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Perfect codes in some products of graphs

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Abstract

A r-perfect code in a graph $G = (V(G), E(G))$ is a subset C of $V(G)$ for which the balls of radius r centered at the vertices of C form a partition of $V(G)$. In this paper, we study the existence of perfect codes in corona product and generalized hierarchical product of graphs where the cardinality of U is equal to one or two. Also, we give some examples as applications of our results.

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1. Introduction

Throughout this paper all graphs considered are finite, simple and connected. The distance $d(u, v)$ between the vertices u and v of a graph G is equal to the length of a shortest path that connects u and v.

Let $G = (V(G), E(G))$ be a graph. Any subset C_G of $V(G)$ is named a code in G. Suppose $S_r(c) = \{u \mid u \in V(G) \text{ and } d(u, c) \leq r\}$, $c \in C_G$. For a positive integer r, we call C_G a r**-perfect code** iff $\cup_{c \in C_G} S_r(c) = V(G)$ and $S_r(c_i) \cap S_r(c_j) = \emptyset$ for each $c_i, c_j \in C_G$, where $i \neq j$. For more information, we recommend the readers to look at [13]. Perfect codes have been used to model the problem of efficient placement of resources in a network. Also, they play a central role in the fast growing of error-correcting codes theory. Perfect codes in direct [11, 20], strong [1] and lexicographic [17] product of graphs have been investigated by many authors. Later, in [6, 7],

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r-perfect codes have been studied over Cartesian products. In this paper, we study r-perfect codes over generalized hierarchical and corona product of graphs.

There are more than twenty graph operations. One can use these graph operations for computing invariants of big graphs in terms of invariants of their factors. Also, some graph operations have some applications in other fields. For example, generalized hierarchical product has some applications in computer science and chemistry [14, 15]. This graph operation have been introduced by Barriére et al. [4, 5] as follows:

A graph G with a specified vertex subset $U \subseteq V(G)$ is denoted by $G(U)$. Suppose G and H are graphs and $U \subseteq V(G)$. The **generalized hierarchical product**, denoted by $G(U) \sqcap H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices (g, h) and (g', h') are adjacent if and only if $g = g' \in U$ and $hh' \in E(H)$ or, $gg' \in E(G)$ and $h = h'$. If $|U| = |\{z\}| = 1$, then we have hierarchical product defined as follows:

The **hierarchical product** $G \sqcap H$ is the graph with vertices the $2-$tuples x_2x_1, $x_1 \in V(H)$ and $x_2 \in V(G)$, and edges defined by the following adjacencies:

$$x_2x_1 \sim \begin{cases} x_2y_1, & \text{if } y_1 \sim x_1 \text{ in } H, \\ y_2x_1, & \text{if } y_2 \sim x_2 \text{ in } G \text{ and } x_1 = z. \end{cases}$$

Let G and H be two graphs. The **corona product** $G \circ H$, is obtained by taking one copy of G and $|V(G)|$ copies of H; and by joining each vertex of the i-th copy of H to the i-th vertex of G, where $1 \leq i \leq |V(G)|$. This product was first introduced by Harary in 1969 [9]. We encourage the readers to consult [2, 16, 18, 19] for more information on corona product.

The **eccentricity** $e(u)$ of a vertex u in a graph G is defined as the largest distance between u and other vertices of G. The **radius** of G, denoted by $\mathrm{rad}(G)$, is the minimum eccentricity among the vertices of G. The **diameter** of G, denoted by $diam(G)$ is the maximum eccentricity among the vertices of G. A vertex v is a **central vertex** if $e(v) = \mathrm{rad}(G)$. The central vertex set of G is denoted by $Z(G)$.

The **hypercube** Q_n is a graph in which vertices are n-tuples $(v_1, v_2, \ldots, v_n)$ that $v_i \in \{0, 1\}$ and two vertices are adjacent when their n-tuples differ in exactly one coordinate. We denote the **path** and **cycle** graphs of order n by P_n and C_n, respectively. The **empty graph** on n vertices is the graph complement of the complete graph K_n, and is denoted by $\bar{K}_n$. The graph K_1 is also called the **trivial graph**. All other graphs are nontrivial. Our other notations are standard and taken mainly from the standard books of graph theory.

