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New measures of graph irregularity

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Abstract

In this paper, we define and compare three new measures of graph irregularity. We use these measures to tighten upper bounds for the chromatic number and the Colin de Verdière parameter. We also strengthen the concise Turán theorem for irregular graphs and investigate to what extent Turán's theorem can be similarly strengthened for generalized r-partite graphs. We conclude by relating these new measures to the Randić index and using the measures to devise new normalised indices of network heterogeneity.

Keywords: graph irregularity, clique, chromatic number, Randic index, network heterogeneity
Mathematics Subject Classification : 68R10

1. Introduction

Many results in extremal graph theory are exact only for some regular graphs. In this paper we strengthen various bounds, using two degree based measures of irregularity and a spectral measure of irregularity, so that they also become exact for some irregular graphs.

Let G be a simple and undirected graph with vertex set V with $|V| = n$, edge set E with $|E| = m$, t triangles, clique number ω, chromatic number χ and vertex degrees $\Delta = d_1 \geq d_2 \geq ... \geq d_n = \delta$. Let μ denote the largest eigenvalue of the adjacency matrix of G and let d denote the average degree.

Existing measures of irregularity include the following. Collatz and Sinogowitz [7] proposed a spectral measure, namely $\mu - d$. Bell [2] proposed a variance measure, namely $\mathrm{var}(G) = \sum(d_i - d)^2/n = \sum(d_i^2/n) - d^2$ and identified the most irregular graphs for both measures. He also showed

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that the measures are incomparable for some pairs of graphs. Albertson [1] used the measure $\sum_{ij \in E} |d_i - d_j|$, which has found applications in chemical graph theory. Nikiforov [27] used the measure $s(G) = \sum_i |d_i - d|$. These measures are all greater than or equal to zero, with equality for regular graphs, and can be described as *additive* measures of irregularity. It is worth noting that $\mathrm{var}(G) = \mathrm{var}(\overline{G})$ and $s(G) = s(\overline{G})$, where $\overline{G}$ denotes the complement of G. The measures defined in this paper are all greater than or equal to one, with equality for regular graphs, and can be described as *multiplicative* measures of irregularity.

In Section 2 we define and compare the new measures; in Section 3 we use the measures to strengthen upper bounds for the chromatic number; in Section 4 we strengthen Turán's Theorem for irregular graphs; in Section 5 we apply the new measures to generalised $r-$partite graphs; in Section 6 we bound a graph's radius, Harmonic index and Randić index; and we conclude with bounds for the new measures and new indices of network heterogeneity.

2. Measures of irregularity

Our first measure of irregularity, ν, was introduced by Edwards [13]. He defined a parameter c_v, which he termed the "vertex degree coefficient of variation" as follows:

$$\nu = 1 + c_v^2 = \frac{n \sum_{i \in V} d_i^2}{4m^2}.$$

Edwards [13] showed that $c_v = 0$ if and only if a graph is regular, so $\nu \geq 1$, with equality only for regular graphs. c_v is the ratio of the standard deviation to the mean of the vertex degrees, which follows the usual definition of a coefficient of variation.

Our second measure of irregularity, ϵ, is defined similarly using an "edge degree coefficient of variation" as follows:

$$\epsilon = 1 + c_e^2 = \frac{n \sum_{ij \in E} \sqrt{d_i d_j}}{2m^2}.$$

It follows from Proposition 2.8 in Favaron, Mahéo and Saclé [15] that $\epsilon \geq 1$, with equality only for regular graphs.

Finally we define a spectral measure of irregularity as follows:

$$\beta = \frac{\mu}{d} = \frac{\mu n}{2m}.$$

It is well known that $\mu \geq d$, with equality only for regular graphs. Therefore $\beta \geq 1$, with equality only for regular graphs.

