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Fault-tolerant designs in lattice networks on the Klein bottle

Ayesha Shabbir

*Abdus Salam School of Mathematical Sciences, GC University,
68-B, New Muslim Town, Lahore, Pakistan*

ashinori@hotmail.com

Abstract

In this note, we consider triangular, square and hexagonal lattices on the flat Klein bottle, and find subgraphs with the property that for any j vertices there exists a longest path (cycle) avoiding all of them. This completes work previously done in other lattices.

Keywords: fault-tolerance; Klein bottle; longest path or cycle; lattice network
Mathematics Subject Classification : 05-02, 05C35, 05C38.

1. Introduction

Fault-tolerance allows a system to continue function after a component failure without any interruption. It is one of the key criteria in deciding the structures of interconnection networks for parallel and distributed systems. A fault-tolerant system follows a pattern or design (along with a set of instructions), which is usually modeled as a graph (see e.g. [1], [5]), in which vertices and edges correspond to processing units and communication links, respectively. Here, we are presenting some fault-tolerant designs in the form of $\mathbf{P}_k^j$-graphs or $\mathbf{C}_k^j$-graphs, in which n processing units are interlinked, and l of these n units forming a chain or cycle of maximal length are used to solve some task. They can tolerate the failure of up to j components or communication links, keeping constant performance, due to which the designs presented here can be used to construct perfectly stable systems, while the concept of j-fault hamiltonicity (see e.g. [8], [9]) does not provide too stable systems, as the (maximal) length of the used circuit changes when failure occurs.

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A finite $\mathbf{P}_k^j$-*graph* ($\mathbf{C}_k^j$-*graph*) is a k-connected graph such that for every choice of j vertices there exists a longest path (cycle) that misses the j chosen vertices. The smallest known $\mathbf{P}_1^1$-graph and planar $\mathbf{P}_1^1$-graph are shown in Fig 1(a) and (b), respectively. The first was found by H.-J. Voss and H. Walther [16] and T. Zamfirescu independently [18], while the second was presented by W. Schmitz in [10]. See [18] and [19], for several small examples of $\mathbf{P}_k^j$- and $\mathbf{C}_k^j$-graphs ($k = 1, 2, 3$; $j = 1, 2$).

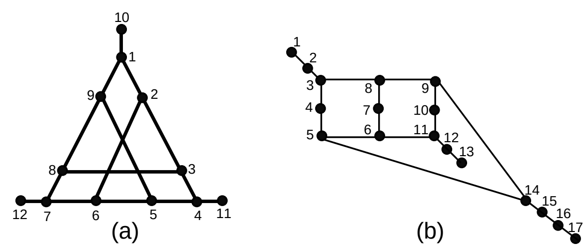


Figure 1.

In 1966, T. Gallai raised the question about the existence of $\mathbf{P}_1^1$-graphs, and in 1972, Zamfirescu refined it to $\mathbf{P}_k^j$- and $\mathbf{C}_k^j$-graphs [17]. A similar question was also asked for planar graphs.

The question raised in 2001, about the existence of such graphs in lattices (see [19]) also received a positive answer. See [2], [3], [7], [13] and [12], where the existence of $\mathbf{P}_k^1$- and $\mathbf{C}_k^j$-graphs ($k = 1, 2$; $j = 1, 2$) in all three (infinite) regular lattices, $\mathcal{T}$ (triangular), $\mathcal{L}$ (square) and $\mathcal{H}$ (hexagonal) is shown. Their orders are considerably larger. Comparatively smaller examples of $\mathbf{P}_k^1$- and $\mathbf{C}_2^j$-graphs for $k = 1, 2$ and $j = 1, 2$ in toroidal and Möbius strip lattices are given in [11] and [13].

To complete the previous investigation, in this note we consider (finite) triangular, square and hexagonal lattices on another interesting surface, the Klein bottle, and construct $\mathbf{P}_1^1$-, $\mathbf{P}_2^1$-, $\mathbf{C}_2^1$- and $\mathbf{C}_2^2$-graphs there. The graphs presented here are smaller than those obtained in other mentioned lattices (see the table given at the end). This was achieved in several cases not by embedding new graphs, but just by using the topology of the Klein bottle. The examples have not been found using computers; instead, modifications of known examples suitable for the new frame have been employed.

