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Note on chromatic polynomials of the threshold graphs

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Abstract

Let G be a threshold graph. In this paper, we give, in first hand, a formula relating the chromatic polynomial of $\overline{G}$ (the complement of G) to the chromatic polynomial of G. In second hand, we express the chromatic polynomials of G and $\overline{G}$ in terms of the generalized Bell polynomials.

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1. Introduction

Recall that for a given graph $G = (V, E)$ of order n, a λ-coloring of G, $\lambda \in \mathbb{N}$, is a mapping $f : V \to \{1, 2, \ldots, \lambda\}$ where $f(u) \neq f(v)$ whenever the edge $uv \in E$. If such mapping f exists, the graph G is said to be λ-colorable, the chromatic number of G, denoted by $\chi(G)$, is the minimal value of λ for which the graph G is λ-colorable and the number of λ-colorings of G is called the chromatic polynomial $P(G, \lambda)$, see [4, 8, 9]. This paper is concerned with the chromatic polynomials and the sigma polynomials of threshold graphs. These graphs was introduced by Chvátal et al. [3] and Henderson et al. [5] and have numerous applications, see for example [6]. They can be constructed from an isolated vertex by repeated applications to addition a vertex to be an isolated vertex or a dominating vertex to the graph. From this definition, it follows that the complement graph $\overline{G}$ of G is also a threshold graph. The object of our investigations in this paper is, in first hand, to deduce for a given threshold graph G the chromatic polynomial $P(\overline{G}, \lambda)$

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from the chromatic polynomial $P(G,\lambda)$, and, in second hand to express the chromatic polynomials of G and $\overline{G}$ in terms of the generalized Bell polynomials $\mathcal{B}_{\mathbf{r},\mathbf{s}}(x)$ defined by Carlitz and studied extensively by Blasiak, Penson and Solomon, see [1, 2]. Below, we use the following notation:
$(\lambda)_n = \lambda(\lambda-1)\cdots(\lambda-n+1)$ if $n \geq 1$ and $(\lambda)_0 = 1$,
G_n is a graph of order n and without edges with the convention $P(G_0,\lambda) = 1$.

2. Chromatic polynomials of threshold graphs

Upon using the definition of threshold graphs G and $\overline{G}$, the following theorem gives simple expressions for their chromatic polynomials.

Theorem 2.1. *Let $(G_n, n \geq 1)$ be a sequence of threshold graphs and G_n has n vertices. Then*

$$\begin{aligned} P(G_n,\lambda) &= \lambda(\lambda - i_{n-1})\cdots(\lambda - i_{n-1} - \cdots - i_1), \\ P(\overline{G}_n,\lambda) &= \lambda(\lambda - j_{n-1})\cdots(\lambda - j_{n-1} - \cdots - j_1), \end{aligned}$$

for some integers $i_1,\ldots,i_{n-1},j_1,\ldots,j_{n-1} \in \{0,1\}$ such that $i_k + j_k = 1,\ \ k = 1,2,\ldots,n-1$. Furthermore, we have

$$\begin{aligned} P(G_n,\lambda) &= \lambda^{r_0}(\lambda-1)^{r_1}\cdots(\lambda-n+1)^{r_{n-1}}, \\ P(\overline{G}_n,\lambda) &= \lambda^{s_0}(\lambda-1)^{s_1}\cdots(\lambda-n+1)^{s_{n-1}}, \end{aligned}$$

where

$$r_k = \sum_{l=1}^{n-1} = \delta_{(k,0)} + \delta_{(i_{n-l}+\cdots+i_{n-1},k)},\ \ s_k = \sum_{l=1}^{n-1} = \delta_{(k,0)} + \delta_{(j_{n-l}+\cdots+j_{n-1},k)},\ \ k = 0,\ldots,n-1,$$

with $i_0 = j_0 = 0$ and δ is the Kronecker delta, i.e. $\delta_{(i,j)} = 1$ if $i = j$ and $\delta_{(i,j)} = 0$ if $i \neq j$.

