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On the super edge-magic deficiency of join product and chain graphs

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Abstract

A graph G of order $|V(G)| = p$ and size $|E(G)| = q$ is called *super edge-magic* if there exists a bijection $f : V(G) \cup E(G) \to \{1, 2, 3, \cdots, p+q\}$ such that $f(x) + f(xy) + f(y)$ is a constant for every edge $xy \in E(G)$ and $f(V(G)) = \{1, 2, 3, \cdots, p\}$. Furthermore, the *super edge-magic deficiency* of a graph G, $\mu_s(G)$, is either the minimum nonnegative integer n such that $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such integer n. In this paper, we study the super edge-magic deficiency of join product of a graph which has certain properties with an isolated vertex and the super edge-magic deficiency of chain graphs.

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1. Introduction

Let G be a finite and simple graph, where $V(G)$ and $E(G)$ are its vertex set and edge set, respectively. Let $p = |V(G)|$ and $q = |E(G)|$ be the number of the vertices and edges of G, respectively. Kotzig and Rosa [12] introduced the concepts of an edge-magic labeling and an edge-magic graph as follows: An *edge-magic labeling* of a graph G is a bijection $f : V(G) \cup E(G) \to \{1, 2, 3, \cdots, p+q\}$ such that $f(x) + f(xy) + f(y)$ is a constant k, called the *magic constant* of

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f, for every edge xy of G. A graph that admits an edge-magic labeling is called an *edge-magic graph*. Motivated by the concept of an edge-magic labeling, Enomoto et al. [6] introduced the concept of a super edge-magic labeling and a super edge-magic graph as follows: A *super edge-magic labeling* of a graph G is an edge-magic labeling f of G with the additional property that $f(V(G)) = \{1, 2, 3, \cdots, p\}$. Thus, a *super edge-magic graph* is a graph that admits a super edge-magic labeling. The next lemma proved by Figueroa-Centeno *et al.* [7] provides necessary and sufficient conditions for a graph to be a super edge-magic graph.

Lemma 1.1. [7] *A graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \to \{1, 2, \cdots, p\}$ such that the set $S = \{f(x) + f(y) : xy \in E(G)\}$ consists of q consecutive integers. In this case, f can be extended to a super edge-magic labeling of G with the magic constant $p + q + min(S)$.*

The next lemma proved by Enomoto *et al.* [6] gives sufficient condition for non-existence of super edge-magic labeling of a graph.

Lemma 1.2. [6] *If G is a super edge-magic graph, then $q \leq 2p - 3$.*

In addition to these two lemmas, the notion of dual labeling will also appear frequently in the next sections. A *dual* labeling of a super edge-magic labeling f is defined as

$$f'(x) = p + 1 - f(x), \text{ for all } x \in V(G),$$

and

$$f'(xy) = 2p + q + 1 - f(xy), \text{ for all } xy \in E(G).$$

It has been proved in [4] that the dual of a super edge-magic labeling is also a super edge-magic labeling.

Kotzig and Rosa [12] also proved that for every graph G there exists a nonnegative integer n such that $G \cup nK_1$ is an edge-magic graph. This fact motivated them to introduced the concept of edge-magic deficiency of a graph. The *edge-magic deficiency* of a graph G, $\mu(G)$, is defined as the minimum nonnegative integer n such that $G \cup nK_1$ is an edge-magic graph. Motivated by Kotzig and Rosa's concept of edge-magic deficiency, Figueroa-Centeno *et al.* [8] introduce the concept of super edge-magic deficiency of a graph. The *super edge-magic deficiency* of a graph G, $\mu_s(G)$, is defined as either the minimum nonnegative integer n such that $G \cup nK_1$ is a super edge-magic graph or $+\infty$ if there exists no such n.

There have been a number of papers dealing with super edge-magic deficiency of graphs. In [1], Ahmad *et al.* studied the super edge-magic deficiency of some families related to ladder graphs and In [2], Ahmad *et al.* studied the super edge-magic deficiency of unicyclic graphs. In [11], Ichishima and Oshima investigated the super edge-magic deficiency of complete bipartite graphs and disjoint union of complete bipartite graphs. Other results can be found in [8, 9] and the latest developments in these and other types of graph labelings can be found in the survey paper of graph labelings by Gallian [10]. In this paper, we study the super edge-magic deficiency of join product graphs as well as the super edge-magic deficiency of some classes of chain graphs.

2. Super edge-magic deficiency of join product graphs

Let G and H be vertex disjoint graphs. *Join product* of G and H, denoted by $G + H$, defined as a graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in E(H)\}$. Thus $G + H$ is a graph of order $p_1 + p_2$ and size $q_1 + q_2 + p_1p_2$, where $p_1 = |V(G)|, p_2 = |V(H)|$, $q_1 = |E(G)|$ and $q_2 = |E(H)|$. In this section, we study the super edge-magic deficiency of join product of a graph G which has certain properties with isolated vertices. Our first result gives necessary conditions for $G + K_1$ to have zero super edge-magic deficiency.

