Electronic Journal of Graph Theory and Applications **9** (1) (2021), 53–64

Electronic Journal of
Graph Theory and Applications

Some new results on the b-domatic number of graphs

Mohammed Benatallah[a], Mustapha Chellali[b], Noureddine Ikhlef Eschouf[c]

[a]*RECITS Laboratory, Faculty of Sciences, UZA Djelfa, Algeria*
[b]*LAMDA-RO Laboratory, Department of Mathematics, University of Blida, B.P. 270, Blida, Algeria*
[c]*Faculty of Sciences, Department of Mathematics and Computer Science, University of Yahia Fares, Medea, Algeria*

m_benatallah@yahoo.fr, m_chellali@yahoo.com, nour_echouf@yahoo.fr

Abstract

A domatic partition $\mathcal{P}$ of a graph $G = (V, E)$ is a partition of V into classes that are pairwise disjoint dominating sets. Such a partition $\mathcal{P}$ is called b-maximal if no larger domatic partition $\mathcal{P}'$ can be obtained by gathering subsets of some classes of $\mathcal{P}$ to form a new class. The b-domatic number $bd(G)$ is the minimum cardinality of a b-maximal domatic partition of G. In this paper, we characterize the graphs G of order n with $bd(G) \in \{n-1, n-2, n-3\}$. Then we prove that for any graph G on n vertices, $bd(G) + bd(\overline{G}) \leq n + 1$, where $\overline{G}$ is the complement of G. Moreover, we provide a characterization of the graphs G of order n with $bd(G) + bd(\overline{G}) \in \{n + 1, n\}$ as well as those graphs for which $bd(G) = bd(\overline{G}) = n/2$.

Keywords: domatic number, b-domatic number, Nordhaus-Gaddum inequalities
Mathematics Subject Classification: 05C69, 05C70
DOI:10.5614/ejgta.2021.9.1.5

1. Introduction

Throughout this paper, G denotes a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N_G(v) = N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *private neighborhood* of a vertex $v \in S$ with respect to S is the set $pn\,[v, S] = \{u \in V\,(G) \mid N\,[u] \cap S = \{v\}\}$. For any $S \subseteq V$, we denote the subgraph of G induced by S with $\langle S \rangle$. The *degree* of a vertex v, denoted by $d_G(v)$, is the number of vertices

Received: 28 March 2018, Revised: 15 June 2020, Accepted: 6 October 2020.

adjacent to v. We denote by $\Delta(G) = \Delta$ and $\delta(G) = \delta$ the *maximum degree* and the *minimum degree* in $V(G)$, respectively. A *universal vertex* is a vertex that is adjacent to all other vertices of the graph, that is a vertex whose degree is exactly $n-1$.

The *complement* $\overline{G}$ of G is the graph with vertex set $V(G)$ and with exactly the edges that do not belong to G. The *complete graph* of order n is denoted by K_n, and K_1 is called the *trivial graph*. The *complete bipartite graph* with partition sets X, Y such that $|X| = p$ and $|Y| = q$ is denoted by $K_{p,q}$. We write P_n for the *path* of order n and C_n for the *cycle* of length n. If G is any graph, the *prism graph* of G is the the graph obtained by taking two copies of G, say G_1 and G_2, with the same vertex labelings and joining each vertex of G_1 to the vertex of G_2 having the same label by an edge; in other words, the *prism graph* of G is the Cartesain product $G\Box K_2$. The *join* of two simple graphs G and H, written $G \vee H$ is the graph obtained by taking the disjoint union of G and H and adding all edges $\{xy \mid x \in V(G), y \in V(H)\}$.

A *dominating set* of a graph G is a set D of vertices such that every vertex in $V \setminus D$ is adjacent to some vertex in D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

In 1977, Cockayne and Hedetniemi [3] introduced the concept of domatic partition as a partition of V into dominating sets. They defined the *domatic number* $d(G)$ as the largest number of sets in a domatic partition of G. For related works in this area see, for instance, [1, 2, 8, 9]. In 2013, Favaron [4] introduced the b-domatic number as follows. A domatic partition $\mathcal{P} = \{C_1, C_2, ..., C_p\}$ is *b-maximal* if there do not exist p subsets $C_i' \subset C_i$ (among them $p-1$ are possibly empty) such that the partition $\mathcal{P}' = \{C_1 \setminus C_1', C_2 \setminus C_2', ..., C_p \setminus C_p', C_1' \cup C_2' \cup ... \cup C_p'\}$ is domatic. The *b-domatic number* of G, denoted $bd(G)$, is the minimum cardinality of a b-maximal domatic partition of G. A $bd(G)$-*domatic partition* of a graph G is a b-maximal domatic partition of G of cardinality $bd(G)$. On the basis of these definitions, $bd(G) \leq d(G)$ for every graph G.

In this paper, we first characterize the graphs G of order n with $bd(G) \in \{n-1, n-2, n-3\}$. Then we prove that for any graph G on n vertices, $bd(G) + bd(\overline{G}) \leq n+1$. Moreover, we characterize all graphs G with $bd(G) = bd(\overline{G}) = n/2$ as well as those graphs for which $bd(G) + bd(\overline{G}) \in \{n+1, n\}$.

2. Known results

In this section, we list some known results that will be useful in our investigations.

Proposition 2.1 ([3])**.** *For any graph G of order n,* $d(G) \leq \min\{\delta(G)+1, n/\gamma(G)\}$.

Theorem 2.1 ([4])**.** *Let $G_1, \ldots, G_k$ be the components of a disconnected graph G without isolated vertices. Then* $bd(G) = \min\{bd(G_i) \mid 1 \leq i \leq k\}$.

Since the vertex set of a graph G is the unique domatic partition if and only if $\delta(G) = 0$, the following lower bound is immediate.

