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Large degree vertices in longest cycles of graphs, II

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Abstract

In this paper, we consider the least integer d such that every k-connected graph G of order n (and of independent number α) has a longest cycle containing all vertices of degree at least d. We completely determine the d when $k = 2$. We propose a conjecture for those k-connected graph, where $k \geq 3$.

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1. Introduction

We use $\mathcal{G}(k,n)$ to denote the class of k-connected graph of order n. In [9], we considered a problem involving large degree vertices in longest cycles, *i.e.,* to determine the least integer d such that for every graph $G \in \mathcal{G}(k,n)$,

(a) every longest cycle of G contains all vertices of degree at least d.

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Since a longest cycle does not always contain all large degree vertices, it would be also an interesting problem to ask whether there is a (longest) cycle containing special vertices (all large degree vertices from some degree). For examples that there is a (longest) cycle contains some special vertices in a graph, see [1], [7], [11] and [12], [14]; for paths, see [8], [13], for digraphs, see [6].

In this paper, we consider a similar problem to determine the least integer d such that for every graph $G \in \mathcal{G}(k,n)$,

(b) there exists a longest cycle in G containing all vertices of degree at least d.

In order to avoid the discussions of petty cases, we only consider 2-connected graphs. In the following when we say a graph is k-connected, we always assume that $k \geq 2$.

Following [9], we denote by $\varphi(k,n)$ the least integer d such that every longest cycle of a graph $G \in \mathcal{G}(k,n)$ contains all vertices of degree at least d. We now denote by $\phi(k,n)$ the least integer d such that every graph $G \in \mathcal{G}(k,n)$ has a longest cycle containing all vertices of degree at least d. Clearly, if $\varphi(k,n)$ and $\phi(k,n)$ exist, we have $\phi(k,n) \leq \varphi(k,n)$.

By definition if a graph G is Hamiltonian, then every longest cycle of G contains all vertices of G. Thus for integers k and n, $\varphi(k,n)$ or $\phi(k,n)$ always exists if and only if there is a non-Hamiltonian graph $G \in \mathcal{G}(k,n)$. This requires $n \geq 2k+1$ by Dirac's theorem [5]. On the other hand, if $n \geq 2k+1$, then the graph $K_{k,n-k}$ is a non-Hamiltonian graph in $\mathcal{G}(k,n)$. So in this paper we assume that $n \geq 2k+1$. In [9], we determined all values $\varphi(k,n)$:

$$\varphi(k,n) = \max\left\{\left\lceil\frac{n}{2}\right\rceil, n-3k+2\right\}, \text{ for } n \geq 2k+1. \tag{1'}$$

In this paper, our first aim is to determine $\phi(k,n)$. As we will show bellow, we have the following formula:

$$\phi(k,n) = \max\left\{\left\lceil\frac{n}{2}\right\rceil, n-4k+2\right\}, \text{ for } n \geq 2k+1. \tag{1}$$

We can see that $\phi(k,n) = \varphi(k,n)$ if $n \leq 6k-4$. This means the result from [9] together with some extremal graphs is enough to determine $\phi(k,n)$ when $n \leq 6k-4$. However, we will completely prove the following Theorem 1.1, since to prove Theorem 1.1 only in the case '$n \geq 6k-3$' cannot reduce the length of the proof.

Theorem 1.1. *Let G be a k-connected graph of order n. Then G has a longest cycle containing all vertices of degree at least*

$$d = \max\left\{\left\lceil\frac{n}{2}\right\rceil, n-4k+2\right\}.$$

The following result is an obvious corollary of Theorem 1.1, when the graphs G of order n are k-connected with $n \leq 8k-4$.

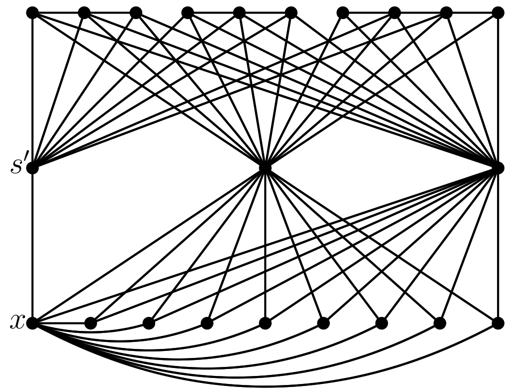

Figure 1. Graph $L(3, 22)$.

Theorem 1.2. *(Shi, [14]) Let G be a 2-connected graph of order n. Then G has a (not necessarily longest) cycle containing all vertices of degree at least $\frac{n}{2}$.*

From Theorem 1.1, we know that $\phi(k, n) \leq \max\{\lceil n/2\rceil, n - 4k + 2\}$. As we pointed out above, this is in fact the exact value of $\phi(k, n)$ when $n \geq 2k + 1$. To prove this, we construct some extremal graphs to show that if $n \geq 2k + 1$, then the bound on d in Theorem 1.1 is sharp.

For the case $n \geq 8k - 4$, we construct a graph as follows: Let $R = 2P_3 \cup (k-2)P_4$, $S = kK_1$ and $T = (n - 5k + 1)K_1$ are vertex-disjoint. Let R' be the subset of $V(R)$ each vertex of which is either a vertex of a P_3 or an interior vertex of a P_4 in R, let s' be a fixed vertex of S and x a vertex not in $R \cup S \cup T$. Let $L(k, n)$ be the graph with vertex set $\{x\} \cup V(R) \cup V(S) \cup V(T)$, and edge set

$$E(R) \cup \{r's', rs, s'x, sx, st, xt : r' \in R', r \in V(R), s \in V(S)\backslash\{s'\}, t \in V(T)\}.$$

For an example, see Fig. 1. One can check that $L(k, n) \in \mathcal{G}(k, n)$ and every longest cycle of $L(k, n)$ excludes the vertex x of degree $n - 4k + 1$.

For the case $2k + 1 \leq n \leq 8k - 5$, we let

$$L(k, n) = \begin{cases} K_{(n-1)/2,(n+1)/2}, & \text{if } n \text{ is odd;} \\ K_{n/2-1,n/2+1}, & \text{if } n \text{ is even.} \end{cases}$$

Note that $L(k, n) \in \mathcal{G}(k, n)$ and every longest cycle of $L(k, n)$ excludes some vertices of degree $\lceil n/2\rceil - 1$.

Thus we get the complete formula as (1).

Now we will consider this kind of problem on those graphs with a given independent number. Let $\mathcal{G}(k, \alpha, n)$ denote the class of k-connected graphs of order n and of independent number α.

We use $\varphi(k,\alpha,n)$ (and $\phi(k,\alpha,n)$, respectively) to denote the least integer such that for every graph $G \in \mathcal{G}(k,\alpha,n)$, every longest cycle of G contains all vertices of degree at least $\varphi(k,\alpha,n)$ (there exists a longest cycle of G containing all vertices of degree at least $\phi(k,\alpha,n)$).

Similarly as the definitions for $\varphi(k,n)$ and $\phi(k,n)$, we should take the triple (k,α,n) such that there exist some non-Hamiltonian graphs in $\mathcal{G}(k,\alpha,n)$. This required $\alpha \geq k+1$ by Chvátal-Erdős' theorem [4], and $\alpha \leq n-k$ since every k-connected graph of order n has independent number at most $n-k$ (every independent set excludes all the neighbors of some vertex). On the other hand, if $k+1 \leq \alpha \leq n-k$, then there indeed exists some non-Hamiltonian graphs in $\mathcal{G}(k,\alpha,n)$ (e.g., $K_k \vee (K_{n-k-\alpha+1} \cup (\alpha-1)K_1)$). So we assume that $k+1 \leq \alpha \leq n-k$. By the definition of $\phi(k,n)$, we have

$$\phi(k,n) = \max\{\phi(k,\alpha,n) : k+1 \leq \alpha \leq n-k\},$$

and it is easy to see that

$$\phi(k,\alpha,n) \leq \varphi(k,\alpha,n), \text{ for } k+1 \leq \alpha \leq n-k.$$

In [9], we proved that

$$\varphi(k,k+1,n) = \left\lfloor \frac{n+1}{k+1} \right\rfloor + k - 1, \text{ for } n \geq 2k+1. \tag{2'}$$

This implies every graph $G \in \mathcal{G}(k,\alpha,n)$ with $n \geq 2k+1$ has a longest cycle containing all vertices of degree at least

$$d = \left\lfloor \frac{n+1}{k+1} \right\rfloor + k - 1.$$

On the other hand, every longest cycle of the graph

$$L(k,k+1,n) = kK_1 \vee (rK_{q+1} \cup (k+1-r)K_q),$$

where $n-k = q(k+1)+r,\ 0 \leq r \leq k$, excludes some vertices of degree

$$q+k-1 = \left\lfloor \frac{n-k}{k+1} \right\rfloor + k - 1 = \left\lfloor \frac{n+1}{k+1} \right\rfloor + k - 2.$$

Thus we conclude that

$$\phi(k,k+1,n) = \left\lfloor \frac{n+1}{k+1} \right\rfloor + k - 1, \text{ for } n \geq 2k+1. \tag{2}$$

Note that we determined all the values $\phi(k,\alpha,n)$ for $\alpha = k+1$. However, we do not know the exactly values $\phi(k,\alpha,n)$ for general case $\alpha \geq k+2$. In the following, we will determine all the values $\phi(k,\alpha,n)$ for $k=2$.

