Electronic Journal of Graph Theory and Applications **9** (1) (2021), 95–103

Electronic Journal of Graph Theory and Applications

Roman domination in oriented trees

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Abstract

Let $D = (V, A)$ be a digraph of order $n = |V|$. A *Roman dominating function* of a digraph D is a function $f : V \longrightarrow \{0, 1, 2\}$ such that every vertex u for which $f(u) = 0$ has an in-neighbor v for which $f(v) = 2$. The weight of a *Roman dominating function* is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a *Roman dominating function* of a digraph D is called the *Roman domination number* of D, denoted by $\gamma_R(D)$. In this paper, we characterize oriented trees T satisfying $\gamma_R(T) + \Delta^+(T) = n + 1$.

Keywords: Roman domination, digraph, oriented tree
2010 Mathematics Subject Classification: 05C20, 05C69
DOI: 10.5614/ejgta.2021.9.1.9

1. Introduction

The *digraph* $D = (V, A)$ of order $n = |V|$ considered here has no loops and no multiple arcs (but pairs of opposite arcs are allowed). If $(u, v) \in A$, then we write $u \to v$ and we say that v is an *out-neighbor* (*successor*) of u, (or u dominates v) and u is an *in-neighbor* (*predecessor*) of v, and if $(u, v) \notin A$, we write $u \nrightarrow v$. If $u \to v$ and $v \to u$, we say that (u, v) is a *symmetrical* arc and we write $v \longleftrightarrow u$. If $u \to v$ and $v \nrightarrow u$, we say that (u, v) is an *asymmetrical* arc and we write $u \longmapsto v$. Also, if u and v are non *adjacent* ($u \nrightarrow v$ and $v \nrightarrow u$), then we write $u \nleftrightarrow v$. Let $S \subseteq V$ be a non-empty set and u a vertex in $V - S$. If u is in-neighbor of each vertex of S, then we use the notation $u \Longrightarrow S$.

A digraph $H = (U, B)$ is the *subdigraph* of D whenever $U \subseteq V(D)$ and $B \subseteq A(D)$, the subdigraph induced by U is denoted by $\langle U \rangle$. If $U = V(D)$, the subdigraph is said to be *spanning*.

Received: 3 March 2018, Revised: 14 December 2020, Accepted: 30 December 2020.

An *oriented graph* is a digraph $D = (V, A)$ containing no symmetric pair of arcs. That can be obtained from a graph G by assigning a direction to each edge of G. The resulting digraph D is called an orientation of G. Thus, if D is an oriented graph, then for every pair u and v of distinct vertices of D, either (u, v) or (v, u) is an arc of D, but not both.

A *tree* is a connected graph without cycles. Also, an *oriented tree* is a connected oriented graph without oriented cycle. Note that an oriented tree with n vertices has $n - 1$ arcs.

Define the out-neighborhood of a vertex $v \in V$ as $N_D^+(v) = \{u \in V : v \to u\}$, the in-neighborhood of v as $N_D^-(v) = \{w \in V : w \to v\}$. We define $N_D^+[v] = N_D^+(v) \cup \{v\}$ and $N_D^-[v] = N_D^-(v) \cup \{v\}$. Also, for a subset $S \subseteq V$, $N_D^+(S) = \cup_{v \in S} N_D^+(v)$ and $N_D^+[S] = N_D^+(S) \cup S$. The definition of $N_D^-(S)$, $N_D^-[S]$ are similar. The out-degree of a vertex v in D is defined as $d_D^+(v) = |N_D^+(v)|$. The maximum (respectively, minimum) out-degree of D is given by $\Delta^+(D) = \max\left\{d_D^+(v) : v \in V\right\}$ (respectively, $\delta^+(D) = \min\left\{d_D^+(v) : v \in V\right\}$). Similarly, the in-degree of v is $d_D^-(v) = |N_D^-(v)|$ and maximum (respectively, minimum) in-degree of D, $\Delta^-(D) = \max\left\{d_D^-(v) : v \in V\right\}$ (respectively, $\delta^-(D) = \min\left\{d_D^-(v) : v \in V\right\}$). For a vertex v in the set S, the out-private neighbors of v with respect to S is the set $opn[v, S] = N_D^+[v] - N_D^+[S - \{v\}]$. For the terminology and notations not defined here, we refer the reader to the book by Haynes et al. [6].