2. Perfect codes in hierarchical product

We find the following notation useful for results of this section. Let G be a graph and H be a rooted graph with the root vertex z. In following, G' denotes the copy of G and H_i denotes the i-th copy of H, corresponding to $x_i \in V(G')$, with the root vertex $z_i = (x_i, z)$ in $G \sqcap H$. Note that by the previous notation, $x_i = z_i$ in $G \sqcap H$. Also, when H is a graph, G has an r-perfect code and $r' = \min\{\mathrm{rad}(G), r\}$, then W_{rj} denotes the ball of radius $(r - j)$ centered at the vertex z in H, for each $j \in \{0, 1, \ldots, r'\}$. Moreover, the notation A_{rj} is used to denote the subgraph of H whose vertex set is $V(H) \backslash W_{rj}$ and whose edge set consists of all edges of H which have both ends in $V(H) \backslash W_{rj}$, for each $i \in \{0, 1, \cdots, r'\}$. In other words, $A_{rj} = H[V(H) \backslash W_{rj}]$. Furthermore, let

$C = \{C_H \mid C_H$ is a r-perfect code in $H\}$ and $D = (d_1, d_2, \ldots, d_n)$ be a decreasing sequence such that $d_i = \min\{d_H(x, z) \mid x \in C_H$ and $C_H \in C\}$.

Proposition 2.1. *If $G \sqcap H$ is an r-perfect graph, then G or H is an r-perfect graph.*

Proof. Let $C_{G\sqcap H}$ be an r-perfect code in $G \sqcap H$. Assume, to the contrary, that G and H do not have any r-perfect code. So $C_{G\sqcap H} \cap G' \neq \emptyset$ and $C_{G\sqcap H} \nsubseteq V(G')$. Since G is not an r-perfect graph, then there exist $x_i \in V(G')$ that $d_{G\sqcap H}(u, x_i) > r$ for each $u \in C_{G\sqcap H} \cap V(G')$. Therefore, there exists a subset $C' \subset C_{G\sqcap H} \cap H_i$ for which the balls of radius r centered at the vertices of C' form a partition of H_i, contrary to the assumption that H does not have any r-perfect code. □

Let X and Y be subsets of $V(G)$. Then distance between two sets X and Y, denoted by $d(X, Y)$, is equal to $\min\{d_G(x, y) \mid x \in X$ and $y \in Y\}$.

Proposition 2.2. *Suppose H is a graph and $C_0, C_1, \cdots, C_{r'}$ are r-perfect codes in $A_{r0}, A_{r1}, \cdots, A_{rr'}$, respectively, such that $d_H(C_i, W_{ri}) > r$, $i = 0, 1, \cdots, r'$. Then $G \sqcap H$ is an r-perfect graph.*

In the following, W^i_{rj} denotes the i-th copy of W_{rj} corresponding to H_i in $G \sqcap H$.

Proposition 2.3. *If G is an r-perfect graph and $C_{G\sqcap H}$ is an r-perfect code in $G \sqcap H$ such that $C_{G\sqcap H} \cap V(G') \neq \emptyset$ and $C_{G\sqcap H} \nsubseteq V(G')$, then $d_{H_i}(C_{H_i}, W^i_{rj}) > r$ for $j = 0, 1, \ldots, r'$ and $i = 1, \ldots, |V(G)|$ where $C_{H_i} = C_{G\sqcap H} \cap H_i$.*

Proposition 2.4. *Suppose G and H are two graphs and $C_{G\sqcap H}$ is an r-perfect code in $G \sqcap H$ and $C_{A_{ri}}$ is an arbitrary r-perfect code of A_{ri}. If A_{ri} does not have any r-perfect code, or there exist $x \in C_{A_{ri}}$ and $y \in W_{ri}$ that $d_H(x, y) \leq r$ for at least one $i \in \{0, 1, \cdots, r'\}$, then all elements of $C_{G\sqcap H}$ are either in G', or in $\cup_{i=1}^{|V(G)|}(V(H_i) \setminus \{z_i\})$.*