If $\alpha = 3$, then the value $\phi(2, 3, n)$ was determined in (2). So we assume that $\alpha \geq 4$. Also recall that by our assumption $n \geq \alpha + 2$. We first deal with the two basic cases $n = \alpha + 2$ and $n = \alpha + 3$.

From [9], we know that

$$\varphi(2, \alpha, \alpha + 2) = 3;\ \varphi(2, \alpha, \alpha + 3) = 4,\ \text{for } \alpha \geq 4. \tag{3$'$}$$

On the other hand, the graphs

$$L(2, \alpha, \alpha + 2) = K_{2,\alpha} \text{ and } L(2, \alpha, \alpha + 3) = K_{3,\alpha}$$

has no longest cycle containing all vertices of degree at least 2 and 3, respectively. This implies that

$$\phi(2, \alpha, \alpha + 2) = 3;\ \phi(2, \alpha, \alpha + 3) = 4,\ \text{for } \alpha \geq 4. \tag{3}$$

Next we deal with the case $n = \alpha + 4$. We will prove the following result.

Theorem 1.3. *Let G be a 2-connected graph of independent number $\alpha \geq 7$ and of order $n = \alpha + 4$. Then G has a longest cycle containing all vertices of degree at least*

$$d = \left\lceil \frac{\alpha}{3} \right\rceil + 2.$$

For $\alpha \geq 7$, let q, r be defined by

$$\alpha - 1 = 3q + r,\ 0 \leq r < 3.$$

Let $T = rK_{1,q+1} \cup (3 - r)K_{1,q}$ be the union of three stars, X be the set of the centers of the three stars, and s, s' be two vertices not in T. Let $L(2, \alpha, \alpha + 4)$ be the graph with vertex set $\{s, s'\} \cup V(T)$ and edge set

$$E(T) \cup \{st, s'x : t \in V(T), x \in X\}.$$

For an example, see Fig. 2. Note that the graph $L(2, \alpha, \alpha + 4) \in \mathcal{G}(2, \alpha, \alpha + 4)$ has no longest cycles containing all vertices of degree at least $q + 2 = \lfloor(\alpha - 1)/3\rfloor + 2 = \lceil\alpha/3\rceil + 1$. By Theorem 1.3, we have

$$\phi(2, \alpha, \alpha + 4) = \left\lceil \frac{\alpha}{3} \right\rceil + 2,\ \text{for } \alpha \geq 7, \tag{4}$$

whereas we had from [9] that

$$\varphi(2, \alpha, \alpha + 4) = \alpha,\ \text{for } \alpha \geq 4. \tag{4$'$}$$

The remaining cases of $n = \alpha + 4$ and $4 \leq \alpha \leq 6$ will be dealt with later. Now we consider the case $n \geq \alpha + 5$. We have the following result.

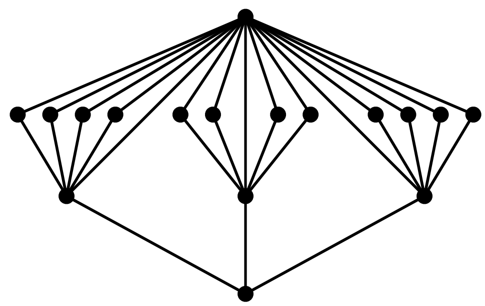

Figure 2. Graph $L(2,13,17)$.

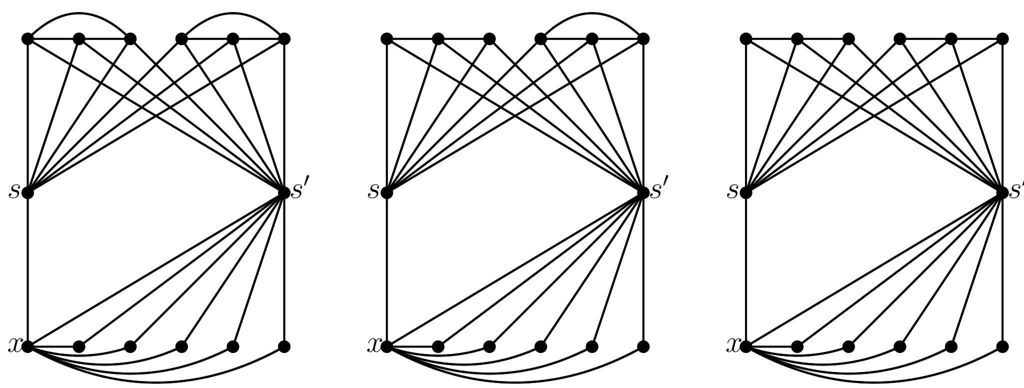

Figure 3. Graphs $L(2,7,14)$, $L(2,8,14)$ and $L(2,9,14)$.

Theorem 1.4. *Let G be a 2-connected graph of independent number $\alpha \geq 4$ and of order $n \geq 12$. Then G has a longest cycle containing all vertices of degree at least*

$$d = \begin{cases} q(\alpha-2)+3, & 0 \leq r \leq 2; \\ q(\alpha-2)+r+1, & 3 \leq r < \alpha, \end{cases} \quad \textit{where } n-7 = q\alpha + r,\ 0 \leq r < \alpha.$$

When $q \geq 1$ (i.e., when $n \geq \alpha + 7$), the bound on d in Theorem 1.4 is sharp. We can construct a graph as follows: If $0 \leq r \leq 2$, then let $R = rK_{q+3} \cup (2-r)K_{q+2}$ and $T = (\alpha-2)K_q$; if $3 \leq r < \alpha$, then let $R = 2K_{q+3}$ and $T = (r-2)K_{q+1} \cup (\alpha-r)K_q$. Let s', s, x be three vertices not in $R \cup T$. Let $L(2,\alpha,n)$ be the graph with vertex set $\{s', s, x\} \cup V(R) \cup V(T)$ and edge set

$$E(R) \cup E(T) \cup \{s'r, sr, s'x, sx, st, xt : r \in V(R), t \in V(T)\}.$$

For an example, see Fig. 3.

One can check that $L(2,\alpha,n) \in \mathcal{G}(2,\alpha,n)$, but every longest cycle of $L(2,\alpha,n)$ excludes the vertex x of degree $d-1$.

Note that when $q=0$, the above graph has independent number less than α. However, for the case $n=\alpha+5$ or $n=\alpha+6$, the bound on d in Theorem 1.4 is also sharp. We can construct two extremal graphs as follows: If $n=\alpha+5$, then let $R=2P_3$; if $n=\alpha+6$, then let $R=P_3\cup K_3$. Let $T=(n-9)K_1$, and let s',s,x be three vertices not in $R\cup T$. Let $L(2,\alpha,n)$ be a graph with vertex set $\{s',s,x\}\cup V(R)\cup V(T)$ and edge set

$$E(R)\cup\{s'r, sr, s'x, sx, st, xt : r\in V(R), t\in V(T)\}.$$

Clearly $L(2,\alpha,n)\in\mathcal{G}(2,\alpha,n)$, and every longest cycle of $L(2,\alpha,n)$ excludes the vertex x of degree $d-1$.