Note that the connectivity of any finite connected graph in $\mathcal{T}$ is at most 3, while in $\mathcal{L}$ or $\mathcal{H}$ it is at most 2. Moreover, it is worth mentioning that no $\mathbf{P}_1^j$- or $\mathbf{C}_2^j$-graph at all (whether planar or not) is known for $j \geq 3$ (see [14], p. 79).

The following two lemmas are of great help.

Let G and H be graphs homeomorphic to the graphs G' and H' in Fig. 2 and Fig. 3(a), respectively. The graph H contains ten subgraphs isomorphic to the graph of Fig. 3(b). The variables v, x, y, z, t and w denote the number of vertices of degree 2 on paths corresponding to edges shown on the respective figures as well.

Lemma 1.1. [7] *Let $v \leq x \leq w$. The longest paths of G have empty intersection if $x + v = y + 2z + w + 1$.*

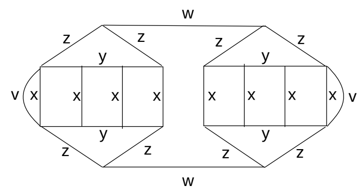


Figure 2. The graph G'.

Lemma 1.2. [13] *Any two vertices of H are missed by some longest cycle of H if $2x \geq y + 2z + 1$ and $2t = x + y + 3z + 3w + 8$.*

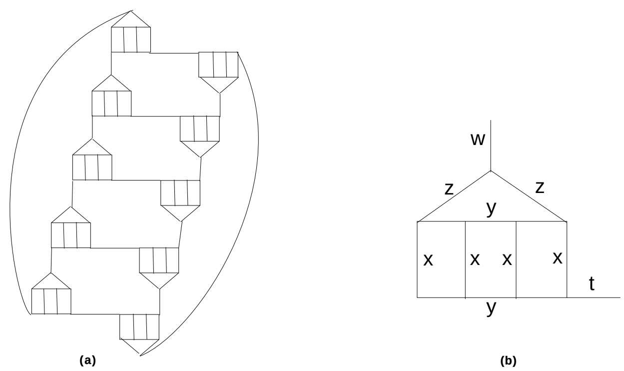


Figure 3. The graph H'.

2. Main Results

We start with the definition of lattices on the Klein bottle.

We consider an $(m+1) \times (n+1)$ parallelogram in $\mathcal{T}$. Then by identifying opposite vertices on the boundary as indicated on Fig. 4(b), we obtain the *$\mathcal{T}$-lattice* $\mathcal{T}^K_{m,n}$ on the Klein bottle. It has mn vertices. Similarly, we define the *$\mathcal{L}$-lattice* $\mathcal{L}^K_{m,n}$ and the *$\mathcal{H}$-lattice* $\mathcal{H}^K_{m,n}$, according to Fig. 4(a) and (c), respectively.

We now state our results, which are distributed in three sections. In next section, $\mathbf{P}^1_1$-, $\mathbf{C}^1_2$- and $\mathbf{C}^2_2$-subgraphs of $\mathcal{H}$-lattices are given. In the second and third sections, we consider $\mathcal{L}$- and $\mathcal{T}$-lattices, respectively, and describe $\mathbf{P}^1_k$- and $\mathbf{C}^j_2$-subgraphs ($k = 1, 2$; $j = 1, 2$) of these lattices.

2.1. $\mathcal{H}$-lattices

Theorem 2.1. *The lattice $\mathcal{H}^K_{6,2}$ contains a spanning $\mathbf{C}^1_2$-graph.*

Proof. In [6], the authors present a $\mathbf{C}^1_2$-graph of order 12 (see, Fig. 5(a)), attributed to Zamfirescu. Fig. 5(b) illustrates an embedding of it in $\mathcal{H}^K_{6,2}$.