We can compare these bounds as follows. Hofmeister [19] proved that $\mu^2 \geq \sum_{i \in V} d_i^2/n$ and Favaron *et al* [15] have proved that $\mu \geq \sum_{ij \in E} \sqrt{d_i d_j}/m$. It is therefore straightforward that:

$$\beta^2 \geq \nu \text{ and } \beta \geq \epsilon.$$

We can also show that $\nu \geq \epsilon$, as follows:

$$\nu = \frac{n\sum_{i\in V} d_i^2}{4m^2} = \frac{n\sum_{ij\in E}(d_i + d_j)}{4m^2} \geq \frac{n\sum_{ij\in E}\sqrt{d_i d_j}}{2m^2} = \epsilon.$$

Finally, for most but not all irregular graphs, $\epsilon^2 > \nu$.

3. Upper bounds for the Chromatic Number

Theorem 3.1. *Let G be a graph with irregularity ν. Then*

$$\chi(G) \leq \frac{n}{\nu}.$$

Proof. Our proof follows that in Deng et al [10], which uses contradictions for $\chi(G) = 2$, $\chi(G) = 3$ and $\chi(G) \geq 4$.

Case 1: $\chi(G) = 2$.

Note that:

$$\frac{n}{\nu} = \frac{4m^2}{\sum_{i\in V} d_i^2} = \frac{4m^2}{\sum_{ij\in E}(d_i + d_j)} < \chi(G) = 2 = \frac{4m^2}{\sum_{ij\in E}(m + m)}$$

which implies some degrees are greater than m, a contradiction.

Case 2 : $\chi(G) = 3$.

Let pq be an edge which has the largest weight of $(d_i + d_j)$ for $ij \in E$. Then:

$$\chi(G) = 3 > \frac{n}{\nu} = \frac{4m^2}{\sum_{ij\in E}(d_i + d_j)} \geq \frac{4m}{(d_p + d_q)} \geq \frac{4(d_p + d_q - 1)}{d_p + d_q} = 4 - \frac{4}{d_p + d_q}.$$

Deng et al [10] demonstrate that this inequality leads to a contradiction.

Case 3 : $\chi(G) \geq 4$.

Let pq be an edge which has the largest weight of $(d_i + d_j)$ for $ij \in E$ and consequently the smallest weight of $\frac{1}{d_i + d_j}$. Deng et al [10] have proved that:

$$\frac{2}{d_p + d_q} > \frac{1}{\chi(G) - 1}.$$

Therefore:

$$\chi(G) > \frac{n}{\nu} = \frac{4m^2}{\sum_{ij\in E}(d_i + d_j)} \geq \frac{4m}{(d_p + d_q)} > \frac{2m}{\chi(G) - 1}.$$

This implies $2m < \chi(G)(\chi(G) - 1)$. However for all graphs $2m \geq \chi(G)(\chi(G) - 1)$, since there must be at least one edge between each pair of color classes.

□

This bound strengthens a bound due to Deng et al [10], who recently proved that $\chi(G) \leq 2H(G)$, where H(G) is the Harmonic index. We discuss this index and the Randić index, R(G), in Section 6.

This bound is sometimes better than Wilf's well known bound ($\chi(G) \leq 1 + \mu$). For example for the Star graph on n vertices, the Wilf bound equals $1 + \sqrt{n-1}$ and this new bound equals $4 - 4/n$.

Hansen and Vukicević [18] proved that $\chi(G) \leq 2R(G)$. We can demonstrate that this bound is never better than Wilf's bound because:

$$\chi(G) \leq \mu + 1 \leq \frac{n}{\beta} \leq \frac{n}{\epsilon} \leq 2R(G).$$

We show that $n/\epsilon \leq 2R(G)$ in Section 6. The only new inequality is therefore that:

$$\mu + 1 \leq \frac{n}{\beta} = \frac{2m}{\mu}.$$

We can prove this inequality using the following lemma.

