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On the Steiner antipodal number of graphs

S. Arockiaraj[a], R. Gurusamy[b], KM. Kathiresan[c]

[a]*Department of Mathematics, Government Arts & Science College, Sivakasi 626124, Tamil Nadu, India*
[b]*Department of Mathematics, Mepco Schlenk Engineering College, Sivakasi 626005, Tamil Nadu, India*
[c]*Center for Research & Post Graduate Studies in Mathematics, Ayya Nadar Janaki Ammal College, Sivakasi 626 124,Tamil Nadu, India*

psarockiaraj@gmail.com, sahama2010@gmail.com, kathir2esan@yahoo.com

Abstract

The Steiner n-antipodal graph of a graph G on p vertices, denoted by $SA_n(G)$, has the same vertex set as G and any $n(2 \leq n \leq p)$ vertices are mutually adjacent in $SA_n(G)$ if and only if they are n-antipodal in G. When G is disconnected, any n vertices are mutually adjacent in $SA_n(G)$ if not all of them are in the same component. $SA_n(G)$ coincides with the antipodal graph $A(G)$ when $n = 2$. The least positive integer n such that $SA_n(G) \cong H$, for a pair of graphs G and H on p vertices, is called the Steiner A-completion number of G over H. When $H = K_p$, the Steiner A-completion number of G over H is called the Steiner antipodal number of G. In this article, we obtain the Steiner antipodal number of some families of graphs and for any tree. For every positive integer k, there exists a tree having Steiner antipodal number k and there exists a unicyclic graph having Steiner antipodal number k. Also we show that the notion of the Steiner antipodal number of graphs is independent of the Steiner radial number, the domination number and the chromatic number of graphs.

Keywords: n-radius, n-diameter, Steiner n-antipodal graph, Steiner A-completion number, Steiner antipodal number
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1. Introduction

This paper considers finite simple undirected graphs. Let G be a graph on p vertices and S a set of vertices of G. The *Steiner distance* of S in G, denoted by $d_G(S)$, is defined as the minimum number of edges in a connected subgraph of G that contains S. Such a subgraph is essentially a tree and is called a *Steiner tree* for S in G [5]. The *Steiner n-eccentricity* $e_n(v)$ of a vertex v in a graph G is defined as $e_n(v) = \max\{d_G(S) : S \subseteq V(G) \text{ with } v \in S \text{ and } |S| = n\}$. The *$n$-radius* $rad_n(G)$ of G is described as the smallest Steiner n-eccentricity among the vertices of G and the *n-diameter* $diam_n(G)$ of G is the largest Steiner n-eccentricity. The notion of Steiner distance was further evolved in [11].

KM. Kathiresan et al. [10] initiated the concept of Steiner radial number of a graph G. The idea of antipodal graph was introduced by Singleton [13] and was further developed by R. Aravamudhan and B. Rajendran [1, 2] and E. Prisner [12].

Based on the above literature, we introduce a new concept called Steiner antipodal number of a graph. Any n vertices of a graph G are said to be n-antipodal to each other if the Steiner distance between them is equal to the n-diameter of the graph G. The *Steiner n-antipodal graph* of a graph G, denoted by $SA_n(G)$, has the vertex set as in G and n $(2 \leq n \leq p)$ vertices are mutually adjacent in $SA_n(G)$ if and only if they are n-antipodal in G. If G is not connected, any n vertices are mutually adjacent in $SA_n(G)$ if not all of them are in the same component. For the edge set of $SA_n(G)$, draw K_n corresponding to each set of n-antipodal vertices. $SA_n(G)$ coincides with $A(G)$ by taking $n = 2$.

Take the graph G which is given in Figure 1. If we let $n = 4$, we get that $diam_4(G) = 4$ and that $S_1 = \{v_1, v_2, v_4, v_5\}, S_2 = \{v_1, v_2, v_4, v_6\}$, $S_3 = \{v_1, v_3, v_4, v_5\}$ and $S_4 = \{v_1, v_3, v_4, v_6\}$ are the sets of 4-antipodal vertices of graph G. The Steiner 4-antipodal graph of G is given in Fig. 1.