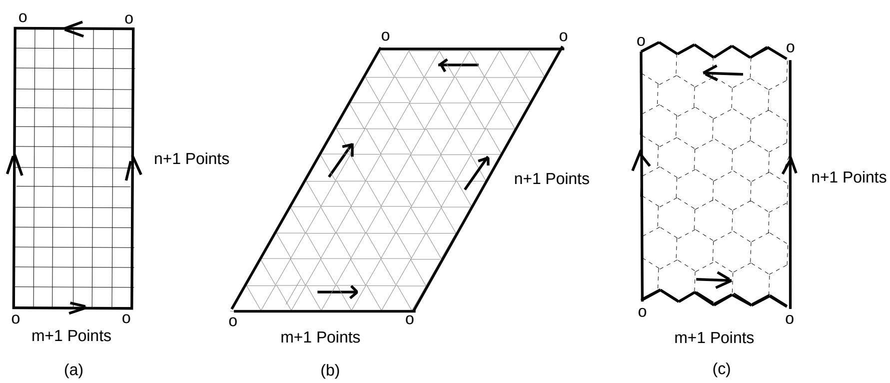


Figure 4.

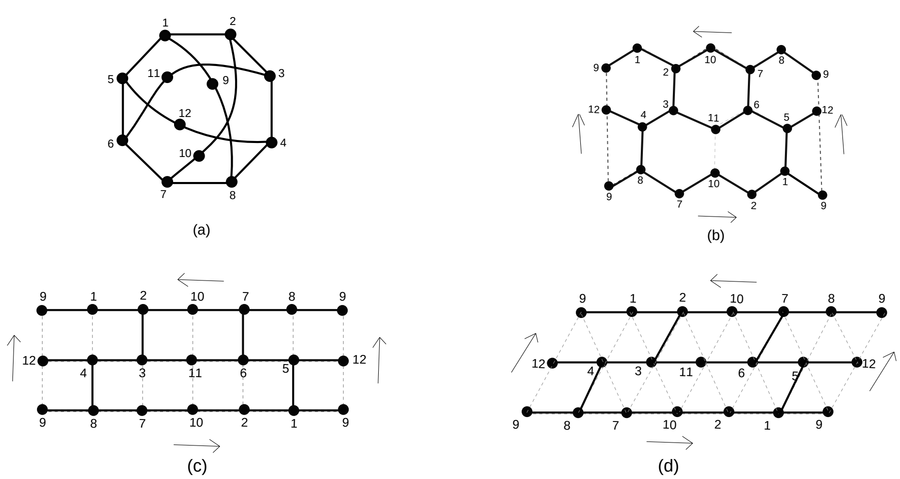
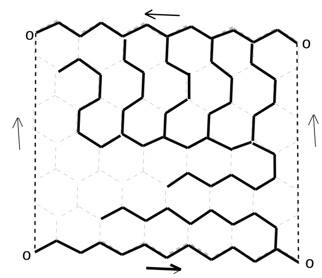


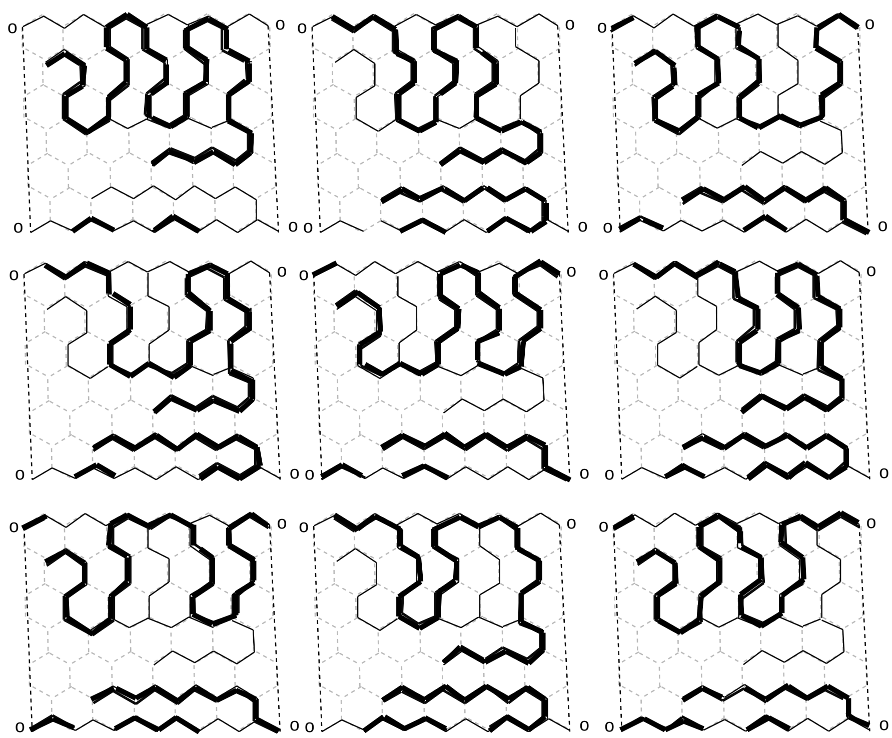
Figure 5.