Lemma 3.1. *Let G be a graph with chromatic number χ. Then:*

$$\chi(\chi - 1) \leq \mu(\mu + 1) \leq 2m.$$

Proof. We have noted above that $\chi(\chi - 1) \leq 2m$ and $\chi \leq 1 + \mu$. In Section 3.1 below we use that $\mu^2 \leq 2m(\omega - 1)/\omega$. Therefore:

$$\mu^2 \leq \frac{2m(\omega - 1)}{\omega} = \frac{2m\omega(\omega - 1)}{\omega^2} \leq \frac{2m\chi(\chi - 1)}{\omega^2} \leq \frac{4m^2}{\omega^2}.$$

Therefore:

$$\mu(\mu + 1) \leq \frac{2m(\omega - 1)}{\omega} + \frac{2m}{\omega} = 2m.$$

□

3.1. Colin de Verdière parameter

The Colin de Verdière parameter, $\lambda(G)$, is the basis for the profound conjecture that $\chi(G) \leq 1 + \lambda(G)$. There is extensive literature on this conjecture, for example by Holst et al [20] and Goldberg [16]. Several upper bounds for $\chi(G)$ are not upper bounds for $1 + \lambda(G)$. For example, the Petersen graph demonstrates that $\lambda \not\leq \mu$ and $K_{4,5}$ demonstrates that $\lambda \not\leq n - \alpha$, where α denotes the independence number, which is the size of the maximum set of vertices, no two of which are adjacent.

We can, however, use β to create a new upper bound for λ as follows.

Theorem 3.2. *Let G be a connected graph with irregularity β. Then:*

$$\lambda(G) \leq \frac{n}{\beta} - 1 = \frac{2m}{\mu} - 1.$$

Proof. One of the deep properties of λ is that it is minor-monotone, from which it follows immediately that $\omega \leq 1 + \lambda$. (A graph parameter $\phi(G)$ is called minor-monotone if $\phi(H) \leq \phi(G)$ for any minor H of G.)

Pendavingh [30] has proved that if $G \neq K_{3,3}$ is a connected graph, then

$$\lambda(\lambda + 1) \leq 2m.$$

We therefore need to consider two options. If $G = K_{3,3}$ then (eg see Goldberg) $\lambda = 4 < (2m/\mu) - 1 = 18/3 - 1 = 5$.

If $G \neq K_{3,3}$ then we use a result due to Nikiforov [26], and conjectured by Edwards and Elphick [14], that:

$$\mu^2 \leq \frac{2m(\omega - 1)}{\omega}.$$

Therefore:

$$\mu^2 \leq \frac{2m(\omega - 1)}{\omega} \leq \frac{2m\lambda}{\lambda + 1} \leq \frac{4m^2}{(\lambda + 1)^2},$$

and consequently:

$$\lambda \leq \frac{2m}{\mu} - 1 = \frac{n}{\beta} - 1.$$

□

4. Turán's Theorem for irregular graphs

Turán's Theorem, proved in 1941, is a fundamental result in extremal graph theory. In its concise form it states that:

$$2m \leq \frac{(\omega - 1)n^2}{\omega}.$$

Observe that the result, due to Nikiforov above, that $\mu^2 \leq 2m(\omega - 1)/\omega$, is equivalent to the following strengthening of the concise Turán theorem:

Theorem 4.1.

$$2m \leq \frac{(\omega - 1)n^2}{\omega\beta^2}.$$

Due to the bounds $\beta^2 \geq \nu$ and $\beta \geq \epsilon$ we obtain:

$$\text{(i) } 2m \leq \frac{(\omega - 1)n^2}{\omega\nu} \text{ and (ii) } 2m \leq \frac{(\omega - 1)n^2}{\omega\epsilon^2}.$$

We provide a non-spectral proof of the bound (i) because it leads to a corollary. Before presenting this proof we explain briefly the intuition underlying the above inequalities. Theorem 4.1 is unusual because it involves m on both sides. A useful way to interpret the theorem is that β, ν and ϵ are measures of graph irregularity. Therefore all graphs with a given clique number and, for example, irregularity as measured by $\nu \geq 2$ have a maximum number of edges that is at most half of the number implied by Turán's Theorem.

