Electronic Journal of Graph Theory and Applications **6** (2) (2018), 306–309

Electronic Journal of
Graph Theory and Applications

Parsimonious edge-coloring on surfaces

Sarah-Marie Belcastro[a]

[a]*Mathematical Staircase, Inc., Holyoke MA, and Smith College, Northampton MA, U.S.A.*

smbelcas@toroidalsnark.net

Abstract

We correct a small error in a 1996 paper of Albertson and Haas, and extend their lower bound for the fraction of properly colorable edges of planar subcubic graphs that are simple, connected, bridgeless, and edge-maximal to other surface embeddings of subcubic graphs.

Keywords: edge coloring, graph embeddings, subcubic graphs
Mathematics Subject Classification : 05C10
DOI: 10.5614/ejgta.2018.6.2.9

1. Introduction and Background

Consider a graph G with maximum degree $\Delta(G)$, and assign colors to the edges. It is well-known from Vizing's Theorem [7] that every simple graph with maximum degree $\Delta(G) = k$ may be properly edge-colored using either k or $k + 1$ colors. In general, most of the edges of a graph can be properly colored with $\Delta(G)$ colors, with few (if any) edges needing an additional color. The question then arises: What fraction of the edges of G is guaranteed to be properly $\Delta(G)$-edge colorable?

We will focus on graphs for which $\Delta(G) = 3$, namely the cubic (3-regular) and subcubic (highest degree 3) graphs. Let $|E(G)|$ be the number of edges in G, let $c(G)$ be the number of edges in a three-edge colorable subgraph of G with the largest possible number of edges, and let $y(G) = \dfrac{c(G)}{|E(G)|}$. In 1996, Albertson and Haas [1] gave lower bounds for $y(G)$ for simple cubic and subcubic graphs. They showed that

(a) if G is simple, cubic, and connected, $y(G) \geq \frac{13}{15} \approx .8667$;

Received: 13 November 2017, Revised: 9 March 2018, Accepted: 11 July 2018.

(b) if G is simple, subcubic, and connected, then $y(G) \geq \frac{26}{31} \approx .8387$;

(c) if G is simple, subcubic, planar, bridgeless, connected, and edge-maximal (no edges may be added to G without violating one of the prior-listed conditions), then $y(G) \geq \frac{6}{7} - \frac{2}{35|E(G)|}$ (a value in the range $.8$–$.8571$).

Note that this statement of (c) is a corrected version of Albertson and Haas's original statement; see Section 2.

While the bound in (a) is best possible for general simple cubic G (see the Petersen graph), restricting the class of graphs under consideration produces better bounds. For example, Rizzi [5] shows that if a subcubic graph G is triangle-free (but might have multiple edges), then $y(G) \geq \frac{13}{15}$. Mkrtchyan and Steffen [4] show that if the girth of G is g, then $y(G) \geq \frac{g}{g+1}$ when g is even and $y(G) \geq \frac{g-1}{g}$ when g is odd. (Note that in the subcubic case, the girth of G must be at least 8 in order for this result to give a stricter bound than the $\frac{13}{15}$ fraction for triangle-free graphs in [5].) Kamiński and Kowalik [3] show that a simple subcubic graph G has $y(G) \geq \frac{13}{15}$ unless it is a 5-cycle with two adjacent chords (a result also shown in [2]), and that a subcubic multigraph G has $y(G) \geq \frac{7}{9} \approx .7778$ unless it is K_3 with an arbitrary edge doubled. (This is not much better than the bound of $y(G) \geq \frac{3}{4}$ given for an arbitrary subcubic multigraph in [1].) Restricting to snarks gives better bounds; Steffen [6] showed that the Petersen graph is the only snark with $y(G) = \frac{13}{15}$, and that snarks with at least 16 vertices have $y(G) \geq \frac{11}{12} \approx .9167$.