A *Roman dominating* function (RDF) on a digraph $D = (V, A)$ is a function $f : V \longrightarrow \{0, 1, 2\}$ such that every vertex u for which $f(u) = 0$ is a successor of some vertex v for which $f(v) = 2$. The weight of a *Roman dominating* function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a *Roman dominating* function on a digraph D is called the *Roman domination* number of D, denoted by $\gamma_R(D)$. Let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{v \in V : f(v) = i\}$ for $i = 0, 1, 2$. Note that there exists a 1-1 correspondence between the RDF f and the ordered partition (V_0, V_1, V_2) of V. Thus, we will write $f = (V_0, V_1, V_2)$. So, a function $f = (V_0, V_1, V_2)$ is a *Roman dominating* function (RDF) if $V_0 \subseteq N^+[V_2]$. The weight of f is $f(V) = \sum_{v \in V} f(v) = |V_1| + 2|V_2|$, and we say that a RDF f is a γ_R-function if $f(V) = \gamma_R(D)$. The Roman dominating function for graphs has been introduced by Cockayne et al. [1] and was motivated by an article in Scientific American by Ian Stewart entitled "Defend the Roman Empire" [11]. For more details on Roman domination and its variants, the reader can be referred to [2, 3, 8, 9, 12, 13].

The concept of Roman dominating function for digraphs was introduced by Kamaraj and Jakkammal in [4], in their paper the authors gave the following upper bound for the parameter $\gamma_R(D)$ and some others results.

$$\gamma_R(D) \leq n - \Delta^+(D) + 1. \tag{1}$$

Later, in [4, 10] the authors gave some bounds and characterization of digraphs for small values of $\gamma_R(D)$. In [7], the authors gave extremal oriented k-out-regular graphs with $1 \leq k \leq n - 1$, and tournaments attaining the upper bound in (1), also they showed that the problem of deciding whether an oriented graph D has $\gamma_R(D) = n - \Delta^+(D) + 1$ is $\mathcal{CO} - \mathcal{NP}$-complete. In this paper, we characterize all oriented trees T satisfying $\gamma_R(T) = n - \Delta^+(T) + 1$.

2. Characterization of Oriented Trees T with $\gamma_R(T) = n - \Delta^+(T) + 1$

In this section, we give a characterization of oriented trees T with $\gamma_R(T) = n - \Delta^+(T) + 1$. Recall that the known upper bound in (1) is given by Kamaraj and Jakkammal in [5].

Proposition 2.1. *[5] If D is a digraph of order n, with maximum out-degree $\Delta^+(D)$. Then*

$$\gamma_R(D) \leq n - \Delta^+(D) + 1.$$

Proposition 2.2. *[10] If D is any digraph of order n, then $\gamma_R(D) < n$ if and only if $\Delta^+(D) \geq 2$.*

Proposition 2.3. *If T is any oriented tree of order $n \geq 2$, then $\gamma_R(T) = n$ if and only if $\Delta^+(T) = 1$.*

For the next result, let X be the set of vertices of an oriented tree T with out-degree 2, i.e., $X = \{x \in V : d^+(x) = 2\}$.

Proposition 2.4. *Let T be an oriented tree of order $n \geq 2$ with maximum out-degree $\Delta^+(T)$ and X be a set of vertices of out-degree 2. Then $\gamma_R(T) = n - 1$ if and only if $\Delta^+(T) = 2$, in addition if $|X| \geq 2$ then X has an unique vertex, say z satisfies $N_T^+[x] \cap N_T^+[y] = \{z\}$, for every pair of vertices x, y in X and x or y may be z (see Figure 1).*

Proof. Let T be an oriented tree of order $n \geq 2$. Assume that $\gamma_R(T) = n - 1$, and suppose to the contrary that $\Delta^+(T) \neq 2$. By Observation 2.3, $\Delta^+(T) \geq 3$ and by Proposition 2.1, $\gamma_R(T) \leq n - \Delta^+(T) + 1$, a contradiction. Thus $\Delta^+(T) = 2$ and $|X| \geq 1$. Now, assume to the contrary that $|X| \geq 2$ and there exists at least two vertices, say x and y in X_2 such that $\left|N_T^+[x] \cap N_T^+[y]\right| \neq 1$. Since T is an oriented tree, $N_T^+[x] \cap N_T^+[y] = \emptyset$. The function $f = (V_0, V_1, V_2)$, where $V_1 = V - \left(N_T^+[x] \cup N_T^+[y]\right)$ and $V_2 = \{x, y\}$, is an RDF of T, so $\gamma_R(T) \leq |V_1| + 2|V_2| = n - 2$, a contradiction.