Proof. This non-spectral proof is based on a 1962 proof of the concise Turán Theorem due to Moon and Moser [24], as written up in an award winning paper by Martin Aigner entitled "Turán's Graph Theorem".

Let C_h denote the set of h-cliques in G with $|C_h| = c_h$. So for example, $c_1 = n, c_2 = m, c_3 = t$ etc. For $A \in C_h$ let $d(A)$ equal the number of $(h+1)$ cliques containing A. Moon and Moser [24] proved that:

$$\frac{c_{h+1}}{c_h} \geq \frac{h^2 c_h/c_{h-1} - n}{h^2 - 1}, \quad \text{for } h \geq 2. \tag{1}$$

They also proved that:

$$nc_h + (h^2-1)c_{h+1} \geq \sum_{B \in C_{h-1}} d(B)^2$$

so with $h = 2$ this becomes:

$$\begin{aligned} nm + 3c_3 &\geq \sum_{i=1}^{n} d_i^2, \quad \text{or equivalently} \\ \frac{c_3}{c_2} &= \frac{c_3}{m} \geq \frac{(\sum d_i^2/m) - n}{3}. \end{aligned} \tag{2}$$

Now define θ as follows:

$$\frac{(\theta-2)n}{\theta} = \frac{\sum d_i^2}{m} - n \tag{3}$$

which is equivalent to:

$$2m = \frac{(\theta-1)n^2}{\theta\nu}.$$

This definition of θ differs from that in [24] and enables the strengthening of Moon and Moser's proof. Combining (2) and (3) we have:

$$\frac{c_3}{c_2} \geq \frac{\sum d_i^2/m - n}{3} = \frac{(\theta-2)n}{3\theta}. \tag{4}$$

To prove Theorem 4.1 (i) we need to show that $\theta \leq k-1$ for graphs without k-cliques. Consider the claim:

$$\frac{c_{h+1}}{c_h} \geq \frac{(\theta-h)n}{\theta(h+1)}, \quad \text{for } h \geq 2. \tag{5}$$

For $h = 2$, this is inequality (4). We therefore use induction on h and (1) as follows:

$$\begin{aligned}\frac{c_{h+1}}{c_h} &\geq \frac{h^2 c_h/c_{h-1} - n}{h^2-1} \geq \frac{h^2(\theta - h + 1)n/(\theta h) - n}{h^2-1} \\ &= \frac{(\theta-h)(h-1)n}{\theta(h^2-1)} = \frac{(\theta-h)n}{\theta(h+1)}\end{aligned}$$

as claimed in (5). Now if G contains no k-clique then $c_k = 0$ and we infer $\theta \leq h = k-1$ from (5).

We have not attempted a non-spectral proof of the ϵ bound. □

4.1. Number of k-cliques

Moon and Moser [24] proved that if t is the number of triangles in a graph, then:

$$t \geq \frac{m(4m - n^2)}{3n}.$$

In the following corollary we strengthen this bound for irregular graphs. This corollary is exact for some irregular complete tripartite and Turán graphs.

Corollary 4.1. *Let G be a graph with irregularity ν. Then:*

$$t \geq \frac{m(4m\nu - n^2)}{3n}.$$

Proof. From inequality (4), we know that:

$$t \geq \frac{nm(\theta-2)}{3\theta} = \frac{\sum d_i^2 - nm}{3} = \frac{4\nu m^2 - n^2 m}{3n}.$$

□

This approach can be continued for larger cliques. For example, we know from (5) that:

$$c_4 \geq \frac{tn(\theta-3)n}{4\theta} \geq \frac{m(4m\nu - n^2)}{12}(1 - \frac{3}{\theta}) = \frac{m(4m\nu-n^2)(3m\nu - n^2)}{6n^2}.$$

4.2. Remarks

Theorem 4.1 is exact for all complete bipartite graphs. The full form of Turán's theorem states that $m(G) \leq m(T_r(n))$, where $T_r(n)$ is the complete r-partite graph of order n whose classes differ by at most one, with equality holding only if $G = T_r(n)$. It is not the case that for all irregular graphs $m(G) \leq m(T_r(n))/\nu$ or that $m(G) \leq m(T_r(n))/\epsilon^2$ or that $m(G) \leq m(T_r(n))/\beta^2$.