Lemma 2.1. *Let G be a graph with no cycle and minimum degree one. If $\mu_s(G + K_1) = 0$ then G is a tree or a forest.*

Proof. Let G be a graph of order p and size q. By Lemma 1.2, $p+q \leq 2(p+1)-3$ or $q \leq p-1$. □

This lemma is attainable by stars, paths and friendship graphs. Chen [5] proved that $\mu_s(K_{1,n} + K_1) = 0$ for every $n \geq 1$, Figueroa-Centeno *et al.* [7] proved that $\mu_s(P_n + K_1) = 0$ if and only if $1 \leq n \leq 6$, and Slamin *et al.* [19] proved that $\mu_s(nK_2 + K_1) = 0$ if and only if $n = 3, 4, 5, 7$.

We also able to prove that the join product of some classes of trees and forests with an isolated vertex has zero super edge-magic deficiency as stated in Theorem 2.1.

Theorem 2.1. a). $\mu_s([P_n \cup P_2] + K_1) = 0$ *if and only if* $3 \leq n \leq 5$.
b). $\mu_s([K_{1,n} \cup K_2] + K_1) = 0$ *if and only if* $n = 2$.
c). $\mu_s([nP_2 \cup P_3] + K_1) = 0$ *for* $1 \leq n \leq 6$.
d). $\mu_s([nP_2 \cup P_4] + K_1) = 0$ *for* $1 \leq n \leq 5$.
e). *For every* $n \geq 1$, $\mu_s(\mathrm{DS}_n + K_1) = 0$, *where* DS_n *is a double star.*
f). *For every* $n \geq 1$ *and* $m = 1, 2$, $\mu_s(G(n,m) + K_1) = 0$, *where* $G(n,m)$ *is a graph obtained from* $K_{1,n}$ *by attaching a path with* m *edges to a single leaf of* $K_{1,n}$.

Proof. a). Let $G_n = [P_n \cup P_2] + K_1$ for every $n \geq 2$. Define G_n as a graph with $V(G_n) = \{z, x_1, x_2, y_i : 1 \leq i \leq n\}$ and $E(G_n) = \{zx_1, zx_2, x_1x_2, zy_i : 1 \leq i \leq n\} \cup \{y_iy_{i+1} : 1 \leq i \leq n-1\}$. Hence, G_n is a graph of order n+3 and of size $2n + 2$. First, we show that, for $n = 3, 4, 5$, $\mu_s(G_n) = 0$. For $n = 3, 4, 5$, label $(z, \{x_1, x_2\}, (y_1, y_2, \ldots, y_n))$ as follows: $(2, \{1, 3\}, (6, 4, 5))$, $(2, \{1, 3\}, (6, 4, 7, 5))$ and $(2, \{1, 3\}, (4, 7, 5, 8, 6))$, respectively. These vertex labelings can be extended to a super edge-magic labeling of G_n for $n = 3, 4, 5$. Next, we show that $\mu_s(G_n) > 0$ for each $n \notin \{3, 4, 5\}$. If $n = 2$ then $G_2 = 2K_2 + K_1$ which is not super edge-magic. Suppose that $\mu_s(G_n) = 0$ for each $n \geq 6$. Then there exists a bijection $f : V(G_n) \cup E(G_n) \to \{1, 2, \ldots, 2n+3\}$ such that set $S = \{f(u) + f(v) : uv \in E(H))\}$ is a set of $2n + 2$ consecutive integers. Since G_n is a graph of order $n + 3$ and size $2n + 2$, so there are two possibilities of S, namely $S_1 = \{3, 4, \ldots, 2n + 4\}$ and $S_2 = \{4, 5, \ldots, 2n + 5\}$. Since S_1 and S_2 are dual to each other, it suffices to consider one of them. Let us consider $S = \{3, 4, \ldots, 2n + 4\}$. The sum of all elements in S contains $n + 2$ time of label z and three time of label y_i, $2 \leq i \leq n - 1$, and two time of label of the remaining vertices. Hence,

$$(n + 2)f(z) + 3\sum_{i=2}^{n-1} y_i + 2[f(x_1) + f(x_2) + f(y_1) + f(y_2)] = \sum_{s \in S} s = 2n^2 + 9n + 7$$

or

$$nf(z) + \sum_{i=2}^{n-1} y_i = n^2 + 2n - 5.$$

On the other hand, to get sum 3, 4 and 5 in S the only possibilities are $3 = 1 + 2$, $4 = 1 + 3$ and $5 = 2 + 3$ or $1 + 4$. Then, the vertices of labels 1, 2 and 3 must form a triangle or the vertex of label 1 is adjacent to the vertices of labels 2, 3 and 4. By this fact and the fact that every triangle in G_n share a common vertex z, hence, we have four following cases:

Case 1. $f(z) = 1$.
Then $\sum_{i=2}^{n-1} y_i = n^2 + n - 5$. It is not possible, since $n^2 + n - 5 > \sum_{i=5}^{n+3} i = \frac{1}{2}(n^2 + 7n - 8)$ for every $n \geq 6$.