Conversely. Let $x \in X$ (in case $z \in X$, x may be z). Clearly by construction of T that the function $f = (N_T^+(x), V - N_T^+[x], \{x\})$ is a $\gamma_R(T)$-function with $\gamma_R(T) = |V_1| + 2|V_2| = (n-3) + 2 = n - 1$. □

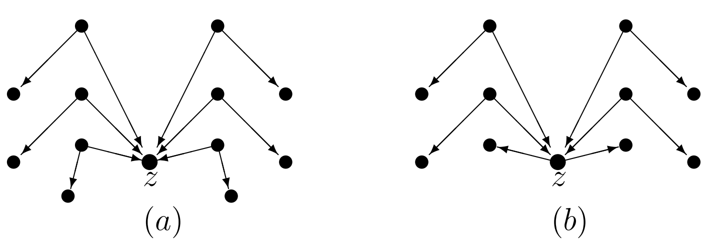

Figure 1. (a) T with $z \notin X$. (b) T with $z \in X$.

In [7], Ouldrabah et al. gave necessary conditions for digraphs D such that $\gamma_R(D) = n - \Delta^+(D) + 1$. Since we are interested in oriented trees, we state their results only for the class of oriented trees. From now, T will be an oriented tree of order n with $\gamma_R(T) = n - \Delta^+(T) + 1$, x will be a vertex of maximum out-degree $\Delta^+(T) \geq 3$ and $\overline{N}^+[x] = V(T) - N^+[x]$.

The next three Lemmas contain main structure properties on oriented tree T with $\gamma_R(T) = n - \Delta^+(T) + 1$, which we will need in order to prove the main results.

Lemma 2.1. *[7] Let T be an oriented tree of order n and let x be a vertex with maximum out-degree $\Delta^+(T) \geq 3$. If $\gamma_R(T) = n - \Delta^+(T) + 1$ then*

1. *every vertex in $\overline{N}^+[x]$ has at most one out-neighbor in $\left\langle \overline{N}^+[x] \right\rangle$, and*
2. *every vertex in $N_T^+(x)$ has at most two out-neighbors in T.*

Lemma 2.2. *Let T be an oriented tree of order n and maximum out-degree $\Delta^+(T) \geq 3$. If $\gamma_R(T) = n - \Delta^+(T) + 1$ then T has a unique vertex with out-degree at least three.*

Proof. Let T be an oriented tree of order n and maximum out-degree $\Delta^+(T) \geq 3$. Suppose there are two vertices x and y in T with out-degree at least 3. Without loss of generality, we can assume that $d^+(x) = \Delta^+(T)$. If $y \in \overline{N}^+[x]$ since T is an oriented tree, then y has at least two out-neighbors that are in $\overline{N}^+[x]$, that is $\left| N^+(y) \cap \overline{N}^+[x] \right| \geq 2$ a contradiction with Lemma 2.1. Thus y must be is in $N^+(x)$. But in this case, since T is an oriented tree, y has at least three out-neighbors vertices in $\overline{N}^+[x]$, that is $\left| N^+(y) \cap \overline{N}^+[x] \right| \geq 3$, again a contradiction with Lemma 2.1. □

Define the following subsets:

$$\begin{array}{r} Y = \{y \in N^+(x) : d^+(y) = 2\}, \\ Z = \{z \in N^-(x) : d^+(z) = 2\}, \\ U = \left\{u \in \overline{N}^+[x] - Z : d^+(u) = 2\right\}. \end{array} \tag{6}$$

It is clear that U, Y and Z form a partition of the set X. Also we define the two following subsets:

$$\begin{array}{r} W = (N^+(x) - Y) \cap N^+(U), \\ R = N^+(x) - (Y \cup W). \end{array} \tag{7}$$

For illustration, see the oriented tree T in Figure 2.