Proof. Suppose G and H are two graphs and $C_{G\sqcap H}$ is an r-perfect code in $G \sqcap H$ and A_{ri} does not have any r-perfect code, or there exist $x \in C_{A_{ri}}$ and $y \in W_{ri}$ that $d_H(x, y) \leq r$ for at least one $i \in \{0, 1, \cdots, r'\}$. Assume, to the contrary, that x_i and v are elements of $C_{G\sqcap H}$ with $x_i \in G'$ and $v \in \cup_{i=1}^{|V(G)|}(V(H_i) \setminus \{z_i\})$. Hence, there exist $x_l \in V(G')$ that $d_{G\sqcap H}(x_i, x_l) \leq r$. This implies that A_{rd_l} does not have any r-perfect code, or there exist $x \in C_{A_{rd_l}}$ and $y \in W_{rd_l}$ that $d(x, y) \leq r$ where $d_l = d(x_l, C_{G\sqcap H})$ and $A_{rd_l} \subseteq H_l$. Therefore, $e(z)$ (the eccentricity of the root vertex z of H) must be less than or equal to r. Now we consider two cases:

Case 1. $d(v, x_i) \leq r$. This case derives a contradiction because $e(z) \leq r$.

Case 2. $d(v, x_i) > r$. Thus, there exists x_t with $d(x_i, x_t) = r$ and so $d(u, x_t)$ must be more than r for a vertex $u \in V(H_k)$ which is contrary to $e(z) \leq r$.

Therefore, all elements of $C_{G\sqcap H}$ are either in G', or in $\cup_{i=1}^{|V(G)|}(V(H_i) \setminus \{z_i\})$, but not both. □

Proposition 2.5. *Suppose G and H are two graphs and $C_{G\sqcap H}$ is an r-perfect code in $G \sqcap H$ such that $C_{G\sqcap H} \cap V(G') \neq \emptyset$. If A_{ri} does not have any r-perfect code, or there exist $x \in C_{A_{ri}}$ and $y \in W_{ri}$ that $d_H(x, y) \leq r$ for at least one $i \in \{0, 1, \cdots, r'\}$, then $|C_{G\sqcap H}| = 1$.*

Proof. Suppose $C_{G\sqcap H}$ is an r-perfect code in $G \sqcap H$ that $C_{G\sqcap H} \cap V(G') \neq \emptyset$. If A_{ri} does not have any r-perfect code, or there exist $x \in C_{A_{ri}}$ and $y \in W_{ri}$ such that $d_H(x,y) \leq r$ for at least one $i \in \{0, 1, \cdots, r'\}$, then by Proposition **??**, $C_{G\sqcap H} \subseteq V(G')$. Assume, to the contrary, that $x_i, x_j \in C_{G\sqcap H}$. Therefore, there exists $x_l \in V(G')$ with $d(x_i, x_l) = r$. So $d(u, x_i) > r$ and $d(u, x_j) > r$ for each $u \in V(H_l) \setminus \{z_l\}$, contrary to the assumption that $C_{G\sqcap H}$ is an r-perfect code. $\square$

Proposition 2.6. *Suppose $C_{G\sqcap H}$ is an r-perfect code in $G \sqcap H$ such that $C_{G\sqcap H} \cap V(G') \neq \emptyset$ and A_i^r does not have any r-perfect code, or there exist $x \in C_{A_i^r}$ and $y \in W_i^r$ that $d_H(x,y) \leq r$ for at least one $i \in \{0, 1, \cdots, r'\}$. Then G is an r-perfect graph iff $G \sqcap H$ has a $(r + e(z))$-perfect code $C'_{G\sqcap H}$ that $C'_{G\sqcap H} \cap V(G') \neq \emptyset$.*

Proposition 2.7. *$G \sqcap H$ has an r-perfect code $C_{G\sqcap H}$ with $C_{G\sqcap H} \cap V(G') = \emptyset$ iff either of the following two conditions holds.*

i. $d_1 = r$.

ii. *There exists $d_i \in D$ that G has a $(r - d_i)$-perfect code and $A_{r-d_i,j}$ has an r-perfect code $C_{A_{r-d_i,j}}$ such that there does not exist $u \in W_{r-d_i,j}$ that $d(x,u) \leq r$ for each $x \in C_{A_{r-d_i,j}}$ and $j \in \{1, 2, \ldots, r'\}$.*

Proof. It is clear that if either of the conditions holds then there is an r-perfect code $C_{G\sqcap H}$ in $G \sqcap H$ that $C_{G\sqcap H} \cap V(G') = \emptyset$.