Case 2. $f(z) = 2$.
Then $\sum_{i=2}^{n-1} y_i = n^2 - 5$ and $n^2 - 5 \leq \frac{1}{2}(n^2 + 7n - 8)$ is possible only for $n = 6$ and $n = 7$. One can check that the condition $f(z) = 2$, for $n \in \{6, 7\}$, do not lead to a super edge-magic labeling of G_6 and G_7, respectively.

Case 3. $f(z) \in \{3, 4\}$.
In this case, the sums $f(z) + n + 4$, $f(z) + n + 5, \ldots, 2n + 3, 2n + 4$ should be the sum of labels of two adjacent vertices in P_n or P_2. To obtain $2n + 4, 2n + 3, 2n + 2$ and $2n + 1$ we only have two possibilities: $(n-1)-(n+2)-n-(n+3)-(n+1)$ or $(n-2)-(n+3)-(n+1)-(n+2)-n$. These constructions fail to get sum $2n$.

Hence, G_n is not super edge-magic for $n \notin \{3, 4, 5\}$. So, $\mu_s(G_n) > 0$ for each $n \notin \{3, 4, 5\}$.

b). Let $H_n = [K_{1,n} \cup K_2] + K_1$ for every $n \geq 1$. H_n is a graph with $|V(H_n)| = n + 4$ and $|E(H_n)| = 2n + 4$. Let $V(H_n) = \{z, c, y_1, y_2, x_i : 1 \leq i \leq n\}$ and $E(H_n) = \{y_1y_2, zc, zy_1, zy_2, cx_i, zx_i : 1 \leq i \leq n\}$. Next, let $\mu_s([K_{1,n} \cup P_2] + K_1) = 0$. By Lemma 1.1, there exists a vertex labeling f such that $S = \{f(u) + f(v) : uv \in E(H)\}$ is a set of $2n + 4$ consecutive integers. Then, there are two possibilities of S, namely $S_1 = \{3, 4, \ldots, 2n + 6\}$ or $S_2 = \{4, 5, \ldots, 2n + 7\}$ and they are dual to each other. If $S = S_1$ then

$$(n + 1)f(z) + (n - 1)f(c) = n^2 + 4n - 2.$$

From this equation, n should be an even integer and both of $f(z)$ and $f(c)$ have the same variety.

By a similar argument as in the proof of part a), the vertices of labels 1, 2 and 3 must form a triangle in H_n or the vertex of label 1 is adjacent to the vertices of labels 2, 3 and 4. By these facts and since all triangles in H_n have a common vertex z, then there are four following cases:

Case 1. $f(z) = 1$, $f(c) = 3$, and $f(x_{i_0}) = 2$ for some $i_0 \in \{1, 2, \ldots, n\}$.
Then $n = 1$. It is well known that $2K_2 + K_1$ is not a super edge-magic graph.

Case 2. $f(z) = 2$ and $\{f(y_1), f(y_2)\} \in \{1, 3\}$.
If $f(z) = 2$ then $f(c) = (n + 3) - \frac{1}{n-1}$. So, $n = 2$ and $f(c) = 4$. Next, set $f(\{x_1, x_2\}) = \{5, 6\}$. This vertex labeling can be extended to a super edge-magic labeling of H_2 with the magic constant 21.

Case 3. $f(z) = 3$, $f(c) = 1$, and $f(x_{i_0}) = 2$ for some $i_0 \in \{1, 2, \ldots, n\}$.
If $f(z) = 3$ and $f(c) = 1$ then $n = 2$. Next, label the remaining vertices in H_2 as follows: $f(\{y_1, y_2\}) = \{4, 6\}$ and $f(\{x_1, x_2\}) = \{2, 5\}$. It can be checked that this vertex labeling can be extended to a super edge-magic labeling of H_2 with the magic constant 21.

Case 4. $n \geq 3$, $f(c) = 1$ and $\{f(x_{i_0}), f(x_{j_0}), f(x_{k_0})\} \in \{2,3,4\}$ for some $i_0, j_0, k_0 \in \{1,2,\ldots,n\}$. If $f(c) = 1$ then $f(z) = (n+2) - \frac{3}{n+1}$. Hence, $n = 2$ and $f(z) = 3$, and it is a contradiction.