Now we have a formula, for $\alpha\geq 4$ and $n\geq\max\{\alpha+5,12\}$,

$$\phi(2,\alpha,n)=\begin{cases} q(\alpha-2)+3, & 0\leq r\leq 2;\\ q(\alpha-2)+r+1, & 3\leq r<\alpha,\end{cases} \quad \text{where } n-7=q\alpha+r,\ 0\leq r<\alpha. \tag{5}$$

The readers can compare the above formula with the one in [9], where we have, for $\alpha\geq 4$ and $n\geq\alpha+5$,

$$\varphi(2,\alpha,n)=\begin{cases} q(\alpha-2)+3, & 0\leq r\leq 2;\\ q(\alpha-2)+r+1, & 3\leq r<\alpha,\end{cases} \quad \text{where } n-5=q\alpha+r,\ 0\leq r<\alpha. \tag{5$'$}$$

Now we have the remaining petty cases excluding in the formulas (3)(4) and (5): $\alpha=4$, $8\leq n\leq 11$; $\alpha=5$, $9\leq n\leq 11$; and $\alpha=6$, $10\leq n\leq 11$. We list the exactly values $\phi(2,\alpha,n)$ for these cases in Table 1. For each case, we also give an extremal graph in $\mathcal{G}(2,\alpha,n)$ which has no longest cycles containing all vertices of degree at least $\phi(2,\alpha,n)-1$.

n	$\alpha=4$	$\alpha=5$	$\alpha=6$
8	4, $K_3\vee(3K_1\cup K_2)$		
9	4, $K_3\vee(2K_1\cup 2K_2)$	5, $K_4\vee 5K_1$	
10	4, $K_3\vee(K_1\cup 3K_2)$	5, $K_4\vee(4K_1\cup K_2)$	5, $K_4\vee 6K_1$
11	5, $K_3\vee 4K_2$	5, $K_4\vee(3K_1\cup 2K_2)$	6, $K_5\vee 6K_1$

Table 1: The values $\phi(2,\alpha,n)$ and extremal graphs $L(2,\alpha,n)$ for petty cases.

It is sufficient to show the up bound for these cases. For the cases $\alpha=4$, $n=8$; $\alpha=5$, $9\leq n\leq 10$; and $\alpha=6$, $10\leq n\leq 11$, we are done from Theorem 1.1. For the case $\alpha=4$, $9\leq n\leq 11$ and $\alpha=5$, $n=11$, we can deduced from the following theorem.

Theorem 1.5. *Let $d = 4$ or 5. Let G be a graph of order at most $d+6$ and of independent number at most d. Then G has a longest cycle containing all vertices of degree at least d.*

On above we gave the values of $\phi(k,\alpha,n)$ when $\alpha = k+1$ or $k = 2$. For the case $k \geq 3$ and $\alpha \geq k+2$, we have the following conjecture.

Conjecture 1. *Let G be a k-connected graph, $k \geq 3$, of independent number $\alpha \geq k+2$ and of order $n \geq \max\{2\alpha+1, \alpha+4k+1\}$. Then G has a longest cycle containing all vertices of degree at least*

$$d = \begin{cases} q(\alpha-k)+k+1, & 0 \leq r \leq k; \\ q(\alpha-k)+k+2, & k+1 \leq r \leq 2k+1; \\ q(\alpha-k)+r-k+1, & 2k+2 \leq r < \alpha+k, \end{cases}$$

where

$$n-3k-1 = q(\alpha+k)+r,\ 0 \leq r < \alpha+k.$$

We remark here that if the above conjecture is true, then the bound on d is sharp. To show this, we construct a graph as follows: If $0 \leq r \leq k$, then let $R = rK_{2q+3} \cup (k-r)K_{2q+2}$ and $T = (\alpha-k)K_q$; if $k+1 \leq r \leq 2k+1$, then let $R = (r-k-1)K_{2q+4} \cup (2k+1-r)K_{2q+3}$ and $T = K_{q+1} \cup (\alpha-k-1)K_q$; if $2k+2 \leq r < \alpha+k$, then let $R = kK_{2q+4}$ and $T = (r-2k)K_{q+1} \cup (\alpha+k-r)K_q$, and let $S = kK_1$. Let x be a vertex not in $R \cup S \cup T$. Let $L(k,\alpha,n)$ be the graph with vertex set $\{x\} \cup V(R) \cup V(S) \cup V(T)$ and edge set

$$E(R) \cup E(T) \cup \{sr, sx, st, xt : r \in V(R), s \in V(S), t \in V(T)\}.$$

One can check that $L(k,\alpha,n) \in \mathcal{G}(k,\alpha,n)$, but every longest cycle of G excludes the vertex x of degree $d-1$.

2. Preliminaries

We will first give some terminology and notation.

Let G be a graph. For a vertex $v \in V(G)$ and a subgraph H of G, we use $N_H(v)$ to denote the set, and $d_H(v)$ the number, of neighbors of v in H. We call $N_H(v)$ the *neighborhood* of v in H and $d_H(v)$ the *degree* of v in H. We use $d_H(u,v)$ to denote the distance between two vertices $u, v \in V(H)$ in H. For two subgraphs H and L of a graph G, we set $N_L(H) = \bigcup_{v\in V(H)} N_L(v)$. When no confusion occurs, we will denote $N_G(v)$ and $d_G(v)$ by $N(v)$ and $d(v)$, respectively. We set $N[x] = N(x) \cup \{x\}$.

Let G be a graph and $x, y \in V(G)$. An x-path is a path with x as one of its end vertices; an (x,y)-path is one connecting x and y. If Y is a subset of $V(G)$, then an (x,Y)-path is one connecting x and a vertex in Y with all internal vertices in $V(G)\backslash Y$; a Y-path is one connecting two vertices in Y with all internal vertices in $V(G)\backslash Y$. For a subgraph H of G, we use the notations (x, H)-path and H-path instead of $(x, V(H))$-path and $V(H)$-path, respectively. It is convenient to denote a path P with end-vertices x, y by $P(x, y)$.

For a cycle C with a given orientation and a vertex x on C, we use x^+ to denote the successor, and x^- the predecessor, of x on C. In the following, we always assume that C has an orientation, $\overrightarrow{C}$. For two vertices x, y on C, $\overrightarrow{C}[x, y]$ or $\overleftarrow{C}[y, x]$ denotes the path from x to y along $\overrightarrow{C}$. Similarly, if x, y are two vertices in a path P, $P[x, y]$ denotes the subpath of P between x and y. For an arbitrary path P or cycle C, we use $l(P)$ or $l(C)$ to denote its length.

The following Lemmas 2.1-2.3 is easy. One can find the proofs in [9].

Lemma 2.1. *Let C be a longest cycle of a graph G, and $P = P(u, v)$ be a C-path. Then $l(\overrightarrow{C}[u, v]) \geq l(P)$.*

Lemma 2.2. *Let C be a longest cycle of a graph G, H be a component of $G - C$ and $P = P(u, v)$ be a C-path of length at least 2 with all internal vertices in H. Then*

$$l(\overrightarrow{C}[u, v]) \geq l(P) + 2|N_C(H) \cap V(\overrightarrow{C}[u^+, v^-])|.$$

Lemma 2.3. *Let G be a graph, C be a longest cycle of G and H be a component of $G - C$.*
(1) If $u \in N_C(H)$, then $u^+, u^- \notin N_C(H)$.
(2) If $u, v \in N_C(H)$, then $u^+v^+, u^-v^- \notin E(G)$.

Let G be a graph, M be a subset of $V(G)$ or a subgraph of G, H be a component of $G - M$ and $x \in V(H)$. If there are two (x, M)-paths such that they have the only vertex x in common, then we say x is *locally 2-connected* to M. If every vertex in H is locally 2-connected to M, then we say that H is *locally 2-connected* to M. Note that if G is 2-connected and M has at least 2 vertices, then every component of $G - M$ is locally 2-connected to M. We will also use the following lemmas from [2] and [10].

Lemma 2.4 (Bondy and Chvátal [2])**.** *Let G be a graph of order n and let $P = P(u, v)$ be a path of G. If $d(u) + d(v) \geq n$, then G has a cycle containing all vertices of P.*

Lemma 2.5 (Li and Zhang [10])**.** *Let $r \geq 2$ be an integer. Let G be a connected graph, $M \subset V(G)$, H be a component of $G - M$, $x \in V(H)$ which is not a cut vertex of G. If H contains no induced copies of $K_{1,r}$ with the center x, then there is an (x, M)-path P such that*

$$|V(P) \cap N_H(x)| \geq \frac{1}{r-1} d_H(x).$$

Lemma 2.6 (Li and Zhang [10]). *Let $r \geq 4$ be an integer. Let G be a graph, $M \subset V(G)$, H be a component of $G - M$ which is locally 2-connected to M, $x \in V(H)$. If G contains no induced copies of $K_{1,r}$ with the center x, then there is an M-path P passing through x such that*

$$|V(P) \cap N_H(x)| \geq \frac{1}{r-2} d_H(x).$$

3. Proofs of main results

In this section, we shall present the proofs of our main results. Through out the proofs, for convenient, we call a vertex *hefty* if it has degree at least d.