Lemma 2.3. *Let T be an oriented tree of order n and maximum out-degree $\Delta^+(T) \geq 3$. If $\gamma_R(T) = n - \Delta^+(T) + 1$ then $|R| \geq 1$, and in addition if $|R| = 1$ then $Z = \emptyset$.*

Proof. Let T be an oriented tree of order $n \geq 2$ with $\gamma_R(T) = n - \Delta^+(T) + 1$, and x a vertex of T satisfying $d^+(x) = \Delta^+(T)$. From Lemma 2.1, we have $d^+(v) \leq 2$ for every vertex v in $T - x$. If $X = \emptyset$, then $\Delta^+(\langle T - x \rangle) \leq 1$, and the condition is done. Assume now that $X = Z \cup Y \cup U \neq \varnothing$. Since $Y \cup W \subseteq N^+(x)$ and T is an oriented tree we have

$$\Delta^+(T) = \left|N^+(x)\right| = |Y| + |W| + |R|,$$

and so

$$\begin{aligned}\left|N^+[U \cup Y]\right| &= \left|N^+[U]\right| + \left|N^+[Y]\right| - \left|N^+(U) \cap Y\right| \\ &= 2|U| + |W| + 3|Y| \\ &= 2|Y| + 2|U| + \Delta^+(T) - |R|.\end{aligned}$$

First, we show that $|R| \geq 1$. Assume to the contrary that $R = \emptyset$, then

$$\Delta^+(T) = \left|N^+(x)\right| = |Y| + |W|.$$

The function $f = (V_0, V_1, V_2)$, where

$$V_1 = V(T) - \left(N^+[U \cup Y]\right) \text{ and } V_2 = U \cup Y$$

is an RDF of T. Hence,

$$\begin{aligned}\gamma_R(T) &\leq |V_1| + 2|V_2| \\ &= |V(T)| - \left(2|Y| + 2|U| + \Delta^+(T)\right) + 2(|U| + |Y|) \\ &= n - \Delta^+(T),\end{aligned}$$

a contradiction.

Now we must show that if $|R| = 1$ then $Z = \emptyset$. Suppose to the contrary that $|R| = 1$ and $Z \neq \emptyset$. Thus $\Delta^+(T) = |Y| + |W| + 1$. The function $f = (V_0, V_1, V_2)$, where

$$V_1 = V(T) - \left(N^+[U \cup Y \cup \{z\}]\right) \text{ and } V_2 = U \cup Y \cup \{z\}$$

is an RDF of T where $z \in Z$. Hence,

$$\begin{aligned}\gamma_R(T) &\leq |V_1| + 2|V_2| \\ &= \left|V(T) - \left(N^+[U \cup Y \cup \{z\}]\right)\right| + 2|U \cup Y \cup \{z\}| \\ &= |V(T)| - \left(2|Y| + 2|U| + \Delta^+(T) - |R| + \left|N^+[z]\right|\right) + 2(|U| + |Y| + |\{z\}|) \\ &= n - \Delta^+(T),\end{aligned}$$

a contradiction. □

In the sequel, we provide a characterization of trees T of order $n \geq 2$ for which $\gamma_R(T) = n - \Delta^+(T) + 1$. For this purpose, we define the following families of trees. Recall that $X = \{x \in V : d^+(x) = 2\}$.

- $\mathcal{F}_1$ the family of all oriented trees T with $\Delta^+(T) = 1$.
- $\mathcal{F}_2$ be the family of all oriented trees T with $\Delta^+(T) = 2$, and T has an unique vertex, say z satisfies $X \subseteq N^-[z]$.
- $\mathcal{F}_3$ be the family of all oriented trees T with $\Delta^+(T) \geq 3$ satisfying the following conditions:

  (a) T has a unique vertex x with out-degree at least three.

  (b) $\Delta^+(\left\langle \overline{N}^+[x] \right\rangle) \leq 1$, and every vertex in $N_T^+(x)$ has at most two out-neighbors in T.

  (c) $|R| \geq 1$ and in addition if $|R| = 1$ then $Z = \emptyset$.

We begin by giving a known result on digraph that will be useful to prove the main result.

Proposition 2.5. *[5] Let $f = (V_0, V_1, V_2)$ be any $\gamma_R(D)$-function of a digraph D. Then*

(a) If $v \in V_1$, then $N^-(v) \cap V_2 = \emptyset$;

(b) Let $H = D[V_0 \cup V_2]$. Then each vertex $v \in V_2$ with $N^-(v) \cap V_2 \neq \emptyset$, has at least two private neighbors with respect to V_2 in the subdigraph H.