Conversly, suppose that $C_{G\sqcap H}$ is an r-perfect code in $G \sqcap H$ that $C_{G\sqcap H} \cap V(G') = \emptyset$. We show that if the second condition does not hold then the first condition must be hold. Assume, to the contrary, that there exist x_i and $u \in H_i \cap C_{G\sqcap H}$ which $d_{G\sqcap H}(x_i, u) < r$. Let $x_{i+1}x_i$ be an edge of G', then $d_{G\sqcap H}(x_{i+1}, u) \leq r$. On the other hand, by our assumption there exists $v \in C_{G\sqcap H} \cap H_{i+1}$ with $d_{G\sqcap H}(x_{i+1}, v) \leq r$, contrary to the assumption that $C_{G\sqcap H}$ is a perfect code. $\square$

Proposition 2.8. *If $C_{G\sqcap H}$ is an r-perfect code in $G \sqcap H$ such that $C_{G\sqcap H} \nsubseteq V(G')$, then $|V(G)| \leq |C_{G\sqcap H}|$.*

Proof. Let $C_{G\sqcap H}$ be an r-perfect code in $G \sqcap H$ that $C_{G\sqcap H} \nsubseteq V(G')$. Now assume, to the contrary, that there exists a copy H_k in $G \sqcap H$ such that $V(H_k) \cap C_{G\sqcap H} = \emptyset$. So, there are two cases as follows:

i. There is a vertex $x_i \in C_{G\sqcap H}$ that $d_{G\sqcap H}(x_i, v) \leq r$, for each $v \in V(H_k)$.

ii. There is a vertex u in $(V(H_l) \setminus \{z_l\}) \cap C_{G\sqcap H}$ that $d_{G\sqcap H}(u, v) \leq r$, for each $v \in V(H_k)$.

In the first case, r must be bigger than or equal to $e(z)$ and since $C_{G\sqcap H} \nsubseteq V(G')$, then there exist x_j that $d_{G\sqcap H}(x_i, x_j) = r$, contrary to the assumption that $C_{G\sqcap H}$ is an r-perfect code. By a similar argument, we have a contradiction in the second case. $\square$

Theorem 2.1. *For each r, $r \in \{e(z) + 1, \ldots, e(z) + \mathrm{rad}(G) - 1\}$, $G \sqcap H$ does not have any r-perfect code.*

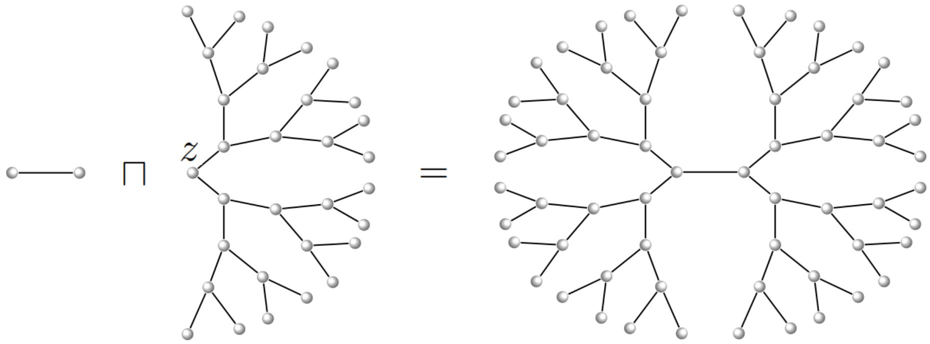


Figure 1. $DD_{2,4}$.