Proof of Theorem 1.1.

Let C be a longest cycle of G containing hefty vertices as many as possible. We assume on the contrary that there is a hefty vertex x in $G - C$. Thus $d(x) \geq d = \max\{\lceil n/2 \rceil, n - 4k + 2\}$.

An (x, C)-*fan* is a collection of (x, C)-paths such that they have the only vertex x in common. Since G is k-connected, there is an (x, C)-fan with $s \geq k$ paths $P_i = P_i(x, z_i)$, $i \in \{1, 2, \dots, s\}$. We choose the (x, C)-fan such that s is as large as possible. We suppose that $z_1, z_2, \dots, z_s$ appear in this order along $\overrightarrow{C}$. Thus

$$l(C) = \sum_{i=1}^{s} l(\overrightarrow{C}[z_i, z_{i+1}]), \tag{6}$$

where the subscripts are taken modulo s.

By Menger's theorem, there is a vertex $y_i \in V(P_i - x)$ such that $S = \{y_i : 1 \leq i \leq s\}$ is a vertex-cut of G separating x and $C - S$. We choose y_i in such a way that $d_{P_i}(x, y_i)$ is as small as possible (note that y_i is possibly equal to z_i). Clearly

$$N_C(x) \subseteq S. \tag{7}$$

Let H be the component of $G - S$ containing x. Then

Claim 1. *For every vertex $y_i \in S$, either $N_H(y_i) = \{x\}$ or $|N_H(y_i)| \geq 2$.*

Proof. Suppose to the contrary that $|N_H(y_i)| = 1$ and $y_i' \neq x$ is the vertex in $N_H(y_i)$. Then y_i' is the neighbor of y_i on $P_i[x, y_i]$. Let $S' = (S \backslash \{y_i\}) \cup \{y_i'\}$. Then S' is a vertex-cut of G separating x and $C - S'$ such that $d_{P_i}(x, y_i') < d_{P_i}(x, y_i)$, contradicting the choice of S. □

If H has only one vertex x, then $d(x) = |S| = s$. By Lemma 2.1, $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 2$ for all $i \in \{1, 2, \dots, s\}$. By (6), $l(C) \geq 2s = 2d(x)$ and

$$n \geq l(C) + 1 \geq 2d(x) + 1 \geq n + 1,$$

a contradiction.

If H has exactly two vertices, then let x' be the vertex in $V(H)\backslash\{x\}$. By Claim 1, every vertex y_i in S is adjacent to x. Hence $s = d(x) - 1$. Note that $d(x') = d_S(x') + 1$ and $d(x') \geq k$ since G is k-connected. We have $d_S(x') \geq k - 1$. By Lemma 2.1, $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 2$ for all i. Moreover, if $x'y_i \in E(G)$, then $P = P_i[z_i, y_i]y_i x' x y_{i+1} P_{i+1}[y_{i+1}, z_{i+1}]$ is a C-path of length at least 3, by Lemma 2.1, $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 3$. Thus

$$\begin{aligned} l(C) &= \sum_{i=1}^{s} l(\overrightarrow{C}[z_i, z_{i+1}]) \\ &\geq 3(k-1) + 2(s-k+1) = 2s + k - 1 \\ &\geq 2d(x) + k - 3 \geq n + k - 3, \end{aligned}$$

and $n \geq l(C) + 2 \geq n + k - 1 \geq n + 1$, a contradiction.

Now it remains to consider the case when H has at least three vertices.

Claim 2. *For every $i \in [1, s]$,*
(a) $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 3$;
(b) if $(N_H(y_i) \cup N_H(y_{i+1}))\backslash\{x\} \neq \emptyset$, then $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 4$;
(c) if there is C-path $P = P(z_i, z_{i+1})$ including x and $l(P) \geq 4$, then $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 5$;
(d) if H has an (x, x')-path P' for some vertex $x' \in N_H(y_i) \cup N_H(y_{i+1})$ and $l(P') \geq 2$, then $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 5$.

Proof. (a) By Lemma 2.1, we have $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 2$ for any i. Suppose now that there exists an integer i such that $l(\overrightarrow{C}[z_i, z_{i+1}]) = 2$, i.e., $z_i^+ = z_{i+1}^-$. If z_i^+ is hefty, then $P' = P_i z_i \overleftarrow{C}[z_i, z_i^+]$ is an (x, z_i^+)-path, and $d(x) + d(z_i^+) \geq n$. By Lemma 2.4, there is a cycle containing all vertices of P', which is longer than C, a contradiction. If z_i^+ is not hefty, then $C' = \overrightarrow{C}[z_{i+1}, z_i]z_i P_i x P_{i+1} z_{i+1}$ is either a cycle longer than C or a longest cycle containing hefty vertices more than C, also a contradiction.

(b) By (a), $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 3$. Suppose now that $l(\overrightarrow{C}[z_i, z_{i+1}]) = 3$. If z_i^+ is hefty, then $P' = P_i z_i \overleftarrow{C}[z_i, z_i^+]$ is an (x, z_i^+)-path, and $d(x) + d(z_i^+) \geq n$. By Lemma 2.4, there is a cycle containing all vertices of P', which is longer than C, a contradiction. This implies that z_i^+, and similarly, z_{i+1}^-, is not hefty. If one of P_i and P_{i+1} has length at least 2, then $C' = \overrightarrow{C}[z_{i+1}, z_i]z_i P_i x P_{i+1} z_{i+1}$ is either a cycle longer than C or a longest cycle containing hefty vertices more than C, a contradiction. Thus we conclude that $l(P_i) = l(P_{i+1}) = 1$, implying that $y_i = z_i$, $y_{i+1} = z_{i+1}$ and $xz_i, xz_{i+1} \in E(G)$. We assume without loss of generality that z_i has a neighbor x' in H other than x. Let P be an (x, x')-path in H. Then $C' = \overrightarrow{C}[z_{i+1}, z_i]z_i x' P x z_{i+1}$ is either a cycle longer than C or a longest cycle containing hefty vertices more than C, a contradiction.

(c) By Lemma 2.1, $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 4$. Suppose now that $l(\overrightarrow{C}[z_i, z_{i+1}]) = 4$. Similarly as (b), we can prove that z_i^+ and z_{i+1}^- are not hefty. If z_i^{++} is hefty, then $P' = P_i z_i \overleftarrow{C}[z_i, z_i^{++}]$ is an (x, z_i^{++})-path, and $d(x) + d(z_i^{++}) \geq n$. By Lemma 2.4, there is a cycle containing all vertices of P', which is either a cycle longer than C or a longest cycle containing hefty vertices more than C, a contradiction. Thus we conclude that z_i^{++} is not hefty. Now $C' = \overrightarrow{C}[z_{i+1}, z_i] z_i P z_{i+1}$ is either a cycle longer than C or a longest cycle containing hefty vertices more than C, a contradiction.

(d) By (c), it suffices to prove that there is a C-path P between z_i and z_{i+1} passing through x and $l(P) \geq 4$. If $l(P_i) + l(P_{i+1}) \geq 4$, then $P = P_i x P_{i+1}$ is a required path. So we assume that $l(P_i) + l(P_{i+1}) \leq 3$. We assume without loss of generality that $x' \in N_H(y_i)$. If $xy_{i+1} \in E(G)$, then $P = P_i[z_i, y_i] y_i x' P' x y_{i+1} P_{i+1}[y_{i+1}, z_{i+1}]$ is a required path. Now we have that $xy_{i+1} \notin E(G)$. This implies that $l(P_{i+1}) = 2$ and $l(P_i) = 1$. Hence $y_i = z_i$, $y_{i+1} = z_{i+1}$ and $xy_i \in E(G)$. Let w be the neighbor of x on P'. By Claim 1, y_{i+1} has a neighbor, say w', in $H \backslash \{w\}$. Let P'' be an (w', P')-path of H, and let w'' be the end-vertex of P'' on P'. If $w'' = x$, then $P = y_i P' x P'' w' y_{i+1}$ is a required path. If $w'' \neq x$, then $P = y_i x P'[x, w''] w'' P'' w' y_{i+1}$ is a required path (note that if P'' is trivial, then $w'' = w' \neq w$ and $l(P'[x, w'']) \geq 2$). Thus we conclude that there is a C-path P between z_i and z_{i+1} passing through x such that $l(P) \geq 4$. Hence the assertion holds. $\square$

By $b(x)$ we denote the number of vertices in $V(G) \backslash N[x]$. Then $b(x) = n - 1 - d(x) \leq 4k - 3$. Hence, by (7),

$$l(C) \leq s + b(x) \leq s + 4k - 3. \tag{8}$$

Claim 3. *Every vertex in $V(H) \backslash \{x\}$ is not a cut-vertex of H.*

Proof. Suppose otherwise that $x' \neq x$ is a cut-vertex of H. Let H_1 and H_2 be two components of $H - x'$ such that $x \in V(H_1)$.