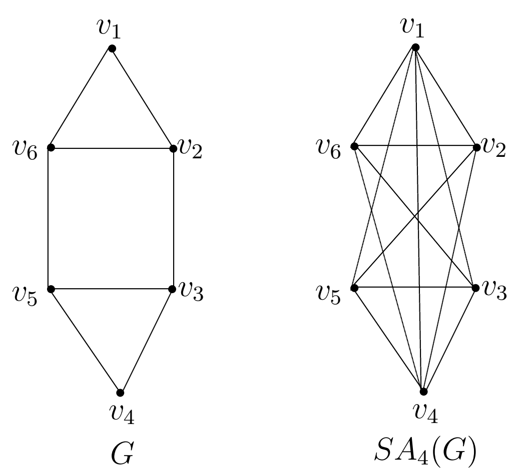

Figure 1. The graph G and its Steiner 4-antipodal graph.

Consider two graphs G and H on p vertices, and H is called a *Steiner A-completion* of G if there exists a positive integer n such that $SA_n(G) \cong H$. The positive integer n is said to be *Steiner A-completion number* of G over H if n is the least positive integer such that $SA_n(G) \cong H$. For instance, the Steiner A-completion number of bistar B_{p_1,p_2} over $K_{p_1+p_2+2} - e$ is $p_1 + p_2 + 1$. If there

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is no such n such that $SA_n(G) \cong H$, then the Steiner A-completion number of G over H is ∞. The Steiner A-completion number of G over H is need not be equal to the Steiner A-completion number of H over G. For the graphs G and H shown in Figure 2, the Steiner A-completion number of G over H is 3 but the Steiner A-completion number of H over G is ∞.

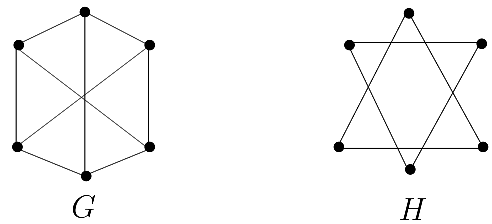

Figure 2. A pair of graphs (G, H) so that Steiner A-completion of G over H is not equal to Steiner A-completion of H over G.

When $H = K_p$, the Steiner A-completion number of G over H is called the Steiner antipodal number of G. In other words, the Steiner antipodal number $a_S(G)$ of a graph G is the least positive integer n such that the Steiner n-antipodal graph of G is complete.

The iterations of radial graph and eccentric graph have been studied to analyze the periodicity of the graph [9, 12]. The iterations of line graph and k^{th} power G^k of a graph G are observed to be complete after certain stage. The Steiner antipodal number of a graph is also one kind of iteration on the number of vertices deals with at a time.

In [7], a subset S of $V(G)$ of a graph G is said to be a *dominating set* if every vertex in $V - S$ is a neighbour of some vertex of S. For a graph G, $V(G)$ itself is a dominating set. The *domination number* is the minimum cardinality of a dominating set in G. The notion of the domination number was introduced to find the minimal dominating set with minimum cardinality. Likewise, if S is taken as the set of all vertices of G, then $SA_p(G) \cong K_p$. The concept of Steiner antipodal number of G is introduced to find the minimum cardinality so that $SA_n(G) \cong K_p$. We determines the Steiner antipodal number of some families of graphs and for any tree. For every positive integer k, there exists a tree having Steiner antipodal number k and there exists a unicyclic graph having Steiner antipodal number k. Also for any pair of positive integers a and b, we prove the existence of a graph such that $r_S(G) = a, a_S(G) = b$; $\chi(G) = a, a_S(G) = b$ and $\gamma(G) = a, a_S(G) = b$. We follow [4] for graph theoretic terminology.

2. Main Results

Observation 2.1. *For any connected graph G on p vertices, $2 \leq a_S(G) \leq p$, which pursues from the definition.*

The sharpness of this observation is given in Theorem 2.2 and Proposition 2.2.

Lemma 2.1. *If G is a graph with $a_S(G) = n$, then $rad_n(G) = diam_n(G)$.*

Proof. If $rad_n(G) \neq diam_n(G)$, then $SA_n(G)$ has isolated vertices whose eccentricity is less than $diam_n(G)$. Hence the result follows. □

The converse of the above lemma needs not be true. For the graph G given in Figure 3, $rad_3(G) = diam_3(G)$ but $a_S(G) = 5$.