Turán's theorem can be further strengthened by using more complex lower bounds for μ. For example, if t_i denotes the sum of the degrees of the vertices adjacent to v_i, then Yu, Lu and Tian [32] have proved that:

$$\mu^2 \geq \frac{\sum t_i^2}{\sum d_i^2} \geq \frac{\sum d_i^2}{n} \geq \frac{4m^2}{n^2}.$$

Thus if we define:

$$\alpha = \frac{n^2 \sum t_i^2}{4m^2 \sum d_i^2} \geq \nu$$

it follows, as above, that:

$$2m \leq \frac{(\omega - 1)n^2}{\omega\alpha} \leq \frac{(\omega - 1)n^2}{\omega\nu}.$$

5. Generalized r-partite graphs

In a series of papers, Bojilov and others have generalized the concept of an r-partite graph. They define the parameter ϕ to be the smallest integer r for which $V(G)$ has an r-partition:

$$V(G) = V_1 \cup V_2 \cup \ldots \cup V_r, \quad \text{such that } d(v) \leq n - |V_i|,$$

for all $v \in V_i$ and for $i = 1, 2, \ldots, r$.

It is notable that ϕ depends only on the degrees of G, and not on the adjacency matrix of G. Indeed, ϕ is defined for any set of n integers a_i, where $0 \leq a_i \leq n - 1$, which may or may not correspond to the degrees of a graph.

Theorem 2.1 in [3] proves that ϕ is a lower bound for the clique number and the greedy Algorithm 1 and Theorem 3.1 in [3] demonstrate that ϕ can be computed in linear time. For d-regular graphs, Theorem 4.4 in [3] proves that:

$$\phi = \left\lceil \frac{n}{n - d} \right\rceil.$$

Khadzhiivanov and Nenov [12] have proved that ϕ satisfies Turán's Theorem:

$$2m \leq \frac{(\phi - 1)n^2}{\phi} \leq \frac{(\omega - 1)n^2}{\omega}. \tag{6}$$

Theorem 4.1 in [3] provides a simpler proof of (6). The study of ϕ has therefore led to a novel proof of the concise version of Turán's Theorem, which also demonstrates that this famous result is in fact a function only of the degrees of a graph rather than its adjacency matrix.

It is of interest to see to what extent (6) can be strengthened in a similar way to Theorem 4.1 For example, Bojilov and Nenov [4] have strengthened (6) as follows:

$$2m \leq \frac{(\phi - 1)n^2}{\phi\sqrt{\nu}}. \tag{7}$$

Inequality (7) is further strengthened in Theorem 5.4 in [3] where it is shown that:

$$\phi \geq \frac{n}{n - d_\phi^*} \geq \frac{n}{n - d_{\phi-1}^*} \geq \ldots \geq \frac{n}{n - d_1^*}$$

where

$$d_r^* = \sqrt[r]{\sum d_i^r / n}.$$

Observe that inequality (7) is equivalent to $r = 2$ in this chain of inequalities.

It is therefore natural to ask whether $2m \leq (\phi - 1)n^2/\phi\nu$? The answer is no, because, for example, the graph in Figure 1 provides a counter-example.

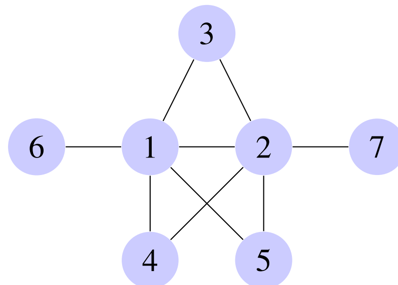

Figure 1. `Graph751` on 7 vertices with degree sequence $(5, 5, 2, 2, 2, 1, 1)$, $\phi = 2$ and $\omega = 3$

There are also various spectral lower bounds for ω of which the simplest, due to Cvetkovic [8], is:

$$\omega \geq \frac{n}{n - \mu}. \tag{8}$$

The graph in Figure 2 is an example of a graph which does not satisfy (8), with ω replaced by ϕ. It also demonstrates that a variety of other spectral lower bounds for ω are not lower bounds for ϕ. Furthermore, ϕ does not satisfy the Motzkin-Straus inequality.