In contrast to the structural approaches given above, Albertson and Haas extended (b) to (c) by restricting to planar graphs, or in other words, by using information about the possible embeddings of G. In Section 3, we generalize using embedding information about G; that is, we take a topological viewpoint. This is of interest because there are often constraints imposed on the colorability of a graph by the genus of that graph.

2. Correction to Albertson and Haas

Albertson and Haas' approach to (c) was as follows: First, vertices of degree one may be eliminated as their incident edges do not affect $y(G)$. Then, examine facial boundaries. Any series of adjacent degree-two vertices may be replaced with a single one without improving the colorability of G. Thus, no face has two adjacent vertices of degree two.

Albertson and Haas additionally assumed that each face has at most one vertex of degree two, reasoning that joining a pair of non-adjacent degree-two vertices by an additional edge increases $|E(G)|$ while leaving $c(G)$ constant and thus decreases $y(G)$. However, an added edge may be colorable so that $c(G)$ and $y(G)$ both increase. (Unfortunately, it is not known whether an example demonstrating this exists: Albertson and Haas conjectured ([1, p. 5]) that every planar bridgeless graph G with $\Delta(G) = 3$ and at least two vertices of degree two has $y(G) = 1$. This problem is still open.) Thus an additional condition must be introduced for their Theorem 3 [1]: the embedding of G must be edge-maximal.

3. Non-planar Embeddings

Result (c) of [1] does not directly generalize to surfaces other than the sphere. Assuming bridgelessness for planar graphs eliminates monofacial edges. This does not hold for higher-genus surfaces; consider a cellular embedding of K_4 on the torus. See Figure 1 at left. It is bridgeless but

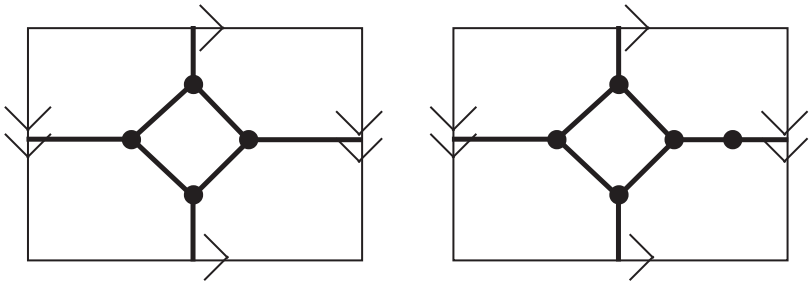

Figure 1. At left, a cellular embedding of K_4 on the torus; at right, the same embedding with a monofacial edge subdivided.

has two monofacial edges. Unless we constrain ourselves to closed 2-cell embeddings, we may have an edge-maximal embedding with a degree-two vertex v incident to two monofacial edges; in this case, v is not shared between two faces, but occurs twice on the same face. (For an example, consider the toroidal embedding of K_4 and subdivide one of the monofacial edges; this is shown at right in Figure 1.)

Theorem 3.1. *If G is simple subcubic and edge-maximally cellularly embedded on a surface S, then $y(G) \geq \frac{17}{20} - \frac{\chi(S)}{20|E(G)|}$, where $\chi(S)$ is the Euler characteristic of the embedding surface.*

Proof. Let V_2 and V_3 be the number of degree-two and degree-three vertices of G, respectively. Additionally, let $E = |E(G)|$ and F be the number of faces in the embedding of G on S. Euler's Formula says that $\chi(S) = V_2 + V_3 - E + F = V_2 + V_3 - \frac{1}{2}(2V_2 + 3V_3) + F = F - \frac{1}{2}V_3$. Each degree-two vertex is on at least one face (but possibly at most one face as well), so $F \geq V_2$. Thus $\chi(S) \geq V_2 - \frac{1}{2}V_3$ or $V_2 \leq \chi(S) + \frac{1}{2}V_3$. We may eliminate V_3 from this equation by noting that $E = V_2 + \frac{3}{2}V_3$ so that $V_2 \leq \frac{3}{4}\chi(S) + \frac{1}{4}E$. From [1](1) we have $c(G) \geq \frac{13}{15}E - \frac{1}{15}V_2$, which becomes $c(G) \geq \frac{17}{20}E - \frac{1}{20}\chi$. Dividing through by E produces the desired result. □