Example 1. *Dendrimers are branched molecules have a high degree of molecular uniformity. The molecular graph of this molecules is constructed from a core and some branches connecting to the core. The notation $DD_{p,r}$ shows the graph of the regular dicentric dendrimer [8]. It is clear that $DD_{p,r} = P_2 \sqcap H$, where H is a tree of progressive degree p and generation r (in other words, the graph H is a complete n-ary Tree, $T(n,k)$, for $k = 2$). The graph depicted in Figure 1 is $DD_{2,4}$. By previous results, this graph is an r-perfect graph where $r \geq 5$ but does not have any r-perfect code where $r < 5$.*

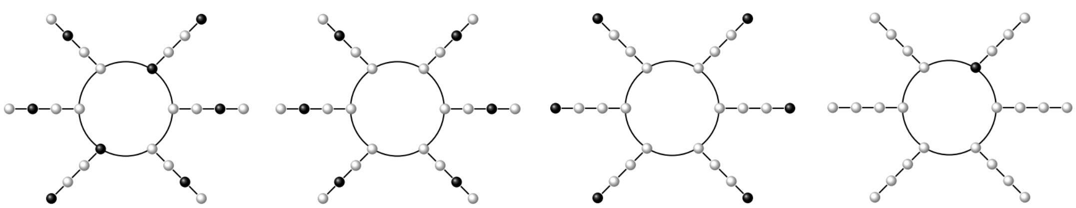


Figure 2. 1-perfect code, 2-perfect code, 3-perfect code and 6-perfect code in $Sun_{6,4}$.

Example 2. *Consider the sun graph $Sun_{m,n} = C_m \sqcap P_n$, introduced by Y.-N. Yeh and I. Gutman [19]. By Proposition 2.5, one can see that $Sun_{m,n}$ has a $\left([\frac{m}{2}] + n - 1\right)$-perfect code. Also, by Proposition 2.7, if $m = 1$ then $Sun_{m,n}$ has an $(n-1)$-perfect code and if $m \neq 1$ and $\frac{n}{2r+1} \in \{0, r+1, \ldots, 2r\}$, then $Sun_{m,n}$ has an r-perfect code. Moreover, by Proposition 2.3, if $r \leq [\frac{m}{2}]$ or $\frac{m}{2r+1} = 0$ then $Sun_{m,n}$ is an r-perfect code, see Figure 2.*

3. Perfect codes in generalized hierarchical product

In this section, we study perfect codes of generalized hierarchical product when $|U| = 2$. To do this, some new notations are needed. Throughout this section, $U = \{u_1, u_2\}$, G_i denotes the i-th copy of G, corresponding to $x_i \in V(H)$, and H_1 and H_2 denote the copies of H corresponding to u_1, u_2 in $G(U) \sqcap H$, respectively. Also, since U is fixed, we simplify the notation $G(U) \sqcap H$ into $G \sqcap H$. Moreover, ${}^aW_i^b = \{x \in V(G) \mid d(u_1, x) \leq d(u_1, a) - i$ or $d(u_2, x) \leq d(u_2, b) - i\}$ where $i \in \{0, 1, \ldots, \max\{r - d(u_1, a), r - d(u_2, b)\}\}$.

Theorem 3.1. *$G \sqcap H$ is an r-perfect code $C_{G\sqcap H}$ with $|C_{G\sqcap H}| = |C_{G\sqcap H} \cap (H_1 \cup H_2)| = 1$ iff $d(u_1, u_2) \le r$ and $\mathrm{rad}(H) + \min\{e(u_1), e(u_2)\} \le r$.*

Proof. Let $x \in Z(H_1)$, $d(u_1, u_2) \le r$ and $\mathrm{rad}(H) + \min\{e(u_1), e(u_2)\} \le r$. Without loss of generality, assume $\min\{e(u_1), e(u_2)\} = e(u_2)$. Since $\mathrm{rad}(H) + e(u_2) \le r$, then $\{x\}$ is a perfect code in $G \sqcap H$.