c). For $1 \leq n \leq 6$, tet $G_n = [nP_2 \cup P_3] + K_1$ and let $V(G_n) = \{z, x_i, y_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq 3\}$ and $E(G_n) = \{x_i y_i : 1 \leq i \leq n\} \cup \{u_1u_2, u_2u_3\} \cup \{zx_i, zy_i : 1 \leq i \leq n\} \cup \{zu_i : 1 \leq i \leq 3\}$. For $1 \leq n \leq 6$, label (z, u_1, u_2, u_3) and $\{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_n, y_n\}\}$ by $(2,6,4,5)$ and $\{\{1,3\}\}$; $(2,7,6,5)$ and $\{\{1,3\},\{4,8\}\}$; $(2,7,9,6)$ and $\{\{1,3\},\{4,10\},\{5,8\}\}$; $(4,5,12,8)$ and $\{\{1,3\},\{2,6\},\{7,11\},\{9,10\}\}$; $(6,7,14,10)$ and $\{\{1,5\},\{2,3\},\{4,8\},\{9,13\},\{11,12\}\}$; $(8,13,15,10)$ and $\{\{1,5\},\{2,6\},\{3,4\},\{7,9\},\{11,16\},\{12,14\}\}$, respectively.

d). Let $H_n = [nP_2 \cup P_4] + K_1$ for $1 \leq n \leq 5$. Let $V(H_n) = \{z, x_i, y_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq 4\}$ and $E(H_n) = \{x_i y_i : 1 \leq i \leq n\} \cup \{u_1u_2, u_2u_3, u_3u_4\} \cup \{zx_i, zy_i : 1 \leq i \leq n\} \cup \{zu_i : 1 \leq i \leq 4\}$. For $1 \leq n \leq 5$, label (z, u_1, u_2, u_3, u_4) and $\{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_n, y_n\}\}$ by $(2,6,4,7,5)$ and $\{\{1,3\}\}$; $(2,7,6,9,5)$ and $\{\{1,3\},\{4,8\}\}$; $(4,2,1,3,5)$ and $\{\{6,10\},\{7,11\}, \{8,9\}\}$; $(6,3,1,4,2)$ and $\{\{5,7\},\{8,12\},\{9,13\},\{10,11\}\}$; $(8,5,1,4,3)$ and $\{\{2,6\},\{7,9\}, \{10,14\},\{11,15\},\{12,13\}\}$, respectively.

e). First, Let $G_n = DS_n + K_1$ for every $n \geq 1$. Next, define vertex and edge sets of G_n as follows: $V(G_n) = \{z, x, y, x_i, y_i : 1 \leq i \leq n\}$ and $E(G_n) = \{xy, zx, zy\} \cup \{xx_i, yy_i, zx_i, zy_i : 1 \leq i \leq n\}$. Next, label (z, x, y), $\{x_i : 1 \leq i \leq n\}$ and $\{y_i : 1 \leq i \leq n\}$ with $(n+2, 1, 2n+3)$, $\{2, 3, \ldots, n+1\}$ and $\{n+3, n+4, \ldots, 2n+2\}$, respectively. By Lemma 1.1, this labeling can be extended to a super edge-magic labeling of G_n with magic constant $6n + 9$.

f). Let $H = G(n,2) + K_1$ for every $n \geq 1$. Define H as a graph with $V(H) = \{z, x, x_i : 1 \leq i \leq n+2\}$ and $E(H) = \{xx_i : 1 \leq i \leq n\} \cup \{x_n x_{n+1}, x_{+1} x_{n+2}\} \cup \{zx, zx_i : 1 \leq i \leq n+2\}$. Label (z, x, x_{n+1}, x_{n+2}) with $(n+2, 1, n+3, n+4)$ and label $\{x_1, x_2, \ldots, x_n\}$ with $\{2, 3, \ldots, n+1\}$. This labeling can be extended to a super edge-magic labeling of H with magic constant $3n + 12$. If x_{n+2} is removed, we get $G(n,1) + K_1$ and the remaining labeling can be extended to a super edge-magic labeling of $G(n,1) + K_1$. □

The open problems relating to these results are as follows:

Problem 1. *Determine if the graphs* $[nP_2 \cup P_3] + K_1$ *for* $n \geq 7$ *and* $[nP_2 \cup P_4] + K_1$ *for* $n \geq 6$ *have zero super edge-magic deficiency.*

As mentioned before, Figueroa-Centeno *et al.* [7] proved that $\mu_s(F_n) = 0$ if and only if $1 \leq n \leq 6$. The natural question arise is what about the super edge-magic deficiency of join product of other trees of order at most six with an isolated vertex? In the next results, we study the super edge-magic deficiency of these graphs.

Lemma 2.2. *For any tree* G *of order* $p \leq 6$ *excluding the tree in Figure 1 (a),* $\mu_s(G) = 0$.

Proof. All trees of order at most six are P_2, P_3, P_4, $K_{1,3}$, P_5, $K_{1,4}$, $G(3,1)$, P_6, $K_{1,5}$, $G(3,2)$, $G(4,1)$ and DS_2. As a direct consequence of results of Chen [5], Figueroa-Centeno *et al.* [7], Theorem 2.1 e) and Theorem 2.1 f), the super edge-magic deficiency of join product of these graphs with an isolated vertex is zero. □

Figure 1. Trees with 6 and 7 vertices

Let $H = G_1 + K_1$, where G_1 is the tree in Figure 1(a). Let $V(H) = \{z, x_i : 1 \leq i \leq 6\}$ and $E(H) = \{x_i x_{i+1} : 1 \leq i \leq 4\} \cup \{x_3 x_6\} \cup \{z x_i : 1 \leq i \leq 6\}\}$. It is not hard to prove that H is not super edge-magic. Furthermore, if we label $z, x_1, x_2, x_3, x_4, x_5, x_6$ with 5, 7, 4, 1, 2, 8, 3, respectively, then this labeling can be extended to a super edge-magic labeling of $H \cup K_1$. So, $\mu_s(H) = 1$. The next result provides a sufficient condition of the join product of a tree of order $p \geq 7$ with an isolated vertex to have nonzero super edge-magic deficiency.