Notice that in Theorem 3.1 we subtract $\chi(S)$ from our lower bound. In effect, this *increases* the lower bound for embeddings on most surfaces, as $\chi(S) < 0$ as long as S is not the projective plane, Klein bottle, torus, or sphere. Even when S is a torus or Klein bottle, the lower bound given in Theorem 3.1 is stronger than given by [1] (b). In the case of a closed 2-cell embedding, we have a result that gives a stronger bound than in [1] (b) and [1] (c) as long as $\chi(S) \leq 0$.

Theorem 3.2. *If G is simple subcubic and edge-maximally closed 2-cell embedded on a surface S, then $y(G) \geq \frac{6}{7} - \frac{\chi(S)}{35|E|}$, where $\chi(S)$ is the Euler characteristic of the embedding surface.*

Proof. We will use the same notation as in the proof of Theorem 3.1. By definition, no face can touch itself at a vertex in a closed 2-cell embedding, so each degree-two vertex is on two faces and thus $F \geq 2V_2$. Then $\chi(S) \geq 2V_2 - \frac{1}{2}V_3$ or $V_2 \leq \frac{1}{2}\chi(S) + \frac{1}{4}V_3$. We may eliminate V_3 from this equation by noting that $E = V_2 + \frac{3}{2}V_3$ so that $V_2 \leq \frac{3}{7}\chi(S) + \frac{1}{7}E$. From [1](1) we have $c(G) \geq \frac{13}{15}E - \frac{1}{15}V_2$, which becomes $c(G) \geq \frac{6}{7}E - \frac{1}{35}\chi$. Dividing through by E produces the desired result. □

The bounds given in Theorems 3.1 and 3.2 are weaker than the various $\frac{13}{15}$ bounds, even for those graphs that embed cellularly on surfaces of high genus. And analogous results to Theorems 3.1 and 3.2 exist for subcubic edge-maximally embedded multigraphs, but give completely useless bounds—the bounds are only better than the $\frac{7}{9}$ bound from [3] when $|E(G)|$ is so small that no corresponding G could embed cellularly on a surface with $\chi(S) < 1$.

Acknowledgement

The author is grateful to Mike Albertson for suggesting that she examine this problem, and to both Mike Albertson and Ruth Haas for discussions (both long-ago and recent) of their paper.

References

[1] M. Albertson and R. Haas, Parsimonious Edge Coloring, *Discrete Math.* **148** (1996), 1–7. https://doi.org/10.1016/0012-365X(94)00254-G

[2] J. Fouquet and J. Vanherpe, On Parsimonious Edge-Colouring of Graphs with Maximum Degree Three, *Graphs Combin.* **29** (2013), 475–487. http://doi.org/10.1007/s00373-012-1145-3

[3] M. Kamiński and Ł. Kowalik, Beyond the Vizing's bound for at most seven colors, *SIAM J. Discrete Math.*, **28** (3) (2014), 1334–1362. https://doi.org/10.1137/120899765

[4] V.V. Mkrtchyan and E. Steffen, Maximum Δ-edge-colorable subgraphs of class II graphs. *J. Graph Theory* **70** (4) (2012), 473–482. http://doi.org/10.1002/jgt.20629

[5] R. Rizzi, Approximating the maximum 3-edge-colorable subgraph problem. *Discrete Math.* **309** (2009), 4166–4170. http://doi.org/10.1016/j.disc.2008.11.017

[6] E. Steffen, Measurements of edge-uncolorability. *Discrete Math.* **280** (2004), 191–214. https://doi.org/10.1016/j.disc.2003.05.005

[7] V.G. Vizing, On an estimate of the chromatic class of a p-graph. *Diskret. Analiz* **3** (1964), 25–30.