Proposition 2.2 ([4])**.** *If G is a graph of minimum degree $\delta(G) \geq 1$, then $bd(G) \geq 2$.*

Proposition 2.3 ([4])**.** $bd(K_n) = n$, $bd(C_n) = 2$ *for* $n \geq 4$, *and* $bd(K_{p,q}) = 2$ $(p \geq q \geq 1)$.

In [5], the authors gave some sufficient conditions for graphs to attain equality in the bound of Proposition 2.2. Recall that a set $S \subseteq V$ is *independent* if no two vertices in S are adjacent.

Theorem 2.2 ([5]). *If G has a vertex whose neighbors form an independent set, then $bd(G) = 2$.*

Proposition 2.4 ([5]). *If G is a prism graph, then $bd(G) = 2$.*

Theorem 2.3 ([5]). *Let $\mathcal{P}$ be a domatic partition of a graph $G = (V, E)$. If there exists a vertex $v \in V$ such that each vertex of $N_G[v]$ is either isolated in its class or has a private neighbor with respect to its class, then $\mathcal{P}$ is b-maximal.*

It has been shown in [5] that if G has a universal vertex v, then $bd(G \setminus v) = bd(G) - 1$. This result can be generalized as follows.

Proposition 2.5. *Let A be the set of universal vertices in a graph G. Then $bd(G) = bd(G \setminus A) + |A|$.*

We note that if G is a graph without universal vertices, then $\gamma(G) \geq 2$. So, the next result follows immediately from Proposition 2.1 and the fact $bd(G) \leq d(G)$.

Corollary 2.1. *If G is a graph of order n without universal vertices, then $bd(G) \leq \frac{n}{2}$.*

3. Graphs with large b-domatic number

In this section, we give a characterization of graphs G of order $n \geq 3$ for which $bd(G) \in \{n-1, n-2, n-3\}$. We recall that graphs G of order n with $bd(G) = n$ have been characterized in [4].

Proposition 3.1 ([4]). *Let G be a graph of order n. Then $bd(G) = n$ if and only if G is isomorphic to K_n.*

Proposition 3.2. *Let G be a graph of order n. Then $bd(G) = n - 1$ if and only if G is isomorphic to graph $K_n - e$, where e is an arbitrary edge of the complete graph K_n.*

Proof. Let $\mathcal{P} = \{U_1, U_2, ..., U_{n-1}\}$ be an $(n-1)$-domatic partition of G. Without loss of generality, we may assume that $U_1 = \{a, b\}$ and $U_i = \{u_i\}$ for each $i \in \{2, ..., n-1\}$. Clearly $d_G(u_i) = n-1$, since each u_i dominates $V(G)$. Now, if $ab \in E$, then $G = K_n$ and by Proposition 3.1, $bd(G) = n$, a contradiction. Hence $ab \notin E$, and thus $G = K_n - e$.

The converse is obvious. □

Proposition 3.3. *Let G be a graph of order $n \geq 3$. Then $bd(G) = n - 2$ if and only if $G \in \{\overline{K}_3, K_2 \cup K_1, P_4, C_4, 2K_2\}$ or G is isomorphic to $G_1 \vee K_{n-3}$ or $G_2 \vee K_{n-4}$, where $G_1 \in \{\overline{K}_3, K_2 \cup K_1\}$ and $G_2 \in \{P_4, C_4, 2K_2\}$.*

Proof. If $n = 3$, then $bd(G) = 1$ and thus G has an isolated vertex. Therefore $G \in \{\overline{K}_3, K_2 \cup K_1\}$. Assume now that $n \geq 4$ and let $\mathcal{P} = \{U_1, U_2, ..., U_{n-2}\}$ be an $(n-2)$-domatic partition of G such that $|U_1| \geq |U_2| \geq ... \geq |U_{n-2}|$. Clearly, either $|U_1| = 3$ and $|U_2| = 1$ or $|U_1| = |U_2| = 2$. Moreover, if $n \geq 5$, then $|U_i| = 1$ for each $i \notin \{1, 2\}$.

Suppose first that $|U_1| = 3$ and $|U_i| = 1$ for each $i \neq 1$. Let $U_i = \{u_i\}$ for each $i \in \{2, ..., n-2\}$. Since each u_i dominates $V(G)$, $G = G_1 \vee K_{n-3}$, where $G_1 = \langle U_1 \rangle$. By Propositions 3.1 and 3.2, $G_1 \notin \{K_3, P_3\}$. Hence $G_1 = K_2 \cup K_1$ or $\overline{K}_3$.

Now suppose that $|U_1| = |U_2| = 2$, and let $G_2 = \langle U_1 \cup U_2 \rangle$. Assume first that $n = 4$. Since U_1 dominates U_2, each vertex of U_1 has a neighbor in U_2, and likewise each vertex of U_2 has a neighbor in U_1. Now using the fact that $G_2 \notin \{K_4, K_4 - e\}$ (by Propositions 3.1 and 3.2) we deduce that $G_2 \in \{P_4, C_4, 2K_2\}$. Assume now that $n \geq 5$ and let $U_i = \{u_i\}$ for each $i \in \{3, ..., n-2\}$. As previously, every u_i dominates $V(G)$, and thus $G = G_2 \vee K_{n-4}$.

For the converse, if $G \in \{\overline{K}_3, K_2 \cup K_1, P_4, C_4, 2K_2\}$, then one can easily check that $bd(G) = n-2$. Now let $G = G_1 \vee K_{n-3}$ or $G = G_2 \vee K_{n-4}$. If A is the set of universal vertices of G, then according to Proposition 2.5, $bd(G) = bd(H) + |A|$, where $H \in \{G_1, G_2\}$. If $H = G_1$, then $bd(G_1) = 1$ and $|A| = n-3$, implying that $bd(G) = n-2$. If $H = G_2$, then $bd(G_2) = 2$ and $|A| = n-4$, implying that $bd(G) = n-2$. □

Let $\mathcal{H}$ be the family of graphs G of order 6 with $\delta(G) \geq 2$ and $3 \leq \Delta(G) \leq 4$, where each vertex is contained in a triangle. We note that $\mathcal{H}$ contains exactly 14 graphs that can be found in [7] (see pages $218-224$).