Theorem 2.2. *In $\mathcal{H}^K_{12,6}$ we have a P^1_1-graph of order 58.*

Proof. The graph of Fig. 6(a) is of order 58 and has the desired property, as its longest paths, of length 40, have empty intersection (see Fig. 6(b)). One can directly check that each vertex is avoided by one of them.



(a)



(b)

Figure 6.

Theorem 2.3. *A planar $\mathbf{C}_2^2$-graph of order* 600 *exists in* $\mathcal{H}_{30,32}^K$.

Proof. The existence of a planar $\mathbf{C}_2^2$-graph of order 600 in $\mathcal{H}_{30,32}^K$ is shown in Fig. 7. This graph is isomorphic to the graph H in Lemma 2 for $y = z = 3$, $x = 8$, $t = 14$ and $w = 0$.

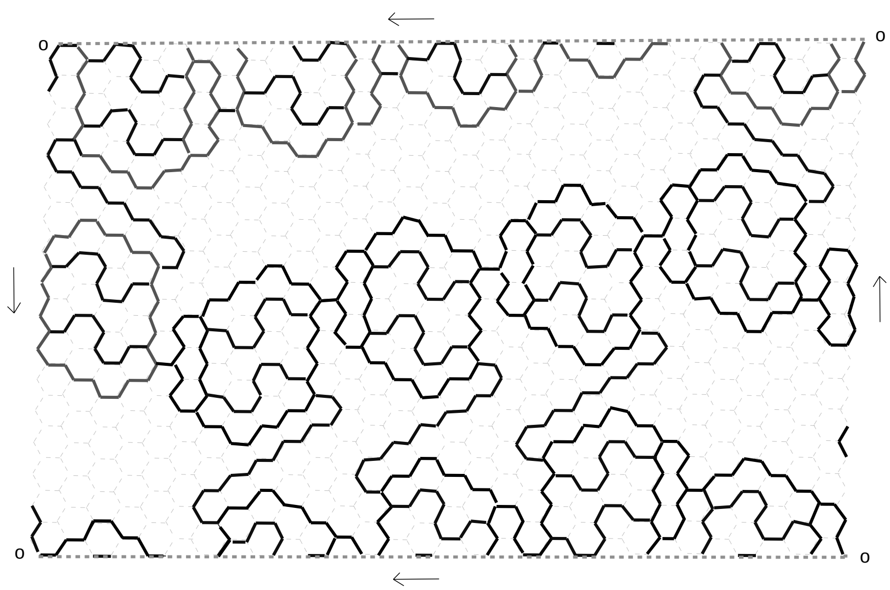


Figure 7.

2.2. $\mathcal{L}$-lattices

As $\mathcal{H} \subset \mathcal{L}$, every subgraph of a $\mathcal{H}$-lattice is a subgraph of some $\mathcal{L}$-lattice, but not vice-versa. The graphs we present next are interesting because they are smaller than the corresponding ones obtained in $\mathcal{H}$-lattices.

Theorem 2.4. *There exists a planar $\mathbf{P}_1^1$-subgraph of $\mathcal{L}_{6,4}^K$ of order* 17.

Proof. The graph shown in Fig. 8, is an embedding of Schmitz's graph of Fig. 1(b) into $\mathcal{L}_{6,4}^K$.

Since Schmitz's graph is conjectured to be the smallest planar $\mathbf{P}_1^1$-graph, it seems reasonable to say that our embedded graph and corresponding $\mathcal{L}$-lattice might be of minimal order too.