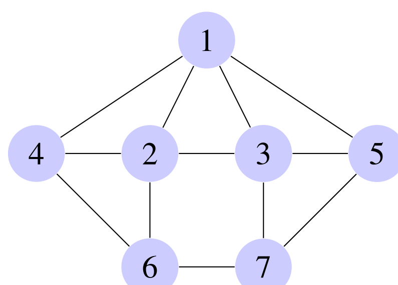

Figure 2. `Graph` on 7 vertices with degree sequence $(4, 4, 4, 3, 3, 3, 3)$, $\phi = 2$, $\mu = 3.503$

Conjecture 1. *We have, however, been unable to find a counter-example to the following conjecture:*

$$2m \leq \frac{(\phi - 1)n^2}{\phi\epsilon}.$$

6. Bounds on graph radius, Harmonic and Randić indices

The Randić index is used in organic chemistry, with bonds between atoms represented by edges in a graph. The Randić index is defined as:

$$R(G) = \sum_{ij \in E} \frac{1}{\sqrt{d_i d_j}}.$$

An alternative to the Randić index is the Harmonic index, which is defined as:

$$H(G) = \sum_{ij \in E} \frac{2}{d_i + d_j}.$$

Using results due to Xu [31] it is straightforward to show that:

$$\frac{n}{2\nu} \leq H(G) \leq R(G) \leq \frac{n}{2}.$$

Liu [23] has recently proved that triangle-free graphs have $H(G) \geq 2m/n$. We can generalise this bound using part (i) of Theorem 4.1 as follows:

$$H(G) \geq \frac{n}{2\nu} \geq \frac{\omega m}{(\omega - 1)n} = \frac{2m}{n} \text{ when } \omega = 2.$$

We can also show that:

$$\frac{m}{\mu} = \frac{n}{2\beta} \leq \frac{n}{2\epsilon} \leq R(G)$$

since using Cauchy-Schwartz, we have $R(G). \sum_{ij \in E} \sqrt{d_i d_j} \geq (\sum_{ij \in E} 1)^2 = m^2$. Note that for Star graphs, $n/2\epsilon = \sqrt{n-1}$, which is the lower bound for $R(G)$ due to Bollobas and Erdos [5]. It is not always the case that $n/2\epsilon \leq H(G)$ or that $n/2\beta \leq H(G)$.

The eccentricity $ecc(v)$ of a vertex v in a connected graph G is the maximum distance between v and any other vertex u of G. The minimum graph eccentricity is the radius, r, of the graph. Xu [31] has proved that if $H(G)$ is the Harmonic index, then:

$$H(G) \geq \frac{m}{n-r}.$$

This bound on r can be strengthened as follows.

Theorem 6.1. *Let G be a connected graph with irregularity ν. Then:*

$$H(G) \geq \frac{n}{2\nu} \geq \frac{m}{n-r}.$$

Proof. Note that for each vertex $i \in V(G)$, we have $d_i \leq n - ecc(i)$. Therefore:

$$\frac{n}{2\nu} = \frac{2m^2}{\sum_{ij \in E}(d_i + d_j)} \geq \frac{2m^2}{\sum_{ij \in E}(2n - ecc(i) - ecc(j))} \geq \frac{2m^2}{2m(n-r)} = \frac{m}{n-r}.$$

□

7. Upper Bounds

Gutman, Furtula and Elphick [17] have proved that:

$$\beta^2 \leq \frac{n^2}{4(n-1)} \text{ ; } \nu \leq \frac{n^2}{4(n-1)} \text{ and } \epsilon^2 \leq \frac{n^2}{4(n-1)}$$

with equality for Star graphs.