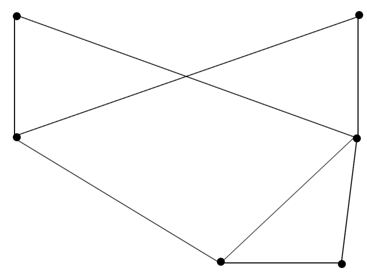

Figure 3. A graph G with $rad_3(G) = diam_3(G)$, but $a_S(G) = 5$.

Proposition 2.1. *For any graph $G, r_S(G) \leq a_S(G)$.*

Proof. Suppose $a_S(G) = n$. Then, n is the least positive integer such that $SA_n(G) \cong K_p$. Therefore by Lemma 2.1, $rad_n(G) = diam_n(G)$. Hence, $SR_n(G) \cong SA_n(G) \cong K_p$. So, by the definition, $r_S(G) \leq n = a_S(G)$. □

Proposition 2.2. *For any star graph $K_{1,p-1}$ with p vertices, $a_S(K_{1,p-1}) = p$.*

Proof. Let v_1 be the vertex of degree $p-1$ and $v_2, v_3, \ldots, v_p$ be the pendant vertices of $K_{1,p-1}$. For any $n, 2 \leq n \leq p-1$, $e_n(v_1) = n-1$ and $e_n(v_i) = n, 2 \leq i \leq p-1$. Hence the n-diameter of $K_{1,p-1}$ is n, for $2 \leq n \leq p-1$. If $n < p$, the vertex v_1 is an isolated vertex of Steiner n-antipodal graph of $K_{1,p-1}$. Hence $a_S(K_{1,p-1}) = p$. □

Proposition 2.3. *For any tree T on p vertices with $m(\neq p-1)$ pendant vertices, $a_S(T) = m+2$.*

Proof. Consider a tree T with m pendant vertices $x_1, x_2, \ldots, x_m$ and the remaining vertices are $v_1, v_2, \ldots, v_{p-m}$. Then $e_{m+1}(x_i) = e_{m+1}(v_i) = p-1$ for all i. Hence $(m+1)$-diameter of T is $p-1$. If v_iv_j is a non-pendant edge in T, then the set $\{v_i, v_j\} \cup X$, where $X \subseteq \{x_1, x_2, \ldots, x_m\}$ with $|X| = m-1$, has Steiner distance less than $p-1$. Therefore, v_i is not adjacent to v_j in Steiner $(m+1)$-antipodal graph of T. Since $(m+2)$-diameter of T is $p-1$ and any set $\{v_i, v_j, x_1, x_2, \ldots, x_m\}$ has Steiner distance $p-1$ for $1 \leq i, j \leq p-m$, the Steiner $(m+2)$-antipodal graph of T is K_p. □

Corollary 2.1. *For every positive integer $k \geq 2$, there exists a tree having Steiner antipodal number k.*

Proof. The result follows from Proposition 2.3 and Proposition 2.2. □

Proposition 2.4. *Let S be the set of all full degree vertices of a graph G. Then, $a_S(G)$ is $p-|S|+1$ when $G-S$ is disconnected and $p-|S|$ when $G-S$ is connected with at least one pendant vertex.*

Proof. When $G-S$ is disconnected, $V(G)-S$ is a $(p-|S|)$-element set having Steiner distance $p-|S|$ as $\langle V(G)-S \rangle$ is disconnected and $\langle (V(G)-S) \cup \{v\} \rangle$ is connected for each $v \in S$. Also every $(p-|S|)$-element set containing at least one element of S has Steiner distance $p-|S|-1$. Therefore, $rad_{p-|S|}(G) = p-|S|-1$ and $diam_{p-|S|}(G) = p-|S|$ and hence by Lemma 2.1,

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$SA_{p-|S|}(G) \ncong K_p$. But every $(p-|S|+1)$-element set has the Steiner distance $p-|S|$. Hence $a_S(G) = p-|S|+1$.