Theorem 2.2. *Let G be a tree of order $p \geq 7$ and let $H = G + K_1$. If $\mu_s(H) = 0$ then either $2K_{1,3}$ or $K_3 \cup K_{1,3}$ is a subgraph of H.*

Proof. Let $\mu_s(H) = 0$ with a super edge-magic labeling f. Since H is a graph of order $p+1$ and size $q = 2p - 1 = 2(p+1) - 3$, then $S = \{f(x) + f(y) : xy \in E(H)\} = \{3, 4, \ldots, 2p+1\}$ and the vertices of labels 1, 2 and 3 must form a triangle or the vertex of label 1 is adjacent to the vertices of labels 2, 3 and 4, respectively. Also, the vertices of labels $p+1$, p and $p-1$ must form a triangle or the vertex of label $p+1$ is adjacent to the vertices of labels p, $p-1$ and $p-2$, respectively. Since H is a graph of order $p \geq 8$, the labels 1, 2, 3, 4, $p+1$, p, $p-1$ and $p-2$ are all distinct. By combining these facts, we obtain either $2K_3$, $K_3 \cup K_{1,3}$ or $2K_{1,3}$ as a subgraph of H. However, $2K_3$ cannot be a subgraph of H since every triangle in H share a common vertex. This completes the proof. □

The converse of Theorem 2.2 is not true. To show this, let us consider the tree G_2 in Figure 1 (b). Define vertex and edge sets of $G_2 + K_1$ as follows: $V(G_2 + K_1) = \{z, x_i : 1 \leq i \leq 5\} \cup \{y_1, y_2\}$, $E(G_2 + K_1) = \{x_i x_{i+1} : 1 \leq i \leq 4\} \cup \{x_3 y_1, x_3 y_2\} \cup \{z x_i : 1 \leq i \leq 5\} \cup \{z y_1, z y_2\}$. It can be checked that $K_3 \cup K_{1,3}$ and $2K_{1,3}$ are subgraphs of $G_2 + K_1$. Assume that $\mu_s(G_2 + K_1) = 0$. Then there exists a vertex labeling f such that $5f(z) + 3f(x_3) + f(x_2) + f(x_4) = 45$. It is easy to check that any solutions of this equation do not lead to a super edge-magic labeling of $G_2 + K_1$. So, $\mu_s(G_2 + K_1) \geq 1$. If we label $z, x_1, x_2, x_3, x_4, x_5, y_1$ and y_2 by 2, 3, 1, 6, 8, 4, 7 and 9, respectively, then this vertex labeling can be extended to a super edge-magic labeling of $[G_2 + K_1] \cup K_1$. So, $\mu_s(G_2 + K_1) \leq 1$. Hence, $\mu_s(G_2 + K_1) = 1$.

Next results provide the super edge-magic deficiency of join product of a tree with $m \geq 2$ isolated vertices.

Lemma 2.3. *Let G a tree of order $p \geq 2$ and $m \geq 2$ be an integer. $\mu_s(G + mK_1) = 0$ if and only if $G = P_2$.*

Proof. Let $\mu_s(G + mK_1) = 0$. Then by Lemma 1.2, $mp + p - 1 \leq 2(p+m) - 3$ or $(p-2)(m-1) \leq 0$ and the desired result. Next we show that $\mu_s(P_2 + mK_1) = 0$. Label the vertices in P_2 with $\{1, m+2\}$ and mK_1 with $\{2, 3, \ldots, m+1\}$. By Lemma 1.2 this labeling can be extended to a super edge-magic labeling of $P_2 + mK_1$. □

Lemma 2.3 show that $\mu_s(G + mK_1) \geq 1$ for all the trees $G \neq P_2$. Next lemma provides the lower bound of its super edge-magic deficiency.

Lemma 2.4. *Let G be a tree of order $p \geq 3$. For every positive integer $m \geq 2$,*

$$\mu_s(G + mK_1) \geq \lfloor \frac{(m-1)(p-2)+1}{2} \rfloor.$$

Proof. This result is a corollary of the result of Ngurah and Simanjuntak [16] (see Lemma 2.2). □

Lemma 2.4 is attainable. It has been proved that $\mu_s(P_4 + mK_1) = m - 1$, $\mu_s(P_6 + mK_1) = 2(m-1)$ [17] and $\mu_s(P_n + 2K_1) = \frac{n-2}{2}$ for any even integer $n \geq 2$ [18].