Proof. By construction, G_n is the graph G_{n-1} plus a vertex x_n such that x_n is an isolated vertex or a dominating vertex. Similarly, by construction, $\overline{G}_n$ is the graph $\overline{G}_{n-1}$ plus a vertex x_n such that x_n is an isolated vertex (if it is a dominating vertex in G_n) or a dominating vertex (if it is an isolated vertex in G_n). Then, for $n \geq 2$ we get

$$P(G_n,\lambda) = \begin{cases} \lambda P(G_{n-1},\lambda), & \text{if } x_n \text{ is an isolated vertex in } G_n, \\ \lambda P(G_{n-1},\lambda-1), & \text{if } x_n \text{ is a dominating vertex in } G_n \end{cases}$$

and

$$P(\overline{G}_n,\lambda) = \begin{cases} \lambda P(\overline{G}_{n-1},\lambda-1), & \text{if } x_n \text{ is an isolated vertex in } G_n, \\ \lambda P(\overline{G}_{n-1},\lambda), & \text{if } x_n \text{ is a dominating vertex in } G_n \end{cases}$$

which can be written as

$$\begin{aligned} P(G_n,\lambda) &= \lambda P(G_{n-1},\lambda - i_{n-1}),\ \ n \geq 1,\ \ \ i_{n-1} \in \{0,1\}, \\ P(\overline{G}_n,\lambda) &= \lambda P(\overline{G}_{n-1},\lambda - j_{n-1}),\ \ n \geq 1,\ \ \ j_{n-1} = 1 - i_{n-1} \in \{0,1\}. \end{aligned}$$

Thus, the desired expressions follow.
Let now r_k be the order of multiplicity of a number k of the zeros of $P(G_n, \lambda)$:

$$\{i_0, i_{n-1}, i_{n-1} + i_{n-2}, \ldots, i_{n-1} + i_{n-2} + \cdots i_1\}.$$

It is obvious that $r_k = \delta_{(k,0)} + \sum_{l=1}^{n-1} \delta_{(i_{n-l}+\cdots+i_{n-1},\, k)}$. Similarly we get $s_k = \delta_{(k,0)} + \sum_{l=1}^{n-1} \delta_{(j_1+\cdots+j_l,\, k)}$.

□

For a given threshold graph G with known chromatic polynomial $P(G, \lambda)$, the following theorem gives the explicit expression of the chromatic polynomial $P(\overline{G}, \lambda)$ of $\overline{G}$.

Theorem 2.2. *Let $(G_n; n \geq 1)$ be a sequence of threshold graphs and G_n has n vertices such that*

$$P(G_n, \lambda) = \lambda^{r_0} (\lambda - 1)^{r_1} \cdots (\lambda - n + 1)^{r_{n-1}},$$

for some non-negative integers $r_0, r_1, \ldots, r_{n-1}$ such that $r_0 + r_1 + \cdots + r_{n-1} = n$.
Then, the following holds

$$P(\overline{G}_n, \lambda) = \lambda (\lambda - r_0 + 1)(\lambda - r_0 - r_1 + 2) \cdots (\lambda - r_0 - \cdots - r_{n-2} + n - 1).$$

Proof. From Theorem 2.1, the chromatic polynomial $P(G_n, \lambda)$ can be written as$P(G_n, \lambda) = \lambda^{r_0} (\lambda - 1)^{r_1} \cdots (\lambda - n + 1)^{r_{n-1}},\ n \geq 1$. We prove that the chromatic polynomial $P(\overline{G}_n, \lambda)$ must be as follows

$$P(\overline{G}_n, \lambda) = \lambda (\lambda - r_0 + 1)(\lambda - r_0 - r_1 + 2) \cdots (\lambda - r_0 - \cdots - r_{n-2} + n - 1).$$

Indeed, by induction on n. The case $n = 1$ is obvious and assume

$$P(\overline{G}_n, \lambda) = \lambda (\lambda - r_0 + 1)(\lambda - r_0 - r_1 + 2) \cdots (\lambda - r_0 - \cdots - r_{n-2} + n - 1).$$