Now suppose that $G-S$ is connected with at least one pendant vertex. Let v be a pendant vertex in $G-S$, adjacent to v' say. As $(p-|S|-1)$-element set not containing v' is of Steiner distance $p-|S|-1$, $e_{p-|S|-1}(u) = p-|S|-1$ for every $u(\neq v') \in V(G)-S$. Since S is the collection of full degree vertices, $e_{p-|S|-1}(u) = p-|S|-2$ for every $u \in S$. Therefore $rad_{p-|S|-1}(G) \neq diam_{p-|S|-1}(G)$ and hence by Lemma 2.1, $a_S(G) > p-|S|-1$. As $G-S$ is connected with $p-|S|$ vertices, every $(p-|S|)$-element set has the Steiner distance $p-|S|-1$ and hence $a_S(G) = p-|S|$. □

Theorem 2.2. *For a graph G, $a_S(G) = 2$ if and only if G is either complete or totally disconnected.*

Proof. When G is complete (respectively a totally disconnected graph), 2-diameter is 1 (respectively ∞) and any pair of vertices has Steiner distance 1 (respectively ∞). Thus $a_S(G) = 2$.

Assume $a_S(G) = 2$ and G is not totally disconnected. If G has at least two components in which one of them is having at least two vertices x and y with $xy \in E(G)$, then by the definition, $xy \notin SA_2(G)$. Therefore G is connected. If G is not complete, then $xy \notin E(G)$ for some vertices x and y in G. Therefore $d(x,y) \geq 2$. Hence $diam_2(G) \geq 2$ and every adjacent vertices of G are non-adjacent in $SA_2(G)$. Hence the result follows. □

Proposition 2.5. *If a graph G is disconnected but not totally disconnected, then $a_S(G) = 3$.*

Proof. Since G is not totally disconnected, G has a component C with at least two vertices. By Theorem 2.2, $a_S(G) > 2$. From, the set of all 3-element sets with exactly two elements in C, every vertex of v in C is adjacent to all the remaining vertices of $V(G)$ in $SA_3(G)$. Also from the set of all 3-element sets with exactly one element in C, every vertex of $u \notin C$ is adjacent to all the remaining vertices of $V(G)$ in $SA_3(G)$. Therefore, $SA_3(G)$ is complete and hence $a_S(G) = 3$. □

Theorem 2.3. *For every positive integer $k \geq 2$, there exists an unicyclic graph having Steiner antipodal number k.*

Proof. Let G be a cycle of length $p = 2m$ with vertices $v_1, v_2, \ldots, v_{2m-1}$ and v_{2m}. For each vertex v_i, $e_n(v_i) = p - \left\lceil \frac{p}{n} \right\rceil$ and hence n-diameter is $p - \left\lceil \frac{p}{n} \right\rceil$, $2 \leq n \leq 2m$. In particular, $e_{m+1}(v_1) = 2m - \left\lceil \frac{2m}{m+1} \right\rceil = 2m-2$ and n-diameter is $2m-2$.

Consider the set $\{v_1, v_3, v_5, \ldots, v_{2m-1}, u\}$ where $u \in \{v_2, v_4, v_6, \ldots, v_{2m}\}$. For $u = v_i, i \in \{2,4,6,\ldots,2m-2\}$, $v_1v_2 \cdots v_{i-1}v_iv_{i+1} \cdots v_{2m-1}$ is a Steiner tree with Steiner distance $2m-2$ and for $u = v_{2m}$, $v_3v_5v_7 \cdots v_{2m-1}v_{2m}v_1$ is a Steiner tree with Steiner distance $2m-2$. Hence v_1 is adjacent to v_i for all $2 \leq i \leq 2m$ in Steiner $(m+1)$-antipodal graph of G.