We can obtain alternative bounds on $\nu(G)$ by using bounds on $\sum d_i^2$, which is often referred to as the first Zagreb index. For example, Das [9] proved that

$$\sum d_i^2 \leq 2m(\Delta + \delta) - n\Delta\delta.$$

Therefore

$$1 \leq \nu = \frac{n \sum d_i^2}{4m^2} \leq \frac{n(\Delta + \delta)}{2m} - \frac{n^2 \Delta\delta}{4m^2} = \frac{\Delta + \delta}{d} - \frac{\Delta\delta}{d^2}.$$

Alternatively, Izumino, Mori and Seo [21] have proved (their Corollary 3.2) that if $0 \leq \delta \leq d_i \leq \Delta$ then:

$$\frac{1}{n} \sum d_i^2 - \left(\frac{\sum d_i}{n}\right)^2 \leq \frac{(\Delta - \delta)^2}{4}.$$

Therefore

$$1 \leq \nu = \frac{n \sum d_i^2}{4m^2} \leq \frac{n^2}{4m^2} \left(\frac{4m^2}{n^2} + \frac{(\Delta - \delta)^2}{4}\right) = 1 + \left(\frac{\Delta - \delta}{2d}\right)^2.$$

We can obtain alternative bounds on $\epsilon(G)$ by using bounds on the generalised Randic index, $R_\alpha(G)$, with $\alpha = 1/2$. For example, Li and Yang [22] proved that for $\alpha \geq 0$:

$$R_\alpha \leq \frac{n(n-1)^{1+2\alpha}}{2}.$$

Therefore

$$1 \leq \epsilon = \frac{nR_{1/2}}{2m^2} \leq \frac{n^2(n-1)^2}{4m^2} = \left(\frac{n-1}{d}\right)^2.$$

Favaron *et al* [15] demonstrate that ν and ϵ^2 are incomparable. However, in practice, $\epsilon^2 \geq \nu$ for almost all graphs. Indeed we have been unable to find a graph amongst the named graphs in Wolfram Mathematica for which $\epsilon^2 < \nu$. Considering the irregular named graphs in Wolfram with 16 vertices, the average value of $\nu = 1.22$, the average value of $\epsilon^2 = 1.27$ and the average value of $\beta^2 = 1.32$. As a specific example, DutchWindmill(5,4) has $\nu = 1.6$, $\epsilon^2 = 1.675$ and $\beta^2 = 1.92$.

The graphs representing some actual networks, such as the World Wide Web, power grids, academic collaborators and neural networks, are highly irregular. For example, Newman [25] calculated that the World Wide Web graph has $c_v = 3.685$, implying $\nu = 14.6$. These high values of irregularity for some actual networks may increase the usefulness of the measures described in this paper.

8. Network heterogeneity indices

Estrada [11] and others have noted that many real-world networks have a power law degree distribution. Estrada has proposed that normalised indices of the heterogeneity of such networks should lie in the range $(0, 1)$, with zero corresponding to regular graphs and unity to Star graphs. Estrada devised the following index, using $R(G)$, which meets these criteria:

$$\rho_n = \frac{n - 2R}{n - 2\sqrt{n-1}}.$$

As discussed above, ν, ϵ and β are minimised for regular graphs and maximised for Star graphs. We can therefore devise the following normalised heterogeneity indices:

$$\nu_n = \frac{n^2 - (n^2/\nu)}{(n-2)^2} \text{ ; } \epsilon_n = \frac{n - (n/\epsilon)}{n - 2\sqrt{n-1}} \text{ and } \beta_n = \frac{n - (n/\beta)}{n - 2\sqrt{n-1}}.$$

It follows from the inequalities in Section 6 above that:

$$0 \leq \rho_n \leq \epsilon_n \leq \beta_n \leq 1.$$

It may be that β_n is the most useful of these indices, perhaps using results due to Chung, Lu and Vu [6] who have investigated the spectrum of random power law graphs.

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