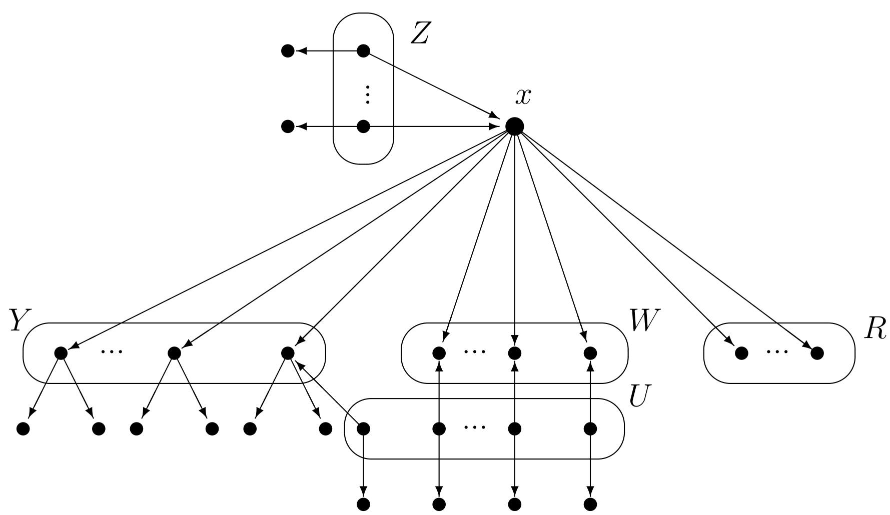

Figure 2. An example of oriented tree T which belongs to $\mathcal{F}_3$. Note that R is not empty, and the set Z must be empty whenever $|R| = 1$.

We now are ready to give our main result.

Theorem 2.1. *Let T be an oriented tree of order $n \geq 2$ with maximum out-degree $\Delta^+(T)$. Then*

$$\gamma_R(T) = n - \Delta^+(T) + 1 \text{ if and only if } T \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3.$$

Proof. Let T be an oriented tree of order $n \geq 2$ with maximum out-degree $\Delta^+(T)$. If $\Delta^+(T) = 1$ or 2, then by Observation 2.3 and Proposition 2.4 $\gamma_R(T) = n - \Delta^+(T) + 1$ if and only if $T \in \mathcal{F}_1$ or $T \in \mathcal{F}_2$, respectively. Hence let $\Delta^+(T) \geq 3$. Then from Lemmas 2.1, 2.2, and 2.3, $T \in \mathcal{F}_3$.

Conversely. Suppose $T \in \mathcal{F}_3$, by Condition (a) of the family $\mathcal{F}_3$, T has a unique vertex, say x with $d^+_T(x) = \Delta^+(T)$ and $\Delta^+(\left[\overline{N}^+[x] \cup R\right]) \leq 1$.

First we will show that there exists a $\gamma_R(T)$-function f with $f(x) = 2$. Suppose to the contrary that every $\gamma_R(D)$-function π, $\pi(x) \neq 2$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(T)$-function, if there exists a vertex v in R such that $f(v) = 0$, then there exists a vertex $w \in \overline{N}^+[x]$ such that $w \to v$ with $f(w) = 2$, and the function

$$g = (V_0 - \{v\}, V_1 \cup \{w, v\}, V_2 - \{w\})$$

is a $\gamma_R(T)$-function with $g(v) = 1$. And if there exists a vertex v in R such that $f(v) = 2$, then there exists a vertex $w \in \overline{N}^+[x]$ such that $v \to w$ with $f(w) = 0$ and the function

$$h = (V_0 - \{w\}, V_1 \cup \{w, v\}, V_2 - \{v\})$$

is a $\gamma_R(T)$-function with $h(v) = 1$. So, we can suppose without loss of generality that $f(v) = 1$ for every vertex v in R. Since $T \in \mathcal{F}$, we deduce from the Condition (c) that $|R| \geq 1$. Since $f(x) \neq 2$, we distinguish tow cases:

Case 1. $f(x) = 1$. If $|R| = 1$, then the function

$$f' = (V_0 \cup R, V_1 - (R \cup \{x\}), V_2 \cup \{x\})$$

is γ_R-function with $f'(x) = 2$, a contradiction with the fact that every $\gamma_R(D)$-function π, $\pi(x) \neq 2$. If $|R| \geq 2$, then f' is an RDF with $f'(V) < f(V)$, a contradiction.