3. Super edge-magic deficiecy of chain graphs

Barrientos [3] defined a *chain graph* as a graph with blocks $B_1, B_2, \cdots, B_k$ such that for every i, B_i and B_{i+1} have a common vertex in such a way that the block-cut-vertex graph is a path. We denote the chain graph with k blocks $B_1, B_2, \cdots, B_k$ by $C[B_1, B_2, \cdots, B_k]$. If $B_1 = \cdots = B_t = B$, we write $C[B_1, B_2, \cdots, B_k]$ as $C[B^{(t)}, B_{t+1}, \cdots, B_k]$. If for every i, $B_i = H$ for a given graph H, then $C[B_1, B_2, \cdots, B_k]$ is denoted by kH-path. Suppose that $c_1, c_2, \ldots, c_{k-1}$ are the consecutive cut vertices of $C[B_1, B_2, \cdots, B_k]$. The *string* of $C[B_1, B_2, \cdots, B_k]$ is $(k-2)$-tuple $(d_1, d_2, \ldots, d_{k-2})$ where d_i is the distance between c_i and c_{i+1}, $1 \leq i \leq k-2$. We will write $(d_1, d_2, \ldots, d_{k-2})$ as $(d^{(t)}, d_{t+1}, \ldots, d_{k-2})$ if $d_1 = \ldots = d_t = d$. Some authors have studied the super edge-magic deficiency of chain graphs. In 2003, Lee and Wang [13] proved that some classes of chain graphs whose blocks are complete graphs are super edge-magic. In other words, they showed that some classes of chain graphs whose blocks are complete graphs have zero super edge-magic deficiency. In [15], Ngurah *et al.* studied the super edge-magic deficiency of kK_3-paths and kK_4-paths.

Let $L_n = P_n \times P_2$ be a ladder. Let TL_n be the graph obtained from the ladder L_n by adding a single diagonal in each rectangle of L_n and let DL_m be the graph obtained from the ladder L_m by adding two diagonals in each rectangle of L_m. It is clear that TL_n is graph of order $2n$ and size $4n-3$ meanwhile DL_m has $2m$ vertices and $5m-4$ edges. In this section, we study the super edge-magic deficiency of chain graphs where its blocks are combination of TL_n and DL_m.

First, we study the super edge-magic deficiency of a chain graph $G = C[B_1, B_2, \cdots, B_k]$ where $B_i = \text{TL}_n$, $n \geq 2$, when i is odd and $B_i = \text{DL}_m$, $m \geq 3$, when i is even. We define vertex and edge sets of B_i, $1 \leq i \leq k$, as follows:
When i is odd, $V(B_i) = \{x_i^j, y_i^j : 1 \leq j \leq n\}$ and $E(B_i) = \{x_i^j y_i^j : 1 \leq j \leq n\} \cup \{x_i^j x_i^{j+1}, y_i^j y_i^{j+1} : 1 \leq j \leq n-1\} \cup \{e_i^j : \text{where } e_i^j \text{ is either } x_i^j y_i^{j+1} \text{ or } y_i^j x_i^{j+1}, 1 \leq j \leq n-1\}$.
When i is even, $V(B_i) = \{u_i^t, u_i^t : 1 \leq t \leq m\}$ and $E(B_i) = \{u_i^t u_i^t : 1 \leq t \leq m\} \cup \{u_i^t u_i^{t+1}, v_i^t v_i^{t+1}, u_i^t v_i^{t+1}, v_i^t u_i^{t+1} : 1 \leq t \leq m-1\}$.
Vertex and edge sets of G are defined as follows: $V(G) = \cup_{i=1}^k V(B_i)$, where $x_i^n = v_{i+1}^1$, $1 \leq i \leq k-1$, and $E(G) = \cup_{i=1}^k E(B_i)$. Under these definitions, $x_i^n = v_{i+1}^1$, $1 \leq i \leq k-1$, are the cut vertices of G. The string of G is $(m-1, d_1, m-1, d_2, m-1, \ldots, d_{(k-3)/2}, m-1)$ when k is odd or $(m-1, d_1, m-1, d_2, m-1, \ldots, d_{(k-2)/2})$ when k is even, where $d_1, d_2, \ldots, d_{\lfloor (k-2)/2 \rfloor} \in \{n-1, n\}$. If $n = m$, G is a $k\text{DL}_m$-path. The super edge-magic deficiency of $k\text{DL}_m$-path has been studied

by Ngurah and Adiwijaya [14]. Here, we study the super edge-magic deficiency of G when n not necessarily equal to m. We found that its super edge-magic deficiency is invariant under n, as we state in the next theorem.