In the sequel, we shall show that all graphs of $\mathcal{H}$, except those depicted in Figure 1, have a b-domatic number equal to 3.

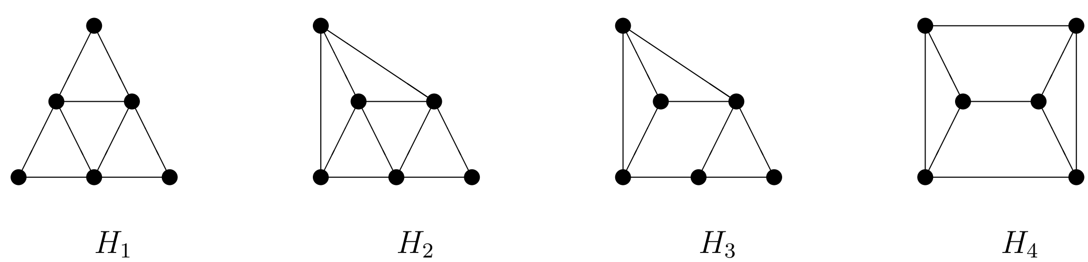

Figure 1. Four graphs of order 6 with b-domatic number 2

Recall that it was shown in [5] that $bd(H_1) = bd(H_4) = 2$.

Proposition 3.4. *The only graphs of* $\mathcal{H}$ *with b-domatic number* 2 *are* H_1, H_2, H_3 *and* H_4.

Proof. Let $G \in \mathcal{H}$, and assume that $bd(G) = 2$. Let $\mathcal{P} = \{U_1, U_2\}$ be a 2-domatic partition of G such that $|U_1| \leq |U_2|$. As G has order 6 and maximum degree at most 4, $3 \leq |U_2| \leq 4$ and so $2 \leq |U_1| \leq 3$. Consider the following two cases.

Case 1. $|U_1| = 2$ and $|U_2| = 4$. Let $U_1 = \{a, b\}$ and $U_2 = \{x, y, z, t\}$. We distinguish between two subcases, depending on whether the edge ab exists or not.

Case 1.1. $ab \notin E$. Since every vertex of G belongs to a triangle, every vertex in U_2 is not isolated in $\langle U_2 \rangle$. If $\langle U_2 \rangle$ does not have two independent edges, then clearly $\langle U_2 \rangle$ is a star $K_{1,3}$, centered, without loss of generality, at x. Note that $\langle U_2 \rangle$ has no triangle. Using the fact that every vertex of G is contained in a triangle and y, z, t form an independent set in $\langle U_2 \rangle$, we deduce that every triangle containing one of y, z and t also contains x. This implies that x is adjacent to both a and b, implying that $d_G(x) = 5$, a contradiction. Hence, we can assume that $\langle U_2 \rangle$ has two independent edges.

Now, let T_a and T_b be two triangles containing a and b, respectively. Since $ab \notin E(G)$, T_a and T_b has at most two common vertices. Suppose first that there is no common vertex between T_a and T_b. Without loss of generality, let $V(T_a) = \{a, x, y\}$ and $V(T_b) = \{b, z, t\}$. In this case, $\{\{a, b\}, \{x, z\}, \{y, t\}\}$ is a domatic partition of G, a contradiction. Suppose now that y is the unique common vertex between T_a and T_b. Without loss of generality, let $V(T_a) = \{a, x, y\}$ and $V(T_b) = \{b, y, z\}$. Since t is dominated by U_1, let $tb \in E$. If $tz \in E$, then $\{a, x, y\}$ and $\{b, z, t\}$ induces two independent triangles and as above we can get a domatic partition of order 3. So $tz \notin E$. Since t belongs to a triangle, we must have $yt \in E$ but then $d_G(y) = 5$, a contradiction. Finally, we may assume that all triangles containing a and b have two common neighbors. Hence let $V(T_a) = \{a, x, y\}$ and $V(T_b) = \{b, x, y\}$. Note that since $\Delta \leq 4$, each of x and y has at most one neighbor in $\{z, t\}$. Also, since U_1 dominates U_2, we may assume that $zb \in E$. Suppose that $zt \in E$. Then $bt \notin E$, for otherwise there are two independent triangles. Therefore $at \in E$ and so $az \notin E$ (else there are two independent triangles). Since each of z and t belongs to a triangle, we have $xz, ty \in E$. But then $\{a, y, t\}$ and $\{b, x, z\}$ are two triangles with no common vertex, a contradiction. Hence $zt \notin E$. Since $\langle U_2 \rangle$ has two independent edges, we can assume that $ty \in E$. Then $at \notin E$ for otherwise $\{a, y, t\}$ and $\{b, x, y\}$ are two triangles with one common vertex, a contradiction. Thus $bt \in E$ but then $\{a, x, y\}$ and $\{b, y, t\}$ are two triangles with one common neighbor, a contradiction.

Case 1.2. $ab \in E(G)$. Clearly since $\Delta(G) \leq 4$, neither a nor b is adjacent to all U_2. Moreover, since $\{U_1, U_2\}$ is a 2-domatic partition of G, we assume without loss of generality, that $at \notin E$ and so $bt \in E$. Likewise $bx \notin E$ and so $ax \in E$. Note that x and t are not isolated in U_2 since $\delta(G) \geq 2$. However, at most one of y, z is isolated in U_2, for otherwise x and t do not belong to any triangle.

