4 \cup \{y_1, y_2, y_3, y_4\}]$ contains an F_l with u_4 as center, a contradiction.

The proof of Theorem 1.3 is completed.

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