Case 2. $f(x) = 0$, then there exists a vertex $u \in N^-_T(x)$, such that $u \to x$ with $f(u) = 2$, and since $x \to v$ with $f(v) = 1$, for every vertex v in E. We have three possibility:

Subcase 2.1. $|R| = 1$. Then

$$f' = (V_0 \cup R - \{x\}, (V_1 - R) \cup \{u\}, (V_2 - \{u\}) \cup \{x\})$$

is a $\gamma_R(T)$-function with $f'(x) = 2$, a contradiction.

Subcase 2.2. $|R| \geq 2$ and $Z = \emptyset$. Then

$$f' = (V_0 \cup R - \{x\}, (V_1 - R) \cup \{u\}, (V_2 - \{u\}) \cup \{x\})$$

is a RDF with $f'(V) < f(V)$, a contradiction.

Subcase 2.3. $|R| \geq 2$ and $Z \neq \emptyset$. For the case $u \notin Z$, like **Subcase 2.2**, we obtain a contradiction. Suppose now that $u \in Z$. If $|R| = 2$, then

$$f' = ((V_0 - \{x\}) \cup R, V_1 - R, V_2 \cup \{x\})$$

is a $\gamma_R(T)$-function with $f'(x) = 2$, a contradiction. And if $|R| > 2$, then f' is a RDF with $f'(V) < f(V)$, a contradiction. Hence, there exists a $\gamma_R(T)$-function $f(V) = (V_0, V_1, V_2)$ such V_2 contains x.

Now, we show that $\gamma_R(T) = n - \Delta^+(T) + 1$. Suppose to the contrary

$$\gamma_R(T) < n - \Delta^+(T) + 1.$$

We have,

$$|V_1| + 2|V_2| < n - \Delta^+(T) + 1 = |V_0| + |V_1| + |V_2| - \Delta^+(T) + 1.$$

This implies that,

$$|V_0| \geq |V_2| + \Delta^+(T). \tag{2}$$

It follows from Proposition 2.5 item (a), $N_D^+(x) \cap V_1 = \emptyset$. We define the two following subsets:

$$P = N_D^+(x) \cap V_2 \text{ and } Q = N_D^+(x) \cap V_0.$$

Let $|P| = p$ and $|Q| = q$, so $p + q = \Delta^+(T)$. Since $V_0 \subseteq N^+[V_2]$, clearly that $|V_2| \geq 2$. Moreover, every vertex in V_2, has at least an out-private neighbor in V_0 with respect to V.

First, assume that $P = \emptyset$. Since $|V_2| \geq 2$ we can deduce from (2) that there exists at least two vertices, say u,v in V_0 dominated by another vertex say x' in V_2 other than x. i.e., $x \nrightarrow u$, $x \nrightarrow v$ and $x' \Longrightarrow \{u, v\}$ which give $\Delta^+(\left\langle \overline{N}^+[x] \right\rangle) > 1$, a contradiction with the Condition (b) of the family $\mathcal{F}_3$, so $P \neq \emptyset$. On the one hand, by Proposition 2.5 item (b), each vertex in P has at least two out-private neighbors in V_0 with respect to V_2, and on the other hand, by Condition (b) of the family $\mathcal{F}$ each vertex in $N_T^+(x)$ has at most two out-neighbors in U, which implies that $|N^+(P) \cap V_0| = 2p$, since T is an oriented tree.

Now, we define the following subsets:

$$F = V_2 - (P \cup \{x\}) \text{ and } E = V_0 - \left(Q \cup \left(N^+(P) \cap V_0\right)\right). \tag{3}$$

So,

$$|E| = |V_0| - |Q| - \left|N^+(P) \cap V_0\right| = |V_0| - q - 2p, \tag{4}$$

It follows from (2), (3) and (4) that

$$\begin{aligned} |F| = |V_2| - (p+1) &\leq |V_0| - \Delta^+(T) - (p+1) \\ &\leq |E| + q + 2p - \Delta^+(T) - (p+1) \\ &= |E| - 1 < |E|. \end{aligned}$$

We have thus shown that $|F| < |E|$. But, since $V_0 \subseteq N^+[V_2]$, thus $F \neq \emptyset$ and $E \subseteq N^+[F]$, which implies that there exists at least a vertex w in F such that $\left|N_D^+(w) \cap E\right| \geq 2$, implying $\Delta^+(\left\langle \overline{N}^+[x] \right\rangle) \geq 2$, a contradiction with the Condition (b) of the family $\mathcal{F}_3$. Hence, $\gamma_R(D) = n - \Delta^+ + 1$. □

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