Firstly, suppose, without loss of generality, that z is isolated in U_2. Then z must be adjacent to both a and b. As each of x and t lies on a triangle, xy and $ty \in E$. Clearly, y has a neighbor in U_1. Assume that y is adjacent to both a, b. If $xt \notin E$, then $G = H_1$, otherwise $G = H_2$. Note that $bd(H_1) = 2$ as proved in [5]. Likewise $bd(H_2) = 2$ by Theorem 2.3 since z is isolated in U_2 and each of a, b has a private neighbor with respect to U_1. Assume now that y is adjacent either to a or to b, but not to both of them. In this case, $tx \in E$ since t belongs to a triangle, whence, $G = H_3$. The above argument applied to z shows that $bd(H_3) = 2$.

Suppose now that U_2 contains no isolated vertex. Since x belongs to a triangle, x must be adjacent to at least one of y, z, say y. By the same argument, t has a neighbor in $\{y, z\}$. Observe that each of a and b has a neighbor in $\{y, z\}$ because each of them belongs to a triangle. Clearly, U_1 dominates y and z. If zt or $zx \in E$, then $\{\{a, b\}, \{x, t\}, \{y, z\}\}$ is a domatic partition of G, a contradiction. Hence $zt, zx \notin E$ implying that $zy \in E$ since z is not isolated in U_2. Therefore $ty \in E$ because t belongs to a triangle. As y has a neighbor in U_1 and $\Delta \leq 4$, y is adjacent to exactly one of a, b. Up to symmetry, let $yb \in E$. Then $ay \notin E$, and thus az and $zb \in E$ since a belongs to a triangle. Since x lies on a triangle, $xt \in E$. In this case, $\{\{a, b\}, \{x, y\}, \{z, t\}\}$ is a domatic partition of G, a contradiction.

Case 2. $|U_1| = |U_2| = 3$. Let $U_1 = \{a, b, c\}$ and $U_2 = \{x, y, z\}$. Here again, we distinguish between four subcases.

Case 2.1. U_1 is an independent set. Clearly every vertex of U_1 is adjacent to at least two vertices of U_2. Suppose that a vertex of U_1, say a is adjacent to all U_2. Since every triangle containing a vertex of U_1 must contain two vertices of U_2, there is a vertex in U_2 adjacent to every vertex of G, which leads to a contradiction since $\Delta \leq 4$. Therefore, every vertex of U_1 has exactly two neighbors in U_2. Now, it is easy to see that U_2 induces a K_3 and so $G = H_1$.

Case 2.2. $\langle U_1 \rangle$ contains exactly one edge. Thus assume that $bc \in E$, and a is isolated in $\langle U_1 \rangle$. Then a is adjacent at least two adjacent vertices of U_2, say x and y. Suppose that zb and $zc \notin E$. Then $za \in E$ and each of b and c has a neighbor in $\{x, y\}$. Since each of b and c belongs to a triangle, we have by or cx, say $by \in E$. Also one of x and y, say y, is adjacent to both b and c. Since z belongs to a triangle, $xz \in E$ ($yz \notin E$ since $\Delta \leq 4$). But then $\{\{a, c\}, \{y, z\}, \{b, x\}\}$ is a domatic partition of G, a contradiction. Hence $N(z) \cap \{b, c\} \neq \emptyset$. Without loss of generality, let $zb \in E$. Clearly $N(c) \cap U_2 \neq \emptyset$. If $cz \in E$, then $\{\{a, b\}, \{y, z\}, \{c, x\}\}$ is a domatic partition of G, a contradiction. Then $cz \notin E$ and thus c must be adjacent to one of x, y. Up to symmetry, let $cy \in E$. If $zx \in E$, then $\{\{a, b\}, \{y, z\}, \{c, x\}\}$ is a domatic partition of G, a contradiction. Hence $zx \notin E$ and therefore $zy \in E$ since z belongs to a triangle. Then $by \notin E$ since $\Delta \leq 4$, which means that $za, bx, cx \in E$ since each of z, b belongs to a triangle. But then again, $\{\{a, b\}, \{x, z\}, \{c, y\}\}$ is domatic partition, a contradiction.

Case 2.3. $\langle U_1 \rangle$ contains exactly two edges. Without loss of generality, let $ba, bc \in E$. Seeing the above situations, $\langle U_2 \rangle = P_3$ or K_3.

Suppose first that $\langle U_2 \rangle$ is a path P_3 centered at y. Assume that $by \in E$. Since $\Delta \leq 4$, one of bx and $bz \notin E$, say $bz \notin E$. Likewise, one of ya and $yc \notin E$. Up to symmetry let $yc \notin E$. Since each of c and z belongs to a triangle, we have $cx, bx \in E$ and $az, ay \in E$. In this case, $\pi = \{\{a, c\}, \{x, z\}, \{b, y\}\}$ is a domatic partition of G, a contradiction. Hence $by \notin E$. Since each U_i is a dominating set of G, we assume, up to isomorphism, that bx and $ya \in E$. If $ax \notin E$, then using the fact that each of a and x belongs to a triangle, we have $az, xc \in E$. But π is a domatic partition of G, a contradiction. Hence $ax \in E$. If $cz \in E$, then π is a domatic partition of G. Hence $cz \notin E$. Therefore $az \in E$ and $cx \in E$ since each of z and c belong to a triangle. Again π is a domatic partition of G, a contradiction.