Theorem 3.1. *Let $k \geq 3$ be an integer. For any integers $n \geq 2$ and odd $m \geq 3$,*

$$\mu_s(G) = \begin{cases} \frac{1}{4}k(m-3)+1, & \text{if } k \text{ is even}, \\ \frac{1}{4}(k-1)(m-3), & \text{if } k \text{ is odd}. \end{cases}$$

Proof. It is clear that, if $k \geq 4$ is even then $|V(G)| = \frac{1}{2}k(2n-1)+\frac{1}{2}k(2m-1)+1$ and $|E(G)| = \frac{1}{2}k(4n-3)+\frac{1}{2}k(5m-4)$. If $k \geq 3$ is odd then $|V(G)| = \frac{1}{2}(k+1)(2n-1)+\frac{1}{2}(k-1)(2m-1)+1$ and $|E(G)| = \frac{1}{2}(k+1)(4n-3)+\frac{1}{2}(k-1)(5m-4)$. By Lemma 1.2, if k is even then G is not super edge-magic for any integers $n \geq 2$ and $m \geq 3$, and if k is odd then G is not super edge-magic for any integers $n \geq 2$ and $m \geq 4$. As we can see later, if k is odd then G is super edge-magic for any $n \geq 2$ and $m = 3$. Again, by Lemma 1.2, it is not hard to prove that $\mu_s(G) \geq \frac{1}{4}k(m-3)+1$ when k is even and $\mu_s(G) \geq \frac{1}{4}(k-1)(m-3)$ when k is odd. To show the upper bound of $\mu_s(G)$, define a vertex labeling f as follows:

$$\begin{aligned} f(x_1^j) &= 2j-1, 1 \leq j \leq n. \\ f(u_2^t) &= \tfrac{1}{2}(4n+5t-3), t \text{ is odd}, 1 \leq t \leq m. \\ f(u_2^t) &= \tfrac{1}{2}(4n+5t-4), t \text{ is even}, 1 \leq t \leq m. \end{aligned}$$

For $1 \leq i \leq \lfloor \frac{1}{2}(k-1) \rfloor$, $f(x_{2i-1}^j) = \frac{1}{2}(4n+5m-7)i + f(x_1^j)$, $1 \leq j \leq n$. For $1 \leq i \leq \lfloor \frac{1}{2}(k-2) \rfloor$, $f(u_{2i+2}^t) = \frac{1}{2}(4n+5m-7)i + f(u_2^t)$, $1 \leq t \leq m$.

For $1 \leq i \leq k$, label the remaining vertices as follows:

$$\begin{aligned} f(y_1^j) &= f(x_1^j)+1, i \text{ is odd}, 1 \leq j \leq n. \\ f(v_i^t) &= f(u_i^t)-2, i \text{ is even}, t \text{ is odd}, 1 \leq t \leq m. \\ f(v_i^t) &= f(u_i^t)-1, i \text{ is even}, t \text{ is even}, 1 \leq t \leq m. \end{aligned}$$

Under the labeling f, one can verify that no labels are repeated, $f(x_i^n) = f(v_{i+1}^1)$, $1 \leq i \leq k-1$, and the largest vertex label used is $\frac{1}{4}(k-2)(4n+5m-7)+\frac{1}{2}(4n+5m-3) = \frac{1}{4}k(m-3)+1+|V(G)|$ when k is even or $\frac{1}{4}(k-1)(4n+5m-7)+2n = \frac{1}{4}(k-1)(m-3)+|V(G)|$ when k is odd. Particularly, if k is odd and $m = 3$ the largest vertex label used is $|V(G)|$. It means that f is a super edge-magic labeling of G when k is odd and $m = 3$.

Next, let $\alpha = \frac{1}{4}k(m-3)+1$ when k is even or $\alpha = \frac{1}{4}(k-1)(m-3)$ when k is odd. Denote the isolated vertices with $\{z_{2i}^l : 1 \leq i \leq \lfloor \frac{k}{2} \rfloor, 1 \leq l \leq \frac{1}{2}(m-3)\} \cup \mathcal{S}$, where $|\mathcal{S}| = 1$ when k is even or $|\mathcal{S}| = 0$ when k is odd. Set $f(z_{2i}^l) = f(y_{2i-1}^l)+5l$ and $f(\mathcal{S}) = f(u_k^m)-1$. It can be checked that f is a bijection from $V(G) \cup \alpha K_1$ to $\{1, 2, \ldots, |V(G)|+\alpha\}$ and $\{f(x)+f(y) : xy \in E(G)\}$ is a set of $|E(G)|$ consecutive integers. By Lemma 1.1, f can be extended to a super edge-magic labeling of $G \cup \alpha K_1$. Hence, $\mu_s(G) \leq \frac{1}{4}k(m-3)+1$ when k is even or $\mu_s(G) \leq \frac{1}{4}(k-1)(m-3)$ when k is odd. This completes the proof. □