If x_{n+1} is an isolated vertex in G_{n+1}, necessarily

$$\begin{aligned} P(G_{n+1}, \lambda) &= \lambda P(G_n, \lambda) \\ &= \lambda^{r_0+1} (\lambda - 1)^{r_1} \cdots (\lambda - n + 1)^{r_{n-1}} \\ &= \lambda^{s_0} (\lambda - 1)^{s_1} \cdots (\lambda - n)^{s_n} \end{aligned}$$

where $s_0 = r_0 + 1,\ s_j = r_j\ (1 \leq j \leq n - 1),\ s_n = 0$.
But since $\lambda - 1 = \lambda - s_0 - \cdots - s_n + n$, we get

$$\begin{aligned} P(\overline{G}_{n+1}, \lambda) &= \lambda P(\overline{G}_n, \lambda - 1) \\ &= \lambda (\lambda - 1)(\lambda - r_0)(\lambda - r_0 - r_1 + 1) \cdots (\lambda - r_0 - \cdots - r_{n-2} + n - 2) \\ &= \lambda (\lambda - s_0 + 1) \cdots (\lambda - s_0 - \cdots - s_{n-2} + n - 1)(\lambda - s_0 - \cdots - s_n + n) \end{aligned}$$

If x_{n+1} is a dominating vertex in G_{n+1}, necessarily

$$\begin{aligned} P(G_{n+1}, \lambda) &= \lambda P(G_n, \lambda - 1) \\ &= \lambda (\lambda - 1)^{r_0} (\lambda - 2)^{r_1} \cdots (\lambda - n)^{r_{n-1}} = \lambda^{s_0} (\lambda - 1)^{s_1} \cdots (\lambda - n)^{s_n} \end{aligned}$$

where $s_0 = 1,\ s_j = r_{j-1}\ (1 \leq j \leq n)$, and since $\lambda = \lambda - s_0 + 1$ we get

$$\begin{aligned} P\left(\overline{G}_{n+1}, \lambda\right) &= \lambda P\left(\overline{G}_n, \lambda\right) \\ &= \lambda^2 (\lambda - r_0 + 1)(\lambda - r_0 - r_1 + 2) \cdots (\lambda - r_0 - \cdots - r_{n-2} + n - 1) \\ &= \lambda (\lambda - s_0 + 1)(\lambda - s_0 - s_1 + 2) \cdots (\lambda - s_0 - s_1 - \cdots - s_{n-1} + n). \end{aligned}$$

So, the induction is true and produces the desired result. □

Corollary 2.1. *Let G be a threshold graph of n vertices. Then*

$$\chi(G_n) + \chi(\overline{G}_n) = n + 1.$$

Proof. It is easy to see that

$$\chi(G_n) = 1 + \sum_{k=1}^{n-1} i_k \text{ and } \chi(\overline{G}_n) = 1 + \sum_{k=1}^{n-1} j_k,$$

which show that $\chi(G_n) + \chi(\overline{G}_n) = 2 + \sum_{k=1}^{n-1} (i_k + j_k) = n + 1.$ □

Corollary 2.2. *Let G be a threshold graph of n vertices. Then, the sum of all zeros of the polynomial $P(G, \lambda) P\left(\overline{G}, \lambda\right)$ equals $\frac{n(n-1)}{2}$.*

Proof. Setting

$$\begin{aligned} P(G, \lambda) &= \lambda^{r_0} (\lambda - 1)^{r_1} \cdots (\lambda - n + 1)^{r_{n-1}}, \\ P\left(\overline{G}, \lambda\right) &= \lambda^{s_0} (\lambda - 1)^{s_1} \cdots (\lambda - n + 1)^{s_{n-1}}, \end{aligned}$$

for some non-negative integers $r_0, r_1, \ldots, r_{n-1}$ and $s_0, s_1, \ldots, s_{n-1}$ such that

$$r_0 + r_1 + \cdots + r_{n-1} = s_0 + s_1 + \cdots + s_{n-1} = n.$$

Since $\sum_{j=0}^{n-1} j r_j$ (resp. $\sum_{j=0}^{n-1} j s_j$) is the sum of all zeros of $P(G, \lambda)$ (resp. $P\left(\overline{G}, \lambda\right)$), then, from Theorem 2.2 the sum of all zeros of the polynomial

$$P(G, \lambda) P\left(\overline{G}, \lambda\right) = \lambda^{r_0 + s_0} (\lambda - 1)^{r_1 + s_1} \cdots (\lambda - n + 1)^{r_{n-1} + s_{n-1}}$$

is to be

$$\begin{aligned} \sum_{j=0}^{n-1} j (r_j + s_j) &= \sum_{j=0}^{n-1} j r_j + 0 + (r_0 - 1) + \cdots + (r_0 + \cdots + r_{n-2} - (n-1)) \\ &= \sum_{j=0}^{n-1} j r_j + \sum_{j=0}^{n-1} (n - 1 - j) r_j - \sum_{j=0}^{n-1} j \\ &= (n-1) \sum_{j=0}^{n-1} r_j - \frac{n(n-1)}{2} \\ &= \frac{n(n-1)}{2}. \end{aligned}$$