Now suppose that $\langle U_2 \rangle$ is a K_3. Since b is adjacent to at least one vertex of U_2 and not to all U_2 because of $\Delta \leq 4$, we may assume, without loss of generality, that $by \in E$ and $bx \notin E$. Likewise, vertex y must be non-adjacent to at least one vertex in U_1. Up to isomorphism, let $ya \notin E$. Now since a lies on a triangle, we must have $az \in E$ and either ax or $bz \in E$. Assume first that $ax \in E$. If $cz \in E$, then $d(z) = 4$, whence, $bz \notin E$ and therefore $cy \in E$ (so that b lies on a triangle). But then $\{\{a, c\}, \{x, y\}, \{b, z\}\}$ is a domatic partition of G, a contradiction. Then $cz \notin E$ and so $cy \in E$ since c belongs to a triangle. As above, we have a domatic partition of order 3, a contradiction. Hence $ax \notin E$, implying that $cx \in E$ since x has at least one neighbor in U_1. Assume now that $bz \in E$. Then $d(z) = 4$, which means that $cz \notin E$. Therefore $cy \in E$ since c lies on a triangle. But then $\{\{a, c\}, \{x, z\}, \{b, y\}\}$ would be a domatic partition of G, a contradiction.

Case 2.4. $\langle U_1 \rangle$ contains exactly three edges, that is $\langle U_1 \rangle = K_3$. Seeing the above situations, $\langle U_2 \rangle = K_3$. Since $\{U_1, U_2\}$ is a 2-domatic partition of G and $\Delta \leq 4$, each vertex of U_1 has either one or two neighbors in U_2. Suppose that $N(a) = \{x, y\}$. Then $za \notin E$ since $\Delta \leq 4$ and therefore $N(z) \cap \{b, c\} \neq \emptyset$. Without loss of generality, assume that $zc \in E$. Suppose that $bz \in E$. Then $\{\{a, c\}, \{x, z\}, \{b, y\}\}$ is a domatic partition of G, a contradiction. Hence $bz \notin E$ and so $N(b) \cap \{x, y\} \neq \emptyset$. By symmetry, assume that $by \in E$. Then $\{\{a, c\}, \{y, z\}, \{b, x\}\}$ is a domatic partition of G, a contradiction. Thus $|N(t) \cap U_2| = 1$ for every $t \in \{a, b, c\}$. Therefore $G = H_4$. As proved in [5], $bd(H_4) = 2$. □

Corollary 3.1. *If $G \in \mathcal{H} \setminus \{H_1, H_2, H_3, H_4\}$, then $bd(G) = 3$.*

Proof. Let $G \in \mathcal{H} \setminus \{H_1, H_2, H_3, H_4\}$. By Propositions 2.2 and 3.4, $bd(G) \geq 3$. Since G has no universal vertex, the equality follows from Corollary 2.1. □

Proposition 3.5. *Let G be a graph of order $n \geq 4$. Then $bd(G) = n - 3$ if and only if G is isomorphic to one of the following graphs.*

i) H *or* $H \vee K_{n-4}$, *where* $H \in \{\overline{K}_4, K_2 \cup \overline{K}_2, P_3 \cup K_1, K_3 \cup K_1\}$,

ii) H *or* $H \vee K_{n-5}$, *where* H *or* $\overline{H} \in \{C_5, P_5, K_{2,3}, P_3 \cup K_2, F_1, F_2, F_3\}$. *($F_1, F_2, F_3$ are given in Figure 2).*

iii) H *or* $H \vee K_{n-6}$, *where* $H = 2K_3$ *or* $H \in \mathcal{H} \setminus \{H_1, H_2, H_3, H_4\}$.

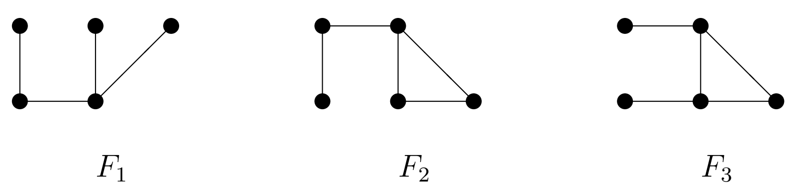

Figure 2. Three graphs of order 5 with b-domatic number 2

Proof. If $n = 4$, then $bd(G) = 1$ and thus G has at least one isolated vertex. Therefore $G \in \{\overline{K}_4, K_2 \cup \overline{K}_2, P_3 \cup K_1, K_3 \cup K_1\}$. Hence we can assume that $n \geq 5$. Then $bd(G) \geq 2$ and thus G has no isolated vertices. Let $\mathcal{P} = \{U_1, U_2, ..., U_{n-3}\}$ be an $(n-3)$-domatic partition of G such that $|U_1| \geq |U_2| \geq ... \geq |U_{n-3}|$. We distinguish between three cases.

Case 1. $|U_1| = 4$ and $|U_i| = 1$ for each $i \neq 1$. It is clear that $G = H \vee K_{n-4}$, where $H = \langle U_1 \rangle$. If H has a universal vertex, say x, then $\{U_1 \setminus \{x\}, U_2, ..., U_{n-3}, \{x\}\}$ is a domatic partition of G of cardinality $n-2$, a contradiction. Hence H has no universal vertices. If $H \in \{P_4, C_4, 2K_2\}$, then according to Proposition 3.3, one can easily see that $bd(G) = n-2$, a contradiction. Consequently, $H \in \{\overline{K}_4, K_2 \cup \overline{K}_2, P_3 \cup K_1, K_3 \cup K_1\}$.

Case 2. $|U_1| = 3, |U_2| = 2$. Let $H = \langle U_1 \cup U_2 \rangle$. Observe that if $n = 5$, then $\mathcal{P} = \{U_1, U_2\}$ and thus $G = H$, while if $n \geq 6$, then $|U_i| = 1$ for each $i \notin \{1, 2\}$ and thus $G = H \vee K_{n-5}$. Since U_1 dominates U_2, each vertex of U_1 has a neighbor in U_2, and likewise each vertex of U_2 has a neighbor in U_1. Hence $\delta(H) \geq 1$. Now, assume that $\Delta(H) = 4$, and let x be a vertex of H

with $d_H(x) = 4$. Then $\mathcal{P}' = \{U_1', U_2', U_3, ..., U_{n-3}\}$ is an $(n-3)$-domatic partition of G, where $U_1' = (U_1 \cup U_2) \setminus \{x\}$ and $U_2' = \{x\}$. But such a case has been already considered (see Case 1). Hence $\Delta(H) \leq 3$. By examining all graphs H of order five with $1 \leq \delta(H) \leq \Delta(H) \leq 3$ listed in [7] (see pages 216–217), we have H or $\overline{H} \in \{C_5, P_5, K_{2,3}, P_3 \cup K_2, F_1, F_2, F_3\}$.