Conversely, suppose $G\sqcap H$ has an r-perfect code $C_{G\sqcap H}$ with $|C_{G\sqcap H}| = |C_{G\sqcap H}\cap(H_1\cup H_2)| = 1$. If $d(u_1, u_2) > r$, then it is clear that $G \sqcap H$ does not have any r-perfect code with one element. On the other hand, if $d(u_1, u_2) \le r$, then it is not difficult to see that $\mathrm{rad}(H) + \min\{e(u_1), e(u_2)\}$ must be smaller than r. □

Theorem 3.2. *If $d(u_1, u_2) > r$ and $G \sqcap H$ is an r-perfect graph, then $C_{G\sqcap H}$ has at least one element in $V(G(U) \sqcap H) \setminus \big(V(H_1) \cup V(H_2)\big)$.*

Proof. Assume, to the contrary, that $C_{G\sqcap H}$ does not have any element in $\big(\cup_{i=1}^{|V(H)|} V(G_i)\big) \setminus \big(V(H_1)\cup V(H_2)\big)$. Since $d(u_1, u_2) > r$, then $C_{G\sqcap H}\cap V(H_1) \ne \emptyset \ne C_{G\sqcap H}\cap V(H_2)$. Thus there exists at least one vertex $h \in V(H)$ such that $(u_1, h), (u_2, h) \in C_{G\sqcap H}$, and so $d_{G\sqcap H}((u_1, h), (u_2, h)) = 2r + 1$. On the other hand, $d_{G\sqcap H}((u_1, h), (u_2, h)) = d_G(u_1, u_2)$ and hence $d_G(u_1, u_2) = 2r + 1$. Now, suppose $G_{h'}$ is the copy of G, corresponding to $h' \in V(H)$, in $G(U) \sqcap H$ where $hh' \in E(H)$. Therefore, there exists a vertex (x, h') in $G_{h'}$ such that $d_{G(U)\sqcap H}((u_1, h), (x, h')) > r$ and $d_{G(U)\sqcap H}((u_2, h), (x, h')) > r$, a contradiction. □

Theorem 3.3. *$G(U) \sqcap H$ has an r-perfect code $C_{G(U)\sqcap H}$ whose elements are in $V(G(U) \sqcap H) \setminus \big(V(H_1) \cup V(H_2)\big)$ if the following conditions hold:*

1. *$\min\{d(u_1, x), d(u_2, x)\} \ge r$ and $G[V(G)\setminus^a W_i^b]$ has a r-perfect code $C_{G[V(G)\setminus^a W_i^b]}$ such that $d_G(x, y) > r$, for each $i \in \{1, 2, \ldots, \max\{r - d(u_1, a), r - d(u_2, b)\}\}$, $x \in C_{G[V(G)\setminus^a W_i^b]}$, $y \in^a W_i^b$ and $a, b \in C_G$ where $d(a, u_1), d(b, u_2) \le r$;*
2. *H has a $(r - d(u_1, a))$-perfect code or a $(r - d(u_2, b))$-perfect code.*

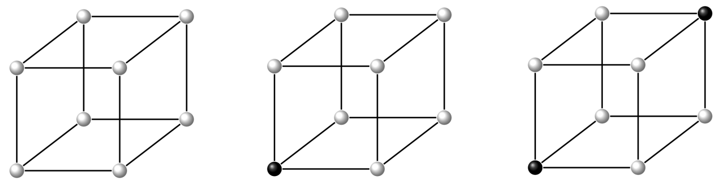
Proof. Suppose there exist $a, b \in C_G$ such that $d(a, u_1) \le r$ and $d(b, u_2) \le r$. Without loss of generality, we may suppose that $d(u_2, a) \ge d(u_1, b)$.

By the first condition, one can see that if $\mathrm{rad}(H) \le r - d(u_2, b)$ or $r - d(u_2, b) < \mathrm{rad}(H) \le r - d(u_1, a)$, then $G(U) \sqcap H$ has an r-perfect code.