Note that for every i with $N_{H_2}(y_i) \neq \emptyset$, H has an (x, x'')-path of length at least 2 for some $x'' \in N_{H_2}(y_i)$. By Claim 2, $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 5$ for those i such that $N_{H_2}(y_i) \neq \emptyset$. Since G is k-connected, $|N_S(H_2)| \geq k - 1$. This implies that there are at least $k - 1$ segments $\overrightarrow{C}[z_i, z_{i+1}]$ with length at least 5. Also note that every segment $\overrightarrow{C}[z_i, z_{i+1}]$ has length at least 3 by Claim 2. Hence we have

$$l(C) \geq 5(k-1) + 3(s - k + 1) = 3s + 2k - 2 \geq s + 4k - 2,$$

contradicting (8). $\square$

By Claim 3, for any two vertices $u, v \in V(H)$, H has a (u, v)-path passing through x; and if $vx \in E(H)$ is a cut-edge of H, then vx is a pendant edge of H.

Claim 4. *H is a star with center x.*

Proof. Suppose, otherwise, there is an end-block B of H which has at least three vertices (possibly B is H itself). For every vertex $v \in V(B)\backslash\{x\}$, H has an (x,v)-path of length at least 2. Thus by Claim 2, $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 5$ for those i such that $y_i \in N_S(B-x)$. Thus we conclude that there are at least $k-1$ segments $\overrightarrow{C}[z_i, z_{i+1}]$ of length at least 5. Hence

$$l(C) \geq 5(k-1) + 3(s-k+1) = 3s + 2k - 2 \geq s + 4k - 2,$$

a contradiction. □

By Claim 4, $H = K_{1,n(H)-1}$. Let $S_0 = \{y_i \in S : N_H(y_i) = \{x\}\}$, $S_1 = \{y_i \in S : |N_H(y_i)\backslash\{x\}| = 1\}$ and $S_2 = S\backslash(S_0 \cup S_1)$. Let $s_i = |S_i|$, $i = 0, 1, 2$. Thus $s = s_0 + s_1 + s_2$.

Let y_{j_i}, $1 \leq j_1 < j_2 < \cdots < j_{s_1+s_2} \leq s$, be the vertices in $S_1 \cup S_2$. Since G is k-connected,

$$s_1 + s_2 \geq |N_S(x')| \geq k - 1 \tag{10}$$

for any $x' \in V(H)\backslash\{x\}$, and

$$s_1 + (n(H)-1)s_2 \geq |E(H-x, S)| \geq (k-1)(n(H)-1). \tag{11}$$

If $s_1 + s_2 = 1$, then without loss of generality we assume that $y_1 \in S_1 \cup S_2$. Note that $\{x, y_1\}$ is a vertex cut of G, implying that $k = 2$. By Claim 2, $l(\overrightarrow{C}[z_1, z_2]) \geq 4$ and $l(\overrightarrow{C}[z_1, z_s]) \geq 4$. Thus

$$l(C) \geq 4 + 4 + 2(s-2) = 2s + 4 \geq s + 4k - 2,$$

a contradiction. Now we conclude that $s_1 + s_2 \geq 2$.

Claim 5. *For every vertex $y_{j_i} \in S_1 \cup S_2$,*

$$l(\overrightarrow{C}[z_{j_i}, z_{j_{i+1}}]) \geq \begin{cases} 4 + 3|N_C(x) \cap V(\overrightarrow{C}[z_{j_i}^+, z_{j_{i+1}}^-])|; & y_{j_i} \in S_1; \\ 5 + 3|N_C(x) \cap V(\overrightarrow{C}[z_{j_i}^+, z_{j_{i+1}}^-])|; & y_{j_i} \in S_2, \end{cases}$$

where the subsubscripts are taken modulo $s_1 + s_2$.

Proof. Suppose first that $N_C(x) \cap V(\overrightarrow{C}[z_{j_i}^+, z_{j_{i+1}}^-]) \neq \emptyset$, i.e., $j_{i+1} \geq j_i + 2$. By Claim 2, we have that $l(\overrightarrow{C}[z_{j_i}, z_{j_i+1}]) \geq 4$, $l(\overrightarrow{C}[z_{j_{i+1}-1}, z_{j_{i+1}}]) \geq 4$ and for every $j \in [j_i+1, j_{i+1}-2]$, $l(\overrightarrow{C}[z_j, z_{j+1}]) \geq 3$. Thus we have the assertion.

Now we assume that $N_C(x) \cap V(\overrightarrow{C}[z_{j_i}^+, z_{j_{i+1}}^-]) = \emptyset$. For any $y_{j_i} \in S_1 \cup S_2$, we let w_{j_i} be a vertex in $N_H(y_{j_i})\backslash\{x\}$.

If $y_{j_i} \in S_1$, then the assertion can be deduced by Claim 2. If $y_{j_i} \in S_2$, then let w'_{j_i} be a vertex in $N_H(y_{j_i})\backslash\{x, w_{j_{i+1}}\}$. Thus $P = P_{j_i}[z_{j_i}, y_{j_i}]y_{j_i}w'_{j_i}xw_{j_{i+1}}y_{j_{i+1}}P_{j_{i+1}}[y_{j_{i+1}}, z_{j_{i+1}}]$ is a C-path of length at least 4 passing through x. By Claim 2, we have the assertion. □

Note that $\sum_{i=1}^{s_1+s_2}|N_C(x)\cap V(\overrightarrow{C}[z_{j_i}^+,z_{j_{i+1}}^-])|=s_0$. By Claim 5,

$$l(C)=\sum_{i=1}^{s_1+s_2} l(\overrightarrow{C}[z_{j_i},z_{j_{i+1}}])\geq 3s_0+4s_1+5s_2=3s+s_1+2s_2.$$

By (10) and (11), we have

$$\begin{aligned} l(C) &\geq 3s+s_1+2s_2\\ &=3s+\frac{n(H)-3}{n(H)-2}(s_1+s_2)+\frac{1}{n(H)-2}(s_1+(n(H)-1)s_2)\\ &\geq 3s+\frac{n(H)-3}{n(H)-2}(k-1)+\frac{n(H)-1}{n(H)-2}(k-1)\\ &=3s+2k-2\geq s+4k-2, \end{aligned}$$

a contradiction.

The proof is complete. □

Proof of Theorem 1.3.

Let C be a longest cycle of G containing hefty vertex as many as possible. We assume on the contrary that there is a hefty vertex x such that $x\in V(G-C)$. Let H be the component of $G-C$ containing x.

If H has only one vertex x, then $d_C(x)=d(x)\geq\lceil\alpha/3\rceil+2$. Note that we assume that $\alpha\geq 7$. We have $d_C(x)\geq 5$. By Lemma 2.1, $l(C)\geq 10$. Hence

$$\alpha(G)\leq\alpha(C)+n(G-C)=\left\lfloor\frac{l(C)}{2}\right\rfloor+n-l(C)=n-\left\lceil\frac{l(C)}{2}\right\rceil\leq n-5,$$

a contradiction. Thus we conclude that H has at least 2 vertices. Let I be a maximum independent set of G. Thus $|V(G)\backslash I|=4$.