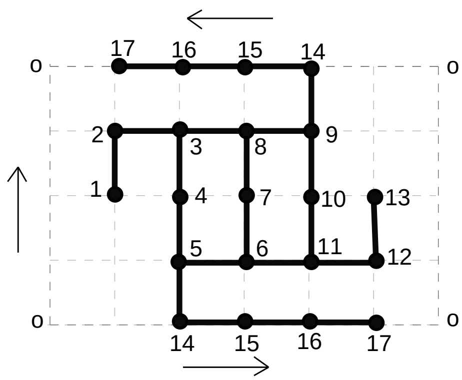


Figure 8.

Theorem 2.5. *In $\mathcal{L}^K_{10,10}$ we have a planar $\mathbf{P}^1_2$-graph of order* 80*.*

Proof. The conditions of Lemma 1 are satisfied if we take $x = v = w = 4$, $y = 1$ and $z = 1$, and the resulting graph G is a planar $\mathbf{P}^1_2$-graph of order 80 which is embeddable in $\mathcal{L}^K_{10,10}$. Fig. 9 presents such an embedding.

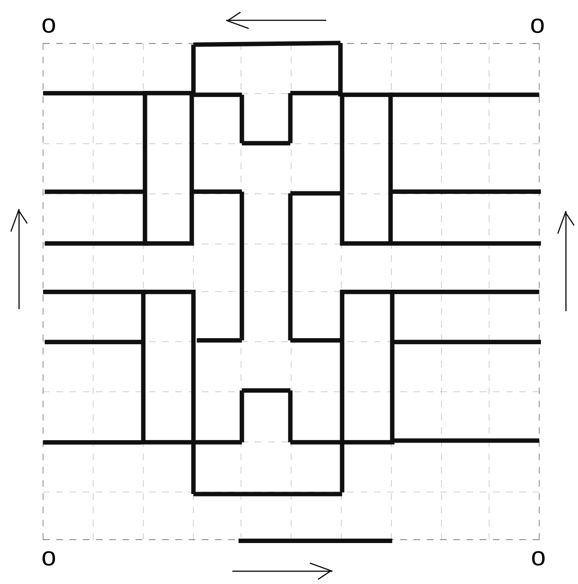


Figure 9.

The last result of this section is as follows.

Theorem 2.6. *We have a planar $\mathbf{C}^2_2$-graph of order* 315 *in $\mathcal{L}^K_{26,18}$.*

Proof. For $y = 1$, $z = 2$, $x = 3$, $t = 9$ and $w = 0$, the conditions of Lemma 2 are valid and the corresponding graph H is a planar $\mathbf{C}^2_2$-graph of order 315. Fig. 10 shows an embedding of H in $\mathcal{L}^T_{26,18}$.

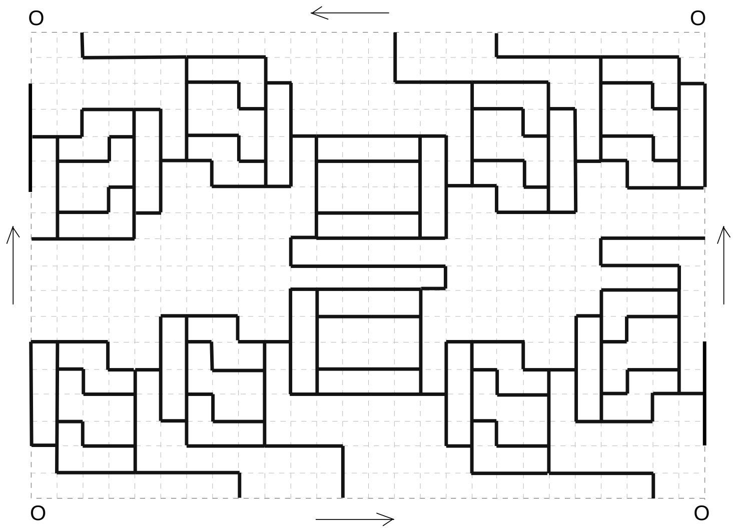


Figure 10.

2.3. $\mathcal{T}$-lattices

Since $\mathcal{H} \subset \mathcal{L} \subset \mathcal{T}$, we may expect to get better results, that is graphs of smaller order, when embedding in $\mathcal{T}$-lattices.