Now, suppose $\mathrm{rad}(H) > r - d(u_1, a)$. Without loss of generality, we may suppose that H has a $(r - d(u_1, a))$-perfect code. Let $c \in C_H$, $H' = H[V']$ and $H'' = H[V'']$ where $V' = \{x \in V(H) \mid d(c, x) \le r - d(u_1, a)\}$ and $V'' = \{x \in V(H) \mid d(c, x) > r - d(u_1, a)\}$. By the previous argument, it is clear that $G \sqcap H'$ and $G \sqcap H''$ have r-perfect codes. Therefore, it is sufficient to prove that $G \sqcap H$ is an r-perfect code if for each $x \in C_{G\sqcap H''}$ and $y \in V(H')$, $d_{G\sqcap H}((x, h), (g, y)) > r$ where h is a vertex of H, corresponding to copy $G[V(G) \setminus^a W_i^b]$. Since $\min\{d(u_1, x), d(u_2, x)\} \ge r$ for each $x \in C_{G[V(G)\setminus^a W_i^b]}$, then $d_{G\sqcap H}((x, h), (g, y)) > r$ for each $x \in C_{G\sqcap H''}$ and $y \in V(H')$. □

Similarly, we can prove the following theorem:

Theorem 3.4. *If $G[V(G)\setminus {}^aW_i^b]$ has an r-perfect code such that $d_G(x,y) > r$ and $\min\{d(u_1,x), d(u_2,x)\} \geq r$ for each $x \in C_{G[V(G)\setminus {}^aW_i^b]}$ and $y \in {}^a W_i^b$, $i \in \{0,1,\cdots,\max\{r-d(u_1,a), r-d(u_2,b)\}\}$, then $G \sqcap H$ is an r-perfect graph.*



(a) (b) (c)

Figure 3. (a) Q_3; (b) the 3-perfect code in Q_3; (c) the 1-perfect code in Q_3.

Example 3. *Consider the hypercube Q_3. It is clear that $Q_3 \cong P_2(U) \sqcap C_4$ where $U = V(P_2)$. Then by the previous results, Q_3 is an r-perfect graph for each $i \in \{1,3\}$ as shown in Figure 3.*

4. Perfect codes in corona product

In this section, we study sufficient conditions for the existence of r-perfect codes in corona product of graphs. To prove the results of this section, we have to present some notations. Let G' denote the copy of G and H_i denote the i-th copy of H, corresponding to $x_i \in V(G')$, in $G \circ H$.

Proposition 4.1. *If $C_{G\circ H}$ is an r-perfect code in $G \circ H$ such that $C_{G\circ H} \cap V(G') \neq \emptyset$, then $|C_{G\circ H}| = 1$.*

Proof. Let $C_{G\circ H}$ is an r-perfect code in $G\circ H$ and $u \in V(G')\cap C_{G\circ H}$. Assume, to the contrary, that $|C_{G\circ H}| > 1$. Then there exists $x_i \in V(G')$ such that $d_{G\circ H}(u, x_i) = r$ and so $d_{G\circ H}(u,w) = r+1$ for each $w \in V(H_i)$. But this is contrary to the hypothesis. □

Corollary 4.1. *If $C_{G\circ H}$ is an r-perfect code in $G \circ H$, then all elements of $C_{G\circ H}$ are either in G', or in $\cup_{i=1}^{|V(G)|} V(H_i)$.*

Proposition 4.2. *If $C_{G\circ H}$ is an r-perfect code in $G \circ H$ such that $C_{G\circ H} \cap V(G') = \emptyset$, then $|C_{G\circ H} \cap V(H_i)| \leq 1$, $i = 1,2,\ldots,|V(G)|$.*

Proof. Assume, to the contrary, that $u, v \in V(H_i)$. Then $d_{G\circ H}(u,x_i) = d_{G\circ H}(v,x_i) = 1$ which is contrary to the hypothesis. □

Proposition 4.3. *$G \circ H$ is a 1-perfect code iff G is the trivial graph, or $\triangle(H) = |V(H)| - 1$.*

Proof. If G is the trivial graph and $u \in V(G')$, it is clear that $\{u\}$ is the 1-perfect code in $G \circ H$. Now suppose u_i be a vertex of H_i, corresponding to the vertex $u \in V(H)$. If u is a vertex of degree $|V(H)| - 1$, then $\cup_{i=1}^{|V(G)|}\{u_i\}$ is the 1-perfect code in $G \circ H$.

For sufficiency, suppose that $G \circ H$ has a 1-perfect code $C_{G\circ H}$. According to Corollary 4.1, there are two cases as follows:

Case 1. $C_{G\circ H} \cap V(G') = \emptyset$. Thus, by Proposition 4.2, $|C