Let x' be a vertex in $N_H(x)$. Since G is 2-connected, there is a C-path $P=P(u,v)$ passing through the edge xx'. We assume that u,x,x',v appear in this order along P. By Lemma 2.1, $l(\overrightarrow{C}[u,v])\geq 3$, $l(\overleftarrow{C}[u,v])\geq 3$, and hence $l(C)\geq 6$. Thus

$$\alpha(G)\leq\alpha(C)+\alpha(G-C)\leq\left\lfloor\frac{l(C)}{2}\right\rfloor+n-l(C)-1=n-\left\lceil\frac{l(C)}{2}\right\rceil-1\leq n-4.$$

This implies that $l(C)=6$, $l(\overrightarrow{C}[u,v])=l(\overleftarrow{C}[u,v])=3$, and $P=uxx'v$. Moreover, $V(G)\backslash I$ consists of one vertex in $\{x,x'\}$ and three pairwise nonadjacent vertices of C (but possibly adjacent in G).

Since every hefty vertex has degree at least 5, it cannot be in I (otherwise I excludes all the neighbors of it and has size at most $n-5$). This implies that $x\in V(G)\backslash I$. We denote the vertices of

C by $\overrightarrow{C} = uyy'vz'zu$. We claim that $V(G)\backslash I = \{x, y, z, v\}$. Otherwise $V(G)\backslash I = \{x, u, y', z'\}$, and $x', v \in I$ are adjacent in G, a contradiction.

Note that y' is not hefty. If y is not hefty either, then $C' = \overleftarrow{C}[u, v]vPu$ is a longest cycle of G containing more hefty vertices than C. This implies that y, and similarly, z, is hefty.

Claim 1. *$xy, xy', x'y, xz, xz', x'z, yz, yz', y'z \notin E(G)$, and any two vertices in $\{x, y, z\}$ have no common neighbors other than u, v.*

Proof. If $xy \in E(G)$, then $C' = uxyy'vz'zu$ is a cycle longer than C, a contradiction; if $xy' \in E(G)$, then $C' = uyy'xx'vz'zu$ is a cycle longer than C, a contradiction. By a similarly analysis, we can see that $xy, xy', x'y, xz, xz'x'z, yz, yz', y'z \notin E(G)$.

If x, y have a common neighbor $w \neq u, v$, then clearly $w \neq x', y', z'$, and $C' = uxwyy'vz'zu$ is a cycle longer than C, a contradiction. Other assertions can be proved similarly. □

Recall that x, y, z are all hefty. By Claim 1,

$$\begin{aligned} n &\geq |N(x)\backslash\{u, v\}| + |N(y)\backslash\{u, v\}| + |N(z)\backslash\{u, v\}| + |\{x, y, z, u, v\}| \\ &\geq 3(d-2) + 5 = 3 \cdot \left\lceil \frac{\alpha}{3} \right\rceil + 5 \geq \alpha + 5, \end{aligned}$$

a contradiction.

The proof is complete. □

Proof of Theorem 1.4.

Suppose firstly that $q = 0$. Then $r = n-7$. Since $n \geq 12$, we have $r \geq 5$ and $d = r+1 = n-6$. By Theorem 1.1, G has a longest cycle containing all hefty vertices. Now we assume that $q \geq 1$. Note that $d \geq q(\alpha - 2) + 3$, and $\alpha \geq 4$. We have

$$d \geq 5, \text{ and if } \alpha \geq 5, \text{ then } d \geq 6. \tag{12}$$

Also note that $2d \geq q(\alpha - 2) + 3 + q(\alpha - 2) + r + 1 \geq n - 3 + q(\alpha - 4)$, that is

$$2d \geq n - 3 \text{ and if } \alpha \geq 5, \text{ then } 2d \geq n - 2. \tag{13}$$

Let C be a longest cycle of G containing hefty vertices as many as possible. We assume on the contrary that there is a hefty vertex x in $G - C$. Let H be the component of $G - C$ containing x.

Set

$$b = n - 1 - d = \begin{cases} 2q + r + 3, & 0 \leq r \leq 2; \\ 2q + 5, & 3 \leq r < \alpha. \end{cases}$$

We use $b(x)$ to denote the number of vertices in $V(G)\backslash N[x]$. Then $b(x)=n-1-d(x)\leq b$,

$$b(x)\leq 2q+5. \tag{14}$$

Suppose first that H has only one vertex x. Recall that $q\geq 1$. Then $d\geq q(\alpha-2)+3\geq\alpha+1$. Note that $N_C^+(x)$ is an independent set of G. This implies that

$$\alpha\geq|N_C^+(x)|=d(x)\geq d\geq\alpha+1,$$

a contradiction. This implies that H has at least two vertices.

Since

$$d-\alpha=\begin{cases}(q-1)(\alpha-2)+1, & 0\leq r\leq 2;\\ (q-1)(\alpha-2)+r-1, & 3\leq r<\alpha.\end{cases}$$

We have $d-\alpha\geq(q-1)(\alpha-2)+1$ and

$$\left\lceil\frac{d-\alpha}{\alpha-2}\right\rceil\geq\left\lceil\frac{(q-1)(\alpha-2)+1}{\alpha-2}\right\rceil=q. \tag{15}$$

Claim 1. *G contains a C-path P passing through x such that*

$$|V(P)\cap N_H(x)|\geq\left\lceil\frac{d_H(x)}{\alpha(H)}\right\rceil.$$

Proof. We first claim that H contains no induced copies of star $K_{1,\alpha(H)+1}$. Otherwise the end-vertices of the star form an independent set of H of order $\alpha(H)+1$. If $d_C(x)\geq 2$, then by Lemma 2.5, there is an (x,C)-path P' such that

$$|V(P')\cap N_H(x)|\geq\left\lceil\frac{d_H(x)}{\alpha(H)}\right\rceil.$$

Let v be the end-vertex of P' on C, and let u be a neighbor of x on C other than v. Then $P=uxP'$ is a required path.

Now we assume that $d_C(x)\leq 1$. Then G does not contain induced copies of star $K_{1,\alpha(H)+2}$; otherwise, the end-vertices of the star contained in H will be an independent set of order at least $\alpha(H)+1$. By Lemma 2.6, there is a C-path P passing through x such that

$$|V(P)\cap N_H(x)|\geq\left\lceil\frac{d_H(x)}{\alpha(H)}\right\rceil,$$

and P is a required path. □

Let P be a C-path as in Claim 1 and let u, v be the two end-vertices of P on C. Then

$$|V(P)| \geq |V(P) \cap N_H(x)| + |\{u, v, x\}| \geq \left\lceil \frac{d_H(x)}{\alpha(H)} \right\rceil + 3.$$

Since $d_H(x) = d(x) - d_C(x) \geq d - |N_C(H)|$, and $\alpha(H) \leq \alpha(G) - |N_C^+(H)| = \alpha - |N_C(H)|$, the above equality implies

$$|V(P)| \geq \left\lceil \frac{d - |N_C(H)|}{\alpha - |N_C(H)|} \right\rceil + 3 = \left\lceil \frac{d - \alpha}{\alpha - |N_C(H)|} \right\rceil + 4 \geq \left\lceil \frac{d - \alpha}{\alpha - 2} \right\rceil + 4.$$

From (15), we have

$$l(P) = |V(P)| - 1 \geq q + 3. \tag{16}$$

If $l(\overrightarrow{C}[u, v]) \leq q + 2$, then $C' = \overrightarrow{C}[v, u]uPv$ is a cycle longer than C, a contradiction. This implies that $l(\overrightarrow{C}[u, v]) \geq q + 3$, and similarly, $l(\overleftarrow{C}[u, v]) \geq q + 3$. Thus $l(C) = l(\overrightarrow{C}[u, v]) + l(\overleftarrow{C}[u, v]) \geq 2q + 6$.

Case 1. $|N_C(H)| = 2$.

In this case, $N_C(H) = \{u, v\}$. If $l(C) \geq 2q+8$, then $b(x) \geq l(C)-2 \geq 2q+6$, a contradiction. This implies that $l(C) \leq 2q + 7$.

We claim that there are no paths from $\overrightarrow{C}[u^+, v^-]$ to $\overleftarrow{C}[u^-, v^+]$ internally disjoint with C. Suppose not. Let P' be such a path and let y, z be the two end-vertices of P', where $y \in V(\overrightarrow{C}[u^+, v^-])$ and $z \in V(\overleftarrow{C}[u^-, v^+])$. Let $C_1 = \overrightarrow{C}[y, v]vPu\overleftarrow{C}[u, z]zP'y$ and $C_2 = \overleftarrow{C}[y, u]uPv\overrightarrow{C}[v, z]zP'y$. Then

$$l(C_1) + l(C_2) - l(C) = 2(l(P) + l(P')) \geq 2q + 8.$$

This implies that either C_1 or C_2 is longer than C, a contradiction. Thus as we claimed there are no paths from $\overrightarrow{C}[u^+, v^-]$ to $\overleftarrow{C}[u^-, v^+]$ internally disjoint with C.