□

3. The generalized Bell polynomials and threshold graphs

To give some connections between the chromatic polynomials and the generalized Bell polynomials (see [7]), let $r_0, \ldots, r_{n-1}$ and $s_0, \ldots, s_{n-1}$ be non-negative integers and set $\mathbf{r} = (r_0, \ldots, r_{n-1})$, $\mathbf{s} = (s_0, \ldots, s_{n-1})$. Recall that the generalized Stirling numbers of the second kind $S_{\mathbf{r},\mathbf{s}}(n,k)$ are defined by

$$(x^{r_{n-1}} D^{s_{n-1}}) \cdots (x^{r_0} D^{s_0}) = x^{d_n} \sum_{k=s_0}^{s_0+\cdots+s_{n-1}} S_{\mathbf{r},\mathbf{s}}(n,k)\, x^k D^k,$$

and the so-called generalized Bell polynomials $\mathcal{B}_{\mathbf{r},\mathbf{s}}(x)$ are to be

$$(x^{r_{n-1}} D^{s_{n-1}}) \cdots (x^{r_0} D^{s_0}) \exp(x) = x^{d_n} \exp(x)\, \mathcal{B}_{\mathbf{r},\mathbf{s}}(x),$$

where

$$\mathcal{B}_{\mathbf{r},\mathbf{s}}(x) = \sum_{k=s_0}^{s_0+\cdots+s_{n-1}} S_{\mathbf{r},\mathbf{s}}(n,k)\, x^k \text{ and } d_n = \sum_{i=0}^{n-1} (r_i - s_i) \geq 0.$$

By choosing $f(x) = x^\lambda$ in the identity

$$(x^{r_{n-1}} D^{s_{n-1}}) \cdots (x^{r_0} D^{s_0}) f(x) = x^{d_n} \sum_{k=s_0}^{s_0+\cdots+s_{n-1}} S_{\mathbf{r},\mathbf{s}}(n,k)\, x^k D^k f(x),$$

we obtain

$$(\lambda + d_1)_{s_1} \cdots (\lambda + d_{n-1})_{s_{n-1}} = \sum_{k=s_0}^{s_0+\cdots+s_{n-1}} S_{\mathbf{r},\mathbf{s}}(n,k)\, (\lambda)_k,$$

where $d_j = \sum_{i=0}^{j-1} (r_i - s_i)$, $j \geq 1$, see a combinatorial proof in [7].

For a given sequence $(G_n, n \geq 1)$ of threshold graphs with G_n has n vertices, we prove in this section that the sequence of the sigma polynomials $(\sigma(G_n, x), n \geq 1)$ can be expressed in terms of the generalized Bell polynomials. The useful representation of the chromatic polynomial of a given graph $G = (V, E)$ used here is

$$P(G, \lambda) = \sum_{k=\chi(G)}^{|V|} \alpha_k(G)\, (\lambda)_k,$$

where $|V|$ is the number of vertices of V and $\alpha_i(G)$ is the number of ways of partitioning V into i nonempty sets. The sigma polynomial $\sigma(G, \lambda)$ of a graph $G = (V, E)$ is defined by

$$\sigma(G, x) = \sum_{k=\chi(G)}^{|V|} \alpha_k(G)\, x^k.$$

Lemma 3.1. *It holds*

$$\sigma(G,x) = \exp(-x) \sum_{j\geq 0} P(G,j) \frac{x^j}{j!}.$$

Proof. From the definition of the chromatic polynomial of G we get

$$\begin{aligned}
\exp(-x) \sum_{j\geq 0} P(G,j) \frac{x^j}{j!} &= \exp(-x) \sum_{j\geq 0} \left(\sum_{k=\chi(G)}^{|V|} \alpha_k(G)(j)_k \right) \frac{x^j}{j!} \\
&= \exp(-x) \sum_{k=\chi(G)}^{|V|} \alpha_k(G) x^k \sum_{j\geq k} \frac{x^{j-k}}{(j-k)!} \\
&= \sum_{k=\chi(G)}^{|V|} \alpha_k(G) x^k \\
&= \sigma(G,x).
\end{aligned}$$

□

The following theorem shows that some generalized Bell polynomials can be interpreted by the chromatic polynomials for the threshold graphs and gives another version of Theorem 2.2.