Problem 2. *For $m \geq 3$ is even, determine $\mu_s(G)$.*

Next, we study the super edge-magic deficiency of $H = C[B_1, B_2, \cdots, B_k]$ where $B_i = \text{TL}_n$, $n \geq 2$, when i is even and $B_i = \text{DL}_m$, $m \geq 3$, when i is odd. We define vertex and edge sets of H as follows: $V(H) = \cup_{i=1}^{k} V(B_i)$, where $u_i^m = y_{i+1}^1$, $1 \leq i \leq k-1$, and $E(H) = \cup_{i=1}^{k} E(B_i)$, where $V(B_i)$ and $E(B_i)$ are defined as before. Under these definitions, $u_i^m = y_{i+1}^1$, $1 \leq i \leq k-1$, are the cut vertices of H. The string of H is $(d_1, m-1, d_2, m-1, \ldots, m-1, d_{(k-1)/2})$ when k is odd or $(d_1, m-1, d_2, m-1, \ldots, d_{(k-2)/2}, m-1)$ when k is even, where $d_1, d_2, \ldots, d_{\lfloor (k-2)/2 \rfloor} \in \{n-1, n\}$. Notice that, when k is even, the chain graph H is isomorphic to G, where G is the chain graph in Theorem 3.1. Hence, $\mu_s(H) = \mu_s(G) = \frac{1}{4}k(m-3)+1$ when k is even. Next theorem gives the upper and lower bounds of the super edge-magic deficiency of H when k is odd.

Theorem 3.2. *Let $k \geq 3$ be an odd integer. For any integers $n \geq 2$ and odd $m \geq 3$, the super edge-magic deficiency of H satisfies*

$$\frac{1}{4}(k+1)(m-1) - \frac{1}{2}(k-1) \leq \mu_s(G) \leq \frac{1}{4}(k+1)(m-1) - \frac{1}{2}(k-3).$$

Proof. H is a graph of order $\frac{1}{2}(k+1)(2m-1)+\frac{1}{2}(k-1)(2n-1)+1$ and size $\frac{1}{2}(k+1)(5m-4)+\frac{1}{2}(k-1)(4n-3)$. By Lemma 1.2, H is not super edge-magic and $\mu_s(H) \geq \frac{1}{4}(k+1)(m-1) - \frac{1}{2}(k-1)$. Next, define a vertex labeling f as follows:

$$\begin{aligned}
f(u_1^t) &= \tfrac{1}{2}(5t-3), t \text{ is odd}, 1 \leq t \leq m-2.\\
f(u_1^t) &= \tfrac{1}{2}(5t-2), t \text{ is even}, 2 \leq t \leq m-1.\\
f(u_1^m) &= \tfrac{1}{2}(5m+1).\\
f(x_2^j) &= \tfrac{1}{2}(5m+4j-5), 1 \leq j \leq n.\\
f(u_3^t) &= \tfrac{1}{2}(5m+4n+5t-6), t \text{ is odd}, 1 \leq t \leq m-2.\\
f(u_3^t) &= \tfrac{1}{2}(5m+4n+5t-7), t \text{ is even}, 2 \leq t \leq m-1.
\end{aligned}$$

For $1 \leq i \leq \frac{1}{2}(k-3)$, $1 \leq j \leq n$ and $1 \leq t \leq m$, label the remaining vertices as follows:

$$\begin{aligned}
f(x_{2i+2}^j) &= \tfrac{1}{2}(5m+4n-7)i + f(x_2^j).\\
f(u_{2i+3}^t) &= \tfrac{1}{2}(5m+4n-7)i + f(u_3^t).
\end{aligned}$$

For $1 \leq i \leq k$, $1 \leq j \leq n$ and $1 \leq t \leq m$, label the remaining vertices as follows:

$$\begin{aligned}
f(v_i^t) &= f(u_i^t)+2, i \text{ and } t \text{ are odd}, t \neq m.\\
f(v_i^t) &= f(u_i^t)+1, i \text{ is odd}, t \text{ is even}.\\
f(v_i^m) &= f(u_i^m)-2, i \text{ is odd}.\\
f(y_i^j) &= f(x_i^j)+1, i \text{ is even}.
\end{aligned}$$

It can be checked that the vertex labeling f constitute a set $\{f(x)+f(y) : xy \in E(H)\}$ of $|E(H)|$ consecutive integers, no labels are repeated and the largest vertex label used is $f(u_k^m) = \frac{1}{4}(k+1)(m-1) - \frac{1}{2}(k-3) + |V(H)|$. Hence, there exist $\frac{1}{4}(k+1)(m-1) - \frac{1}{2}(k-3)$ labels that are not utilized. Thus, for each the number from 1 to $|V(G)|$ that has not been used as a label, we introduce a new vertex with that number as its label which gives $\frac{1}{4}(k+1)(m-1) - \frac{1}{2}(k-3)$ new isolated vertices. By Lemma 1.1, this yields a super edge-magic labeling of $H \cup [\frac{1}{4}(k+1)(m-1) - \frac{1}{2}(k-3)]K_1$. So, $\mu_s(H) \leq \frac{1}{4}(k+1)(m-1) - \frac{1}{2}(k-3)$. □

Problem 3. *Determine the exact value of the $\mu_s(H)$ when $k, m \geq 3$ are odd.*

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