We claim that $|V(G-C-H)| \leq 1$. Otherwise $b(x) \geq |V(C)|-2+2 \geq 2q+6$, a contradiction. Thus as we claimed, there is at most one vertex in $V(G - C - H)$.

Claim 2. *One of the following holds:*
(a) $l(\overrightarrow{C}[u, v]) = q + 3$ *and* $\overrightarrow{C}[u^+, v^-]$ *has no neighbors in* $G - \overrightarrow{C}[u, v]$*;*
(b) $l(\overleftarrow{C}[u, v]) = q + 3$ *and* $\overleftarrow{C}[u^-, v^+]$ *has no neighbors in* $G - \overleftarrow{C}[u, v]$.

Proof. We first suppose that $V(G-C-H) = \emptyset$. Then $\overrightarrow{C}[u^+, v^-]$ has no neighbors in $G - \overrightarrow{C}[u, v]$ and $\overleftarrow{C}[u^+, v^-]$ has no neighbors in $G - \overleftarrow{C}[u, v]$. Note that $l(C) \leq 2q+7$ and $l(C) = l(\overrightarrow{C}[u, v]) + l(\overleftarrow{C}[u, v])$. Also note that $l(\overrightarrow{C}[u, v]) \geq q+3$ and $l(\overleftarrow{C}[u, v]) \geq q+3$. We have either $l(\overrightarrow{C}[u, v]) = q + 3$ or $l(\overleftarrow{C}[u, v]) = q + 3$.

Now we suppose that there is a vertex, say w, in $V(G-C-H)$. If $l(C)=2q+7$, then $b(x)\geq |V(C-\{u,v\})|+|\{w\}|\geq 2q+6$, a contradiction. This implies that $l(C)\leq 2q+6$ and $l(\overrightarrow{C}[u,v])=l(\overleftarrow{C}[u,v])=q+3$. Note that w has no neighbors in $\overrightarrow{C}[u^+,v^-]$ or has no neighbors in $\overleftarrow{C}[u^-,v^+]$; otherwise there will be a path from $\overrightarrow{C}[u^+,v^-]$ to $\overleftarrow{C}[u^-,v^+]$ internally disjoint with C. This implies that either $\overrightarrow{C}[u^+,v^-]$ has no neighbors in $G-\overrightarrow{C}[u,v]$ or $\overleftarrow{C}[u^-,v^+]$ has no neighbors in $G-\overleftarrow{C}[u,v]$. $\square$

By Claim 2, we assume without loss of generality that $|V(\overrightarrow{C}[u^+,v^-])|=q+2$ and $\overrightarrow{C}[u^+,v^-]$ has no neighbors in $G-\overrightarrow{C}[u,v]$. Let y be an arbitrary vertex in $\overrightarrow{C}[u^+,v^-]$. We claim that y is not hefty. If $q=1$, then

$$d(y)\leq |V(\overrightarrow{C}[u,v])|-1=q+3=4<d$$

(by (12)). So we assume that $q\geq 2$. Then $|V(\overleftarrow{C}[u^-,v^+])|\geq 4$, and

$$\begin{aligned} d(x)+d(y) &\leq |V(H-x)|+|\{u,v\}|+|V(\overrightarrow{C}[u,v]-y)| \\ &\leq |V(G)|-|V(\overleftarrow{C}[u^-,v^+])|\leq n-4. \end{aligned}$$

From (13), we also have $d(y)<d$. Thus as we claimed, every vertex in $\overrightarrow{C}[u^+,v^-]$ is not hefty.

Now $C'=\overrightarrow{C}[v,u]uPv$ is a longest cycle of G containing hefty vertices more than C, a contradiction.

Case 2. $|N_C(H)|\geq 3$.

By Lemma 2.2,

$$\begin{aligned} l(C) &= l(\overrightarrow{C}[u,v])+l(\overleftarrow{C}[u,v]) \\ &\geq 2l(P)+2(|N_C(H)|-2)\geq 2q+2+2|N_C(H)|, \end{aligned}$$

and

$$b(x)\geq l(C)-|N_C(H)|\geq 2q+2+|N_C(H)|\geq 2q+5.$$

Recall that $b(x)\leq 2q+5$. This implies that $|N_C(H)|=3$, $l(P)=q+3$ and $l(C)=2q+8$. Moreover, x is adjacent to all vertices in $(V(H)\cup N_C(H))\backslash\{x\}$, and $G-C$ has only one component H.

Let w be the unique vertex in $N_C(H)$ other than u,v. We assume without loss of generality that w is in $\overleftarrow{C}[u^-,v^+]$. Set

$$k_1=l(P[u,x]),\ \text{and}\ k_2=l(P[x,v]),\ \text{where}\ k_1+k_2=q+3.$$

Then $l(\overrightarrow{C}[u,v]) \geq l(P) = q+3$, $l(\overrightarrow{C}[v,w]) \geq l(P[v,x]xw) = k_2+1$ and $l(\overrightarrow{C}[w,u]) \geq l(wxP[x,u]) = k_1+1$. Thus

$$\begin{aligned} l(C) &= l(\overrightarrow{C}[u,v]) + l(\overrightarrow{C}[v,w]) + l(\overrightarrow{C}[w,u]) \\ &\geq q+k_1+k_2+5 \geq 2q+8. \end{aligned}$$

Note that $l(C) = 2q+8$, this implies that $l(\overrightarrow{C}[u,v]) = q+3$, $l(\overrightarrow{C}[v,w]) = k_2+1$ and $l(\overrightarrow{C}[w,u]) = k_1+1$.

Recall that $xw \in E(G)$. Now we claim that $N_H(w) = \{x\}$. Suppose on the contrary that w has a neighbor y in H other than x. If $y \notin V(P)$, then let P' be a path of H from y to $P - \{u,v\}$ and let z be the end-vertex of P' on P. Then

$$\begin{aligned} &l(P[v,z]zP'yw) + l(wyP'zP[z,u]) \geq l(P) + 2l(P'yw) \geq q+7, \text{ and} \\ &l(\overrightarrow{C}[v,w]) + l(\overrightarrow{C}[w,u]) \geq l(P[v,z]zP'yw) + l(wyP'zP[z,u]) \geq q+7, \end{aligned}$$

a contradiction. Now we assume that $y \in V(P)\backslash\{u,v,x\}$. We assume without loss of generality that $y \in V(P[u,x])\backslash\{u,x\}$. Then

$$l(\overrightarrow{C}[v,w]) \geq l(P[v,y]yw) \geq l(P[v,x]) + l(P[x,y]yw) \geq k_2+2,$$

also a contradiction. Thus as we claimed, $N_H(w) = \{x\}$.

By Lemma 2.3, $w \neq v^+, u^-$. We claim that $w \neq v^{++}$. Suppose that $w = v^{++}$, i.e., $w^- = v^+$. Then $k_2 = 1$ and $k_1 = q+2 \geq 3$. Thus $|V(\overrightarrow{C}[w^+,u^-])| \geq 2$. By Lemma 2.3, $v^+u^+, v^+v^-, v^+w^+, v^+u^- \notin E(G)$. Moreover, $v^+u^{++} \notin E(G)$; otherwise $C' = \overrightarrow{C}[u^{++},v]vPu\overleftarrow{C}[u,v^+]v^+u^{++}$ is a cycle longer than C (note that $l(P) \geq q+3 \geq 4$), a contradiction. Thus

$$\begin{aligned} d(x) + d(v^+) &\leq |V(H-x)| + |\{u,v,w\}| + |V(C - \{u^+,u^{++},v^-,v^+,w^+,u^-\})| \\ &\leq n-4. \end{aligned}$$

By (13), $d(v^+) < d$. Thus $C' = \overrightarrow{C}[w,v]vxw$ is a longest cycle of G containing hefty vertices more than C, a contradiction. Thus as we claimed, $w \neq v^{++}$, and similarly, $w \neq u^{--}$, i.e., $l(\overrightarrow{C}[v,w]) \geq 3$ and $l(\overrightarrow{C}[w,u]) \geq 3$. This implies that $k_1, k_2 \geq 2$, and u (and similarly, v) has a neighbor in H other than x.