Theorem 3.1. *It holds*

$$\begin{aligned}
P(G_n,\lambda) &= \sum_{k=\chi(G_n)}^{n} S_{\mathbf{r},\mathbf{s}}(n, k-\chi(G_n))(\lambda)_k, \\
P(\overline{G}_n,\lambda) &= \sum_{k=\chi(\overline{G}_n)}^{n} S_{\mathbf{r},\overline{\mathbf{s}}}(n, k-\chi(\overline{G}_n))(\lambda)_k,
\end{aligned}$$

where

$$\mathbf{r} = (1,\ldots,1),\ \mathbf{s} = (j_0,\ldots,j_{n-1}),\ \overline{\mathbf{s}} = (i_0,\ldots,i_{n-1}).$$

Proof. For $n \geq 1$ we have $P(G_n,\lambda) = \lambda P(G_{n-1}, \lambda - i_{n-1})$. Then, by Lemma 3.1 and the proof of Theorem 2.1 we get $\exp(x)\sigma(G_n,x) = \sum_{k\geq 0} P(G_n,k)\frac{x^k}{k!}$.

For $i_{n-1} = 0$ we get

$$\exp(x)\sigma(G_n,x) = \sum_{k\geq 0} kP(G_{n-1},k)\frac{x^k}{k!} = x\frac{d}{dx}\left(\exp(x)\sigma(G_{n-1},x)\right),$$

and for $i_{n-1} = 1$ we get

$$\exp(x)\sigma(G_n,x) = \sum_{k\geq 0} kP(G_{n-1},k-1)\frac{x^k}{k!} = x\exp(x)\sigma(G_{n-1},x).$$

Then, an equivalent expression is to be

$$\exp(x)\,\sigma(G_n,x) = xD^{j_{n-1}}\left(\exp(x)\,\sigma(G_{n-1},x)\right), \;\; j_{n-1} = 1 - i_{n-1} \in \{0,1\}.$$

and this can be rewritten recursively as

$$\begin{aligned}
\exp(x)\,\sigma(G_n,x) &= xD^{j_{n-1}}\left(\exp(x)\,\sigma(G_{n-1},x)\right)\\
&\vdots\\
&= \left(xD^{j_{n-1}}\right)\left(xD^{j_{n-2}}\right)\cdots\left(xD^{j_1}\right)\left(\exp(x)\,\sigma(G_1,x)\right)\\
&= \left(xD^{j_{n-1}}\right)\left(xD^{j_{n-2}}\right)\cdots\left(xD^{j_1}\right)\left(x\exp(x)\right)\\
&= \left(xD^{j_{n-1}}\right)\left(xD^{j_{n-2}}\right)\cdots\left(xD^{j_0}\right)\exp(x)\\
&= \exp(x)\,x^{d_n}\mathcal{B}_{\mathbf{r},\mathbf{s}}(x)\\
&= \exp(x)\,x^{d_n}\sum_{k=s_0}^{j_0+\cdots+j_{n-1}} S_{\mathbf{r},\mathbf{s}}(n,k)\,x^k\\
&= \exp(x)\,x^{d_n}\sum_{k=0}^{n-\chi(G_n)} S_{\mathbf{r},\mathbf{s}}(n,k)\,x^k
\end{aligned}$$

where $d_n = \sum_{k=0}^{n-1}(1-j_k) = n - \sum_{k=1}^{n-1} j_k = n + 1 - \chi\left(\overline{G}_n\right) = \chi(G_n)$.

So we get $\sigma(G_n,x) = \sum_{k=0}^{n-\chi(G_n)} S_{\mathbf{r},\mathbf{s}}(n,k)\,x^{k+\chi(G_n)}$ which gives

$$P(G_n,\lambda) = \sum_{k=0}^{n-\chi(G_n)} S_{\mathbf{r},\mathbf{s}}(n,k)\,(\lambda)_{k+\chi(G_n)}.$$

The correspondent expression of $P\left(\overline{G}_n,\lambda\right)$ can obtained by symmetry. □

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