Proceeding in this way, each vertex $v_{2i+1}, 1 \leq i \leq m-1$ is adjacent to all the remaining vertices in Steiner $(m+1)$-antipodal graph of G. By considering the set $\{v_2, v_4, v_6, \ldots, v_{2m}, u\}$ where $u \in \{v_1, v_3, v_5, \ldots, v_{2m-1}\}$, each vertex $v_{2i}, 1 \leq i \leq m$ is adjacent to all the remaining vertices in Steiner $(m+1)$-antipodal graph of G. Hence the Steiner $(m+1)$-antipodal graph of G is K_p. For $n \leq m$, there does not exist a set with n elements containing v_1 and v_2 with Steiner distance $p - \left\lceil \frac{p}{n} \right\rceil$. Hence Steiner n-antipodal graph is not complete for $n \leq m$. Therefore, $a_S(G) = m+1$. Also $a_S(K_3) = 2$. □

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Proposition 2.6. *If G is a graph with $a_S(G) = n$, then K_p is the only Steiner m-antipodal graph of G for $m \geq n$.*

Proof. For a graph G, let $a_S(G) = n$ and d be the n-diameter of G. By Lemma 2.1, $rad_n(G) = diam_n(G)$. Therefore, $e_n(v) = d$ for all $v \in V(G)$. Suppose $e_{n+1}(v) > d+1$ for some $v \in V(G)$. Since $e_n(v) = d$, there is a set S having v whose Steiner distance is the maximum distance d. $e_{n+1}(v) > d+1$ implies that there exists a vertex v' in G such that $d(v', S) > 1$. Let u be the vertex in S such that $d(v', u) = d(v', S)$. Therefore, the Steiner distance of the set $(S - \{u\}) \bigcup \{v'\}$ is greater than d. Hence, $e_n(v') > d$ which is a contradiction to $e_n(v') = d$. Hence, $e_{n+1}(v)$ is either d or $d+1$. This implies that $diam_{n+1}(G) = d$ or $d+1$. The result follows if $diam_{n+1}(G) = d$. Suppose $diam_{n+1}(G) = d+1$. Let v_1 and v_2 be two non-adjacent vertices in the Steiner $(n+1)$-antipodal graph of G. Then every set S with $n+1$ elements containing v_1 and v_2 have the Steiner distance less than $d+1$. This implies that $d_G(S) \leq d$ and hence $d_G(S - \{v_2\}) \leq d-1$, for every set S with $n+1$ elements containing v_1 and v_2. Since all the n-element sets $S - \{v_2\}$ containing v_1 are such that $d_G(S - \{v_2\}) \leq d-1, e_n(v_1) \leq d-1$ which is a contradiction to the fact that $e_n(v) = d$. Hence the result follows. □

Theorem 2.4. *For any pair of positive integers $a, b \geq 3$ with $a \leq b$, there exists a graph whose Steiner radial number is a and Steiner antipodal number is b.*

Proof. Let $\{u_1, u_2, \ldots, u_{p_1}\}$ and $\{v_1, v_2, \ldots, v_{p_2}\}$ be a partition of the vetex set of K_{p_1,p_2}, where $p_1 = a-1, p_2 = b-1$ and $p_1 \geq 2$. When $n \leq p_1, e_n(u_i) = n, 1 \leq i \leq p_1$ and $e_n(v_i) = n, 1 \leq i \leq p_2$. Hence $rad_n(K_{p_1,p_2}) = n = diam_n(K_{p_1,p_2})$. In the Steiner n-radial (n-antipodal) graph of G, u_i is not adjacent to v_j, since all the n-element sets containing u_i and v_j have only the Steiner distance $n-1$. Consequently, $r_S(K_{p_1,p_2}) > p_1$.

When $p_1 < n \leq p_2, e_n(u_i) = n-1, 1 \leq i \leq p_1$ and $e_n(v_i) = n, 1 \leq i \leq p_2$. Hence $rad_n(K_{p_1,p_2}) = n-1$ and $diam_n(K_{p_1,p_2}) = n$. In Steiner (p_1+1)-radial graph of G, u_i is adjacent to u_j for $1 \leq i, j \leq p_1$, u_i is adjacent to v_j for all $1 \leq i \leq p_1$, $1 \leq j \leq p_2$ and v_i is adjacent to v_j for all $1 \leq i, j \leq p_2$, since each of the sets $\{u_1, u_2, \ldots, u_{p_1}, v_j\}$ and $\{v_i, v_j, u_2, u_3, \ldots, u_{p_1}\}$ have the Steiner distance p_1 respectively. Thus Steiner (p_1+1) - radial graph of K_{p_1,p_2} is $K_{p_1+p_2}$. Also by Lemma 2.1, $a_S(G) > n$.