Case 3. $|U_1| = |U_2| = |U_3| = 2$. Let $H = \langle U_1 \cup U_2 \cup U_3 \rangle$. Clearly, if $n = 6$, then $G = H$, while if $n \geq 7$, then $|U_i| = 1$ for each $i \notin \{1, 2, 3\}$, and thus $G = H \vee K_{n-6}$. Note that by Proposition 2.5, $bd(H) = 3$, and thus every vertex of H is contained in a triangle (by Theorem 2.3). Therefore $\delta(H) \geq 2$. By a similar argument to that used in Case 2, we shall have $\Delta(H) \leq 4$. Observe that if $\Delta(H) = 2$, then either $H = 2K_3$ or $H = C_6$. However, the case $H = C_6$ is excluded since $bd(C_6) = 2$. For the next, we may assume that H is a graph of order 6 satisfying $\delta(H) \geq 2$ and $3 \leq \Delta(H) \leq 4$ and every vertex is contained in a triangle. Thus $H \in \mathcal{H}$. Using Propositions 2.5 and 3.4, one can see that $H \notin \{H_1, H_2, H_3, H_4\}$. Consequently, $H \in \mathcal{H} \setminus \{H_1, H_2, H_3, H_4\}$.

Conversely, if G is isomorphic to one of the graphs H given in the statement, then $bd(G) = n-3$. Assume now that G is isomorphic to one of the join graphs described in items (i), (ii) or (iii). Let A be the set of universal vertices of G. According to Proposition 2.5, $bd(G) = bd(H) + |A|$. If G fulfills (i), then $bd(H) = 1$ and $|A| = n - 4$, implying that $bd(G) = n - 3$. If G fulfills (ii), then $bd(H) = 2$ (by Theorem 2.1, Proposition 2.3 and Theorem 2.2) and $|A| = n - 5$, implying that $bd(G) = n - 3$. Finally, if G fulfills (iii), then $bd(H) = 3$ (by Corollary 3.1) and $|A| = n - 6$, implying that $bd(G) = n - 3$. □

4. Graphs G of order n with $bd(G) = bd(\overline{G}) = \frac{n}{2}$

Our aim in this section is to characterize the graphs G of order n such that $bd(G) = bd(\overline{G}) = \frac{n}{2}$. To do this, we will use a result by Dunbar et al. [6] who characterized the graphs G of order n such that $d(G)d(\overline{G}) = n^2/4$. Let us first define the family $\mathcal{G}_k$ of graphs given in [6] as follows. For each integer $k \geq 2$, let $I = \{1, 2, ...k\}$. If $G \in \mathcal{G}_k$, then the vertices of the graph G can be labelled $u_1, u_2, ..., u_k, v_1, v_2, ...v_k$ so that each $i \in I$ satisfies one of the following conditions:

(C_1) : For all $l \in I - \{i\}$,
either $u_iu_l, v_iv_l \in E(G)$ and $u_iv_l, v_iu_l \in E(\overline{G})$ or
$u_iu_l, v_iv_l \in E(\overline{G})$ and $u_iv_l, v_iu_l \in E(G)$;

(C_2) : There exists a $j \in I - \{i\}$, such that

(a) For all $l \in I - \{i, j\}$, either
$u_iu_l, v_iv_l \in E(G)$ and $u_iv_l, v_iu_l \in E(\overline{G})$ or
$u_iu_l, v_iv_l \in E(\overline{G})$ and $u_iv_l, v_iu_l \in E(G)$;

(b) $u_iu_j, u_iv_j, v_iu_j \in E(G)$ and $u_iv_i, u_jv_j, v_iv_j \in E(\overline{G})$;

(c) in the graph G,
$N_G(u_i) \setminus V_{ij} = N_G(u_j) \setminus V_{ij}$ and $N_G(v_i) \setminus V_{ij} = N_G(v_j) \setminus V_{ij}$,
where $V_{ij} = \{u_i, u_j, v_i, v_j\}$.

Dunbar et al. [6] showed that for any graph G of order $n \geq 4$, $d(G)d(\overline{G}) \leq n^2/4$, and characterized all graphs achieving this bound as follows.

Theorem 4.1 ([6]). *For every graph G with order $n \geq 4$, $d(G)d(\overline{G}) = \frac{n^2}{4}$ if and only if $G \cong K_4$ or $G \in \mathcal{G}_k$ for some integer $k \geq 2$.*

The proof of Theorem 4.1 was based on some facts which are summarized in the following result.

Proposition 4.1 ([6]). *Let G be a graph of order $n \geq 4$ satisfying $d(G)d(\overline{G}) = \frac{n^2}{4}$. Let $k = \frac{n}{2}$ and $\mathcal{P} = \{U_1, U_2, ..., U_k\}$ be a domatic partition of G of cardinality k such that $\sum_{i=1}^{k} |E(\langle U_i \rangle)|$ is a maximum. Then,*

(i) $k - 1 \leq \delta(G) \leq \Delta(G) \leq k$ and $k - 1 \leq \delta(\overline{G}) \leq \Delta(\overline{G}) \leq k$.

(ii) If U_i is a dominating set of $\overline{G}$, then i satisfies Condition (C_1).

(iii) If U_i is not a dominating set of $\overline{G}$, then i satisfies Condition (C_2).