It is a conjecture that the graph of Fig. 1(a) is the smallest $\mathbf{P}_1^1$-graph [18]. Hence, we may conjecture that our next result is best possible.

Theorem 2.7. *In $\mathcal{T}_{4,3}^K$ we have a spanning $\mathbf{P}_1^1$-graph.*

Proof. A spanning $\mathbf{P}_1^1$-subgraph of $\mathcal{T}_{4,3}^K$ isomorphic to the graph of Fig. 1(a) is shown in Fig. 11.

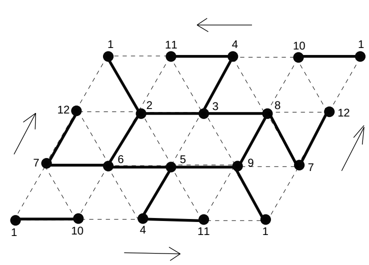


Figure 11.

Theorem 2.8. *There exists a planar $\mathbf{P}_2^1$-graph in $\mathcal{T}_{7,8}^K$ of order 48.*

Proof. The graph of Fig. 12 is a planar graph of order 48 in $\mathcal{T}_{7,8}^K$. It is a $\mathbf{P}_2^1$-graph, being obtained as a particular case of Lemma 1, for $y = 1$, $z = 0$, and $x = v = w = 2$.

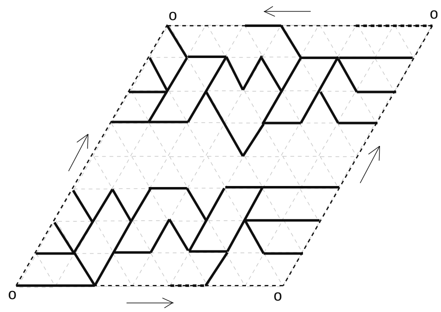


Figure 12.

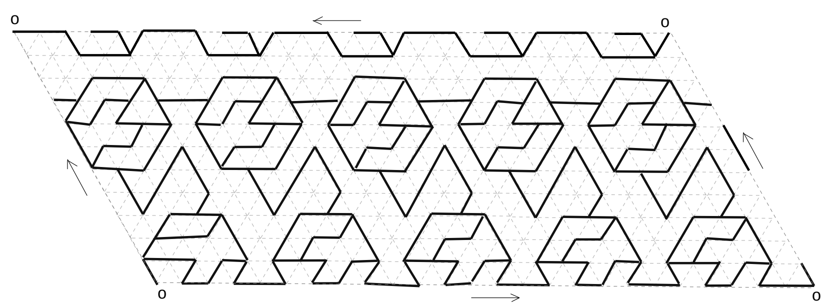


Figure 13.

Theorem 2.9. *There exists a planar* $\mathbf{C}_2^2$*-graph in* $\mathcal{T}_{11,25}^K$ *of order* 235.

Proof. The conditions of Lemma 2 are also verified by $y = 1$, $z = 0$, $x = 2$ and $w = 1$. Hence, the resulting graph H is a planar $\mathbf{C}_2^2$-graph of order 235. Fig. 13 reveals an embedding of it in $\mathcal{T}_{11,25}^K$.

Remark 2.1. The existence of spanning $\mathbf{C}_2^1$-graphs in $\mathcal{L}_{6,2}^K$ and $\mathcal{T}_{6,2}^K$, is shown in Fig. 5(c) and (d), respectively. It is also follows from Theorem 1.

	$\mathbf{P}_1^1$-graphs	$\mathbf{P}_2^1$-graphs	$\mathbf{C}_2^1$-graphs	$\mathbf{C}_2^2$-graphs
Arbitrary	12 [16, 18]	26 [19]	10	75 [18]
$\mathcal{H}$	94 [7]	244 [7]	89 [3]	950 [13]
$\mathcal{L}$	46 [7]	126 [7]	35 [3]	490 [13]
$\mathcal{T}$	30 [2]	92 [2]	33 [3]	375 [12]
$\mathcal{H}_{m,n}^T$	58 [11]	244 [7]	30 [11]	600 [13]
$\mathcal{L}_{m,n}^T$	20 [11]	80 [11]	15 [11]	315 [13]
$\mathcal{T}_{m,n}^T$	12 [12]	48 [12]	15 [12]	235 [12]
$\mathcal{H}_{m,n}^M$	46 [11]	244 [7]	32 [11]	670 [13]
$\mathcal{L}_{m,n}^M$	17 [11]	112 [11]	12 [11]	350 [13]
$\mathcal{T}_{m,n}^M$	17 [12]	64 [12]	12 [12]	235 [12]
$\mathcal{H}_{m,n}^K$	58	244	12	600
$\mathcal{L}_{m,n}^K$	17	80	12	315
$\mathcal{T}_{m,n}^K$	12	48	12	235