When $n > p_2, e_n(u_i) = n-1, 1 \leq i \leq p_1$ and $e_n(v_i) = n-1, 1 \leq i \leq p_2$. Therefore, $diam_n(G) = n-1$. Since every n-element sets must contain at least one u_i and v_j, it is of Steiner distance $n-1$. Hence the Steiner n-antipodal graph of G is complete. Since p_1+1 is the least positive integer such that the Steiner (p_1+1)-radial graph of G is complete and p_2+1 is the least positive integer such that the Steiner (p_2+1)-antipodal graph of G is complete, $r_S(K_{p_1,p_2}) = p_1+1 = a$ and $a_S(K_{p_1,p_2}) = p_2+1 = b$. □

Proposition 2.7. *For any pair of positive integers $a, b \geq 2$, there exists a graph G such that $\chi(G) = a$ and $a_S(G) = b$.*

Proof. Consider the complete a-partite graph $G = K_{n_1,n_2,\ldots,n_a}$ with $n_i = b-1$, $1 \leq i \leq a$. Suppose that $a > 2$ and $b > 2$. Since each partition of G should have different colours, $\chi(G) = a$. If $n \leq b-1, e_n(v) = n$ for each vertex $v \in V(G)$. Hence $diam_n(G) = n$. As $b > 2$, each partition has at least two vertices. Also any n-element set S having at least two vertices of a partition is of

Steiner distance $n-1$. Therefore no two vertices in the same partition are adjacent in $SA_n(G)$. If $n > b-1$, then $e_n(v) = n-1$ for each vertex $v \in V(G)$ and hence $diam_n(G) = n-1$. As every n-element set must contain vertices from different partitions, its Steiner distance is $n-1$ and hence $SA_n(G)$ is complete. Therefore, $a_S(G) = b$. By Proposition 2.2, $a_S(K_{1,b-1}) = b$. Also $\chi(K_{1,b-1}) = 2$. For the graph K_a with $a \geq 2, \chi(K_a) = a$ and $a_S(K_a) = 2$. □

Theorem 2.5. *For any pair of positive integers a and b ($\neq 1$), there exists a graph G such that $\gamma(G) = a$ and $a_S(G) = b$.*

Proof. Let G be a graph obtained by identifying a pendant vertex of the path on $3a-2$ vertices and a pendant vertex of the star graph on $b-1$ vertices. Let $v_1, v_2, \ldots, v_{3a-2}$ be the vertices of the path and $u_1, u_2, \ldots, u_{b-1}$ be the vertices of the star graph in which u_{b-1} is the full degree vertex and u_{b-2} be identified with v_{3a-2}. Then $\gamma(G) = a$ as the set $\{v_2, v_5, v_8, \ldots, v_{3a-4}, u_{b-1}\}$ is a minimal dominating set with minimum cardinality. Since G has $b-2$ number of pendant vertices, by Proposition 2.3, $a_S(G) = b$. For the graph $H = aK_2$, a copies of K_2 where $a \geq 2, \gamma(H) = a$ and $a_S(H) = 3$. For the totally disconnected graph $\overline{K}_a, a \geq 2, \gamma(\overline{K}_a) = a$ and $a_S(\overline{K}_a) = 2$. □

A graph G is called n-connected if G has at least $n+1$ vertices and it is not possible to disconnect G by removing $n-1$ or fewer vertices. The connectivity of G, denoted $k(G)$, is defined to be n if G is n-connected but not $(n+1)$-connected [6].

In [3], the Harary graph $H_{m,n}$ on n vertices with connectivity m was constructed based on the parities of m and n.

Case 1. m is even.

Let $m = 2r$. Then $H_{2r,n}$ is constructed as follows. It has vertices $0, 1, \ldots, n-1$ and two vertices i and j are joined if $i - r \leq j \leq i + r$ (where addition is taken modulo n).

Case 2. m is odd, n is even.

Let $m = 2r+1$. Then $H_{2r+1,n}$ is constructed by first drawing $H_{2r,n}$ and then adding edges joining vertex i to vertex $i + (\frac{n}{2})$ for $1 \leq i \leq \frac{n}{2}$.