If H is a clique, then let y be a neighbor of u in H other than x, P' be a Hamilton path of H from x to y. Then

$$l(\overrightarrow{C}[w,u]) \geq l(wxP'yu) \geq |V(H)| + 1 \geq l(P) = q+3,$$

contradicting to the fact that $l(\overrightarrow{C}[w,u]) = k_2 + 1$ (noting that $k_1 + k_2 = q + 3$). This implies that H is not a clique, i.e., $\alpha(H) \geq 2$.

Note that an independent set of H together with $\{u^+, v^+, w^+\}$ forms an independent set of G. Thus we conclude that $\alpha \geq 5$. From (12) and (13), $d \geq 6$ and $2d \geq n - 2$.

Claim 3. *Let y be an arbitrary vertex in $V(C - \{u,v,w\})$. If there are four vertices in $V(C)\backslash\{y\}$ that are nonadjacent to y, then y is not hefty.*

Proof. Note that all neighbors of y are in C, and $d(y) \leq |V(C)| - 5$.

$$d(x) + d(y) \leq |V(H - x)| + |\{u,v,w\}| + |V(C)| - 5 \leq n - 3.$$

By (13), $d(y) < d$, implying y is not hefty. □

Claim 4. *Let $y, z \in \{u,v,w\}$, $y \neq z$. Then $y^+z^+, y^+z^{++} \notin E(G)$.*

Proof. Note that for any choice of y and z, there is a (y,z)-path P' internally disjoint with C, such that $l(P') \geq 3$. If $y^+z^+ \in E(G)$, then let $C' = \overrightarrow{C}[y^+, z]zP'y\overleftarrow{C}[y, z^+]z^+y^+$; if $y^+z^{++} \in E(G)$, then let $C' = \overrightarrow{C}[y^+, z]zP'y\overleftarrow{C}[y, z^{++}]z^{++}y^+$. Then C' is a cycle longer than C, a contradiction. □

From Claims 3 and 4, we can see that u^+, v^+, w^+ are not hefty.

We claim that every vertex in $\overrightarrow{C}[v^+, w^-]$ is not hefty. Suppose not. Let y be the first hefty vertex in $\overrightarrow{C}[v^+, w^-]$. Then $y \neq v^+$. Let

$$C' = \begin{cases} \overrightarrow{C}[y,u]uPv\overleftarrow{C}[v,u^+]u^+y, & \text{if } yu^+ \in E(G); \\ \overrightarrow{C}[y,u]uPv\overleftarrow{C}[v,u^{++}]u^{++}y, & \text{if } yu^{++} \in E(G); \\ \overrightarrow{C}[y,w]wxP[x,v]v\overleftarrow{C}[v,w^+]w^+y, & \text{if } yw^+ \in E(G); \\ \overrightarrow{C}[y,w]wxP[x,v]v\overleftarrow{C}[v,w^{++}]w^{++}y, & \text{if } yw^{++} \in E(G). \end{cases}$$

Then C' is either a cycle longer than C or a longest cycle of G containing hefty vertices more that C. In any case, we get a contradiction with the choice of C. This implies that any of $\{yu^+, yu^{++}, yw^+, yw^{++}\}$ is not in $E(G)$. By Claim 3, y is not hefty. Thus as we claimed, every vertex in $\overrightarrow{C}[v^+, w^-]$ is not hefty.

Now $C' = \overrightarrow{C}[w,v]vP[v,x]xw$ is a longest cycle of G containing hefty vertices more than C, a contradiction.

The proof is complete. □

Proof of Theorem 1.5.

Let C be a longest cycle of G containing hefty vertices as more as possible. We assume on the contrary that there is a vertex $x \in V(G-C)$ with $d(x) \geq d$. Let H be the component of $G-C$ containing x.

Note that $N_C^+(H) \cup \{x\}$ is an independent set of G, implying that $|N_C(H)| \leq d-1$. Since $N(x) \subseteq N_C(H) \cup (V(H)\backslash\{x\})$, x has a neighbor $x' \in V(H)$. Since G is 2-connected, G has a C-path $P = P(u,v)$ passing through the edge xx'. We assume that u, x, x', v are in this order along P.

First we assume that $|N_C(H)| = 4$ (when $d = 5$). By Lemma 2.2,

$$l(C) \geq 2l(P) + 2|N_C(H)\backslash\{u,v\}| \geq 10,$$

and $n(G) \geq l(C) + n(H) \geq 12$, a contradiction.

Second we assume that $|N_C(H)| = 3$. Since $d(x) \geq d$, we have $d_H(x) \geq d-3$. Let w be the third vertex in $N_C(H)$. We suppose without loss of generality that $w \in V(\overrightarrow{C}[u^+, v^-])$. By Lemmas 2.1 and 2.2, $l(\overrightarrow{C}[u,v]) \geq 5$ and $l(\overleftarrow{C}[u,v]) \geq 3$. Thus

$$n(G) \geq l(C) + d_H(x) + 1 \geq 8 + d - 3 + 1 = d + 6.$$

This implies that $V(G) = V(C) \cup V(H)$, $l(\overrightarrow{C}[u,v]) = 5$ and $l(\overleftarrow{C}[u,v]) = 3$ and $N(x) = (V(H) \cup N_C(H))\backslash\{x\}$. Moreover, we have by Lemma 2.1 that $l(\overrightarrow{C}[u,w]) = 2$ and $l(\overrightarrow{C}[w,v]) = 3$. Thus $C = uu^+ww^+v^-vv^+u^-u$. By Lemma 2.3, u^+ is nonadjacent to every vertex in $\{w^+, v^-, v^+, u^-\}$. This implies that $d(u^+) \leq 3$. Thus $C' = uxw\overrightarrow{C}[w,u]$ is a longest cycle containing vertices of degree at least 4 more than C, a contradiction.

Now we assume that $|N_C(H)| = 2$. Then $d_H(x) \geq d-2$. By Lemma 2.1, $l(\overrightarrow{C}[u,v]) \geq l(P) \geq 3$ and $l(\overleftarrow{C}[u,v]) \geq 3$. On the other hand, Since $n(G) \leq d+6$ and $n(H) \geq d-1$, we have $n(G-H) \leq 7$. Thus $l(C) = 6$ or 7.

If there is a C-path $P' = P'(y,z)$ (say) from $\overrightarrow{C}[u^+, v^-]$ to $\overleftarrow{C}[u^-, v^+]$, then we let $C_1 = \overrightarrow{C}[u,y]P'\overleftarrow{C}[z,v]P$ and $C_2 = \overleftarrow{C}[u,z]P'\overrightarrow{C}[y,v]P$. Note that

$$l(C_1) + l(C_2) - l(C) \geq 2l(P) + 2l(P') \geq 8.$$

This implies either C_1 or C_2 is a cycle longer than C, a contradiction. So we assume that there are no C-path from $\overrightarrow{C}[u^+, v^-]$ to $\overleftarrow{C}[u^-, v^+]$. Recall that $n(G-H) \leq 7$. It is not difficult to see that one of the following is true:

(a) $l(\overrightarrow{C}[u,v]) = 3$ and $\overrightarrow{C}[u^+, v^-]$ has no neighbors other than u and v; or

(b) $l(\overleftarrow{C}[u,v]) = 3$ and $\overleftarrow{C}[u^-, v^+]$ has no neighbors other than u and v.

Without loss of generality we assume that $l(\overrightarrow{C}[u,v]) = 3$ and $\overrightarrow{C}[u^+, v^-]$ has no neighbors other than u and v. Thus $d(u^+) \leq 3$ and $d(v^-) \leq 3$. Thus u^+, v^- are not hefty and $C' = P\overrightarrow{C}[v,u]$ is a longest cycle containing hefty vertices more than C, a contradiction.

The proof is complete. $\square$

4. Concluding remarks

In this paper, we show that there is a longest cycle passing through large degree vertices, while we show that every longest cycle passes through large degree vertices in the sequel of our previous work [9]. These works relate longest cycle and large degree vertices, which are popular in Hamiltonian literature. There would also be interesting to consider the same problem in some special graphs, for example bipartite graph and claw-free graphs and so on. It is wide open in this kind of problems. On the other hand, our results may be applied in the similar problem related longest cycle and large degree vertices.

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