The smallest known orders of the old examples and of the newly obtained graphs in the various lattices considered here are given in the table below, from which it can be easily observed that the topology of the surface carrying the lattice (especially of the Klein bottle) is really useful in reducing the order of our examples (for details see the mentioned references).

We conclude this paper with the following open problem.

Problem. Does any $\mathbf{P}_1^2$-graph exist in $\mathcal{T}$?

References

[1] D. P. Agrawal, Graph theoretical analysis and designs of multistage interconnection networks, *IEEE Transaction on Computers,* **32** (1983).

[2] A. Dino and T. Zamfirescu, On longest paths in triangular lattice graphs, *Util. Math.* **89** (2012), 269–273.

[3] A. Dino, C. T. Zamfirescu and T. I. Zamfirescu, Lattice graphs with non-concurrent longest cycles, *Rendiconti del Seminario Matematico della Università di Padova*, to appear.

[4] T. Gallai, Problem 4, in: *Theory of Graphs*, *Proc. Tihany* 1966 (ed: P. Erdös & G. Katona), Academic Press, New York, 1968, 362.

[5] J. P. Hayes, A graph model for fault tolerent computer systems , *IEEE Transaction on Computers* **34** (1985).

[6] S. Jendrol and Z. Skupień, Exact number of longest cycles with empty intersection, *Europ. J. Combin.* **18** (1974), 575–578.

[7] F. Nadeem, A. Shabbir, T. Zamfirescu, Planar lattice graphs with Gallai's property, *Graphs Combin.* **29** (2013), 1523–1529.

[8] J.-H. Park, H.-C. Kim, H.-S. Lim, Fault-hamiltonicity of hypercube-like interconnection networks, *Proc. of IEEE International Parallel and Distributed Processing Symposium IPDPS 2005*, Denver, Apr. 2005.

[9] J.-H. Park, H.-S. Lim, H.-C. Kim, Panconnectivity and pancyclicity of hypercube-like interconnection networks with faulty elements, *Theoretical Computer Science* **377** (2007), 170–180.

[10] W. Schmitz, Über längste Wege und Kreise in Graphen, *Rend. Sem. Mat. Univ. Padova* **53** (1975), 97–103.

[11] A. Shabbir and T. Zamfirescu, Gallai's property for graphs in lattices on the torus and Möbius strip, *Periodica Mathematica Hungarica*, to appear.

[12] A. Shabbir, C. T. Zamfirescu and T. I. Zamfirescu, Intersecting longest paths and longest cycles: A survey, *Electron. J. Graph Theory Appl.* **1**(1) (2013), 56–76.

[13] A. Shabbir and T. Zamfirescu, Highly non-concurrent longest cycles in lattice graphs, *Discrete Math.* **313**(19) (2013), 1908–1914.

[14] H.-J. Voss, *Cycles and Bridges in Graphs*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1991.

[15] H. Walther, Über die Nichtexistenz eines Knotenpunktes, durch den alle längsten Wege eines Graphen gehen, *J. Combin. Theory* **6** (1969), 1–6.

[16] H. Walther, H.-J. Voss, *Über Kreise in Graphen*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1974.

[17] T. Zamfirescu, A two-connected planar graph without concurrent longest paths, *J. Combin. Theory* B **13** (1972), 116–121.

[18] T. Zamfirescu, On longest paths and circuits in graphs, *Math. Scand.* **38** (1976), 211–239.

[19] T. Zamfirescu, Intersecting longest paths or cycles: A short survey, *Analele Univ. Craiova, Seria Mat. Inf.* **28** (2001), 1–9.