Case 3. m is odd, n is odd.

Let $m = 2r+1$. Then $H_{2r+1,n}$ is constructed by first drawing $H_{2r,n}$ and then adding edges joining vertex 0 to vertices $\frac{(n-1)}{2}$ and $\frac{(n+1)}{2}$ and vertex i to vertex $i + \frac{(n+1)}{2}$ for $1 \leq i \leq \frac{(n-1)}{2}$.

Theorem 2.6. *Let $n \geq 3$ be any positive integer and m be any positive integer less than n such that*

$$m \geq \begin{cases} \frac{2n}{3}, & n \equiv 0, 3 (mod\ 6); \\ \frac{2n-2}{3}, & n \equiv 1, 4 (mod\ 6); \\ \frac{2n+2}{3}, & n \equiv 2, 5 (mod\ 6). \end{cases}$$

Then the Steiner antipodal number of the Harary graph $H_{m,n}$ is $n-m+1$.

Proof. Let $G = H_{m,n}$. Let $v_1, v_2, \ldots, v_n$ be the vertices of G. By the choice of m, every vertex of $H_{m,n}$ is adjacent to at least one of v_1, v_{m+1} and v_{n-m+1}.

Let m and n be even. Construct the set S which contains v_1 and all its non-neighbouring vertices. Then $|S| = n-m$ and $d_G(S) = n-m$. If one of the vertices in S other than v_1 is adjacent to v_1, then its Steiner distance is less than or equal to $n-m$. Hence $e_{n-m}(v_1) = n-m$.

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Similarly $e_{n-m}(v_i) = n-m$, for $2 \leq i \leq n$. Hence $diam_{n-m}(G) = n-m$. But $SA_{n-m}(G) \ncong K_n$, since there is no set with $n-m$ elements containing v_1 and v_{m+1} with Steiner distance $n-m$. Whenever a set with $n-m+1$ elements is taken, its induced subgraph definitely have a Steiner tree with Steiner distance $n-m$ and hence $a_S(H_{m,n}) = n-m+1$.

Let m be odd and n be even. In this case, construct a set S which includes the vertex v_1 and all its non-neighbouring vertices. Then $|S| = n-m$ and $d_G(S) = n-m$. By the same argument, $e_{n-m}(v_i) = n-m$, for $1 \leq i \leq n$ and hence $diam_{n-m}(G) = n-m$. But $SA_{n-m}(G) \ncong K_n$, since there is no set with $n-m$ elements containing v_1 and $v_{\frac{n}{2}+1}$ with Steiner distance $n-m$. Also every set with $n-m+1$ elements has a Steiner tree in its induced subgraph and hence its Steiner distance is $n-m+1$. Therefore $a_S(H_{m,n}) = n-m+1$.

By the same argument given in the first case, it can be shown that $a_S(H_{m,n}) = n-m+1$ when m is even and n is odd.

Let m and n be odd. Construct the set S which contains v_1 and all its non-neighbouring vertices. Let $S_1 = S \cup \{u\}$ where $u \in V(G) - S$. Then $|S_1| = n-m$ and $d_G(S_1) = n-m-1$. As all the $(n-m)$-element sets containing v_1 has the Steiner distance less than or equal to $n-m-1$, $e_{n-m-1}(v_1) = n-m-1$. Construct the set $S_i, 2 \leq i \leq n$ which contains v_i and all its non-neighbouring vertices. Then $|S_i| = n-m$ and $d_G(S_i) = n-m$. Also for each v_i, all the $(n-m)$-element sets containing v_i have the Steiner distance less than or equal to $n-m$. Therefore $e_{n-m}(v_i) = n-m$ for $2 \leq i \leq n$, and hence $rad_{n-m}(G) \neq diam_{n-m}(G)$. Therefore by Lemma 2.1, $a_S(G) > n-m$. Since the induced subgraph of every $(n-m+1)$-element set has a Steiner tree with Steiner distance $n-m$, so $a_S(G) = n-m+1$. □

Conjecture 1. *For any pair of positive integers k and $m(\neq 1)$, there exists a graph which is k-connected whose Steiner antipodal number is m.*

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