According to Proposition 4.1, every graph $G \in \mathcal{G}_k$ is either regular or semi-regular of minimum degree either $n/2 - 1$ or $n/2$.

Theorem 4.2. *For every graph G with order $n \geq 4$,*

$$bd(G) = bd(\overline{G}) = \frac{n}{2}$$

if and only if $G \in \{2K_2, C_4, P_4\}$.

Proof. It is easy to show that if $G \in \{2K_2, C_4, P_4\}$, then $bd(G) = bd(\overline{G}) = n/2$. To prove the necessity, let G be a graph of order $n \geq 4$ with $k = bd(G) = bd(\overline{G}) = \frac{n}{2}$ and let $\mathcal{P} = \{U_1, U_2, ..., U_k\}$ be a b-maximal partition of G of cardinality k. Note that G has no universal vertex for otherwise $\overline{G}$ has an isolated vertex and so $bd(\overline{G}) = 1 < n/2$, a contradiction. Likewise $\overline{G}$ has no universal vertex. Hence $\gamma(G) \geq 2$ and $\gamma(\overline{G}) \geq 2$. It follows that $|U_i| = 2$ for all i since $k = \frac{n}{2}$. Moreover, $d(G) \leq \frac{n}{2}$ and $d(\overline{G}) \leq \frac{n}{2}$ by Proposition 2.1. Let $I = \{1, ..., k\}$ and $U_i = \{u_i, v_i\}$ for each $i \in I$. Since $bd(G) \leq d(G)$ and $bd(\overline{G}) \leq d(\overline{G})$, we obtain $d(G) = d(\overline{G}) = \frac{n}{2}$ and thus $d(G)d(\overline{G}) = \frac{n^2}{4}$. Clearly $G \neq K_4$ which means, by Theorem 4.1, that $G \in \mathcal{G}_k$. Since $d(G) = bd(G)$, each $d(G)$-domatic partiton of G is a $bd(G)$-domatic partition of G. Therefore, we can assume that $\mathcal{P}$ is chosen among all $d(G)$-domatic partitons of G so that $\sum_{i=1}^{k} |E(\langle U_i \rangle)|$ is a maximum. Now, by Proposition 4.1-(i), we have $\frac{n}{2} - 1 \leq \delta(G) \leq \Delta(G) \leq \frac{n}{2}$. It is a routine matter to check that if $n = 4$, then $G \in \{2K_2, C_4, P_4\}$. Hence we can assume that $n \geq 5$, and thus $k = \frac{n}{2} \geq 3$. We distinguish between two cases.

Case 1. U_i is a dominating set of $\overline{G}$ for all $i \in I$.

By Proposition 4.1-(ii), Condition (C_1) is satisfied for all $i \in I$. As U_i is a dominating set of both G and $\overline{G}$, each vertex of any U_j, with $j \neq i$, is adjacent to exactly one vertex of U_i in G. Therefore,

$$\forall x \in U_i, \;\; |pn\,[x, U_i]| = k - 1. \tag{1}$$

Moreover, we claim that

$$\text{for all } i \in I, \; N_G(u_i) \setminus \{v_i\} \text{ induces a complete graph.}$$

Indeed, suppose to the contrary that for a some $p \in I$, there is a vertex $u_p \in U_p$ such that $N_G(u_p) \diagdown \{v_p\}$ contains two non-adjacent vertices. Without loss of generality, let u_q and u_r be the two non-adjacent vertices in $N_G(u_p) \diagdown \{v_p\}$. By Condition (C_1), vertices v_q and v_r are not adjacent in $N_G(v_p) \diagdown \{u_q\}$. Let $U'_q = \{u_q, u_r, v_p\}$, $U'_r = \{v_q, v_r, u_p\}$ and $\mathcal{P}' = (\mathcal{P} \diagdown \{U_p, U_q, U_r\}) \cup \{U'_q, U'_r\}$. Observe that

$$U'_q \text{ and } U'_r \text{ are independent sets in } G. \tag{2}$$

Hence by (1) and (2), $\mathcal{P}'$ satisfies the following: each vertex of G is either isolated in its class or has a private neighbor with respect to its class. Therefore, by Theorem 2.3, $\mathcal{P}'$ is a b-maximal domatic partition of G of cardinality $\frac{n}{2} - 1$, a contradiction, which completes the proof of the claim.

Thus for every $i \in I$, the vertices of $N_G(u_i) \diagdown \{v_i\}$ are pairwise adjacent. Then G is a graph consisting of two disjoint complete graphs each of order $\frac{n}{2}$ to which s $(0 \leq s \leq \frac{n}{2})$ independent edges may be added such that each edge joins a vertex of one $K_{\frac{n}{2}}$ to a one vertex of the other $K_{\frac{n}{2}}$. But then by Theorem 2.2, $bd(\overline{G}) = 2 < bd(G)$, a contradiction.

Case 2. U_i is not a dominating set of $\overline{G}$ for some $i \in I$.

By Proposition 4.1-(iii), i satisfies Condition (C_2) . Let $j \in I - \{i\}$ such that items (a), (b) and (c) of Condition (C_2) are fulfilled. Observe that u_i, v_i, u_j, v_j induce a path P_4 : v_i-u_j-u_i-v_j (by item (b)). Also by item (c), each of the pair u_i, u_j and v_i, v_j have the same neighborhood in $V(G) \diagdown \{u_i, u_j, v_i, v_j\}$. Since $k \geq 3$, let $l \in I - \{i, j\}$ and $\mathcal{P}' = \mathcal{P} \diagdown \{U_i, U_j, U_l\}$. Now, by item (c), either ($u_iu_l, u_ju_l \in E$ and $v_iv_l, v_jv_l \in E$) or ($u_iv_l, u_jv_l \in E$ and $v_iu_l, v_ju_l \in E$). In the former, let $\mathcal{P}_1 = \mathcal{P}' \cup \{\{u_i, u_j, v_l\}, \{v_i, v_j, u_l\}\}$ and in the later let $\mathcal{P}_2 = \mathcal{P}' \cup \{\{u_i, u_j, u_l\}, \{v_i, v_j, v_l\}\}$. Whatever, the partition we shall have, $\mathcal{P}_1$ and $\mathcal{P}_2$ are domatic partitions of G. For the next, we may assume, without loss of generality, that $\mathcal{P}_1$ occurs. To show that $\mathcal{P}_1$ is b-maximal, it suffices to consider Theorem 2.3 on vertex v_l and using the fact that G is regular or semi-regular of (minimum) degree either $n/2 - 1$ or $n/2$. Indeed, v_l is isolated in its class $\{u_i, u_j, v_l\}$ and for any $x \in N_G(v_l)$, vertex $x$is either isolated in its class (when $d_G(x) = n/2 - 1$) or has a private neighbor with respect to its class (when $d_G(x) = n/2$). Therefore $\mathcal{P}_1$ is a b-maximal domatic partition of G of order $|\mathcal{P}'| + 2 = (\frac{n}{2} - 3) + 2 < \frac{n}{2}$, a contradiction. □

5. Nordhaus-Gaddum results

In this section, we present a Nordhaus-Gaddum bound for $bd(G) + bd(\overline{G})$ in terms of the order of the graph G, and we characterize extremal graphs attaining this bound.

Theorem 5.1. *For any graph G of order n, $bd(G) + bd(\overline{G}) \leq n + 1$, with equality if and only if $G \cong K_n$ or $\overline{K}_n$.*

Proof. By Proposition 2.1, we have

$$bd(G) + bd(\overline{G}) \leq \delta(G) + \delta(\overline{G}) + 2. \tag{3}$$

Moreover, since $\delta(\overline{G}) = n - \Delta(G) - 1$, we obtain that

$$bd(G) + bd(\overline{G}) \leq n + 1 + \delta(G) - \Delta(G), \tag{4}$$

and the bound follows since $\delta(G) - \Delta(G) \leq 0$.

Now assume that $bd(G) + bd(\overline{G}) = n + 1$. Then by (4), we have $\delta(G) = \Delta(G)$, that is G is a regular graph. Observe that if neither G nor $\overline{G}$ has universal vertices, then $\gamma(G) \geq 2$ and $\gamma(\overline{G}) \geq 2$. Therefore by Corollary 2.1, $bd(G) \leq n/2$ and $bd(\overline{G}) \leq n/2$, implying that $bd(G) + bd(\overline{G}) \leq n$ which leads to a contradiction. Hence at least one of G and $\overline{G}$ has a universal vertex. Now, if G has a universal vertex, then $bd(\overline{G}) = 1$ and $bd(G) = n$, implying that $G = K_n$. While if $\overline{G}$ has a universal vertex, then $bd(G) = 1$ and $bd(\overline{G}) = n$ implying that $\overline{G} = K_n$.

The converse is obvious. □

Theorem 5.2. *Let G be a graph of order n. If neither G nor $\overline{G}$ is a complete graph, then*

$$bd(G) + bd(\overline{G}) \leq n,$$

with equality if and only if $G \in \{K_n - e, 2K_2, C_4, P_4\}$.

Proof. The bound follows from Theorem 5.1 since neither G nor $\overline{G}$ is a complete graph.

Assume that $bd(G) + bd(\overline{G}) = n$. If G has a universal vertex, then $bd(\overline{G}) = 1$ and $bd(G) = n - 1$. By Proposition 3.2, $G = K_n - e$, where e is an arbitrary edge of K_n. By symmetry if $\overline{G}$ has a universal vertex, then $\overline{G} = K_n - e$, where e is an arbitrary edge of K_n. Hence we can assume that neither G nor $\overline{G}$ has a universal vertex. It follows that $\gamma(G) \geq 2$ and $\gamma(\overline{G}) \geq 2$, and so $n \geq 4$. Now since $bd(G) + bd(\overline{G}) = n$, Corollary 2.1 implies that $bd(G) = bd(\overline{G}) = \frac{n}{2}$, and by Theorem 4.2, $G \in \{2K_2, C_4, P_4\}$.

The converse is obvious. □

References

[1] J. Amjadi, S. Nazari-Moghaddam, and S.M. Sheikholeslami, Total Roman domatic number of a graph, *Asian-Eur. J. Math.* **13** (2020) Article ID: 2050110.

[2] G.J. Chang, The domatic number problem, *Discrete Math.* **125** (1994), 115–122.

[3] E.J. Cockayne and S.T. Hedetniemi, Towards a theory of domination in graphs, *Networks* **7** (1977), 247–261.

[4] O. Favaron, The b-domatic number of a graph. *Discuss. Math. Graph Theory* **33** (2013), 747–757.

[5] M. Benatallah, N. Ikhlef Eschouf, and M. Mihoubi, On the b-domatic number of graphs. *Discuss. Math. Graph Theory* **39** (2019), 313–324.

[6] J.E. Dunbar, T.W. Haynes, and M.A. Henning, Nordhaus-Gaddum type results for the domatic number of a graph. *In Combinatorics, Graph Theory, and Algorithms, vols. I, II, New Issues Press, Kalamazoo* (1999), 303–312.

[7] F. Harary, *Graph Theory*, Addison-Wesley Publishing Company, 1969.

[8] S.M. Sheikholeslami and L. Volkmann, The signed Roman domatic number of a digraph, *Electron. J. Graph Theory Appl.* **3** (1) (2015), 85–93.

[9] L. Volkmann, The Roman $\{2\}$-domatic number of graphs. *Discrete Appl. Math.* **258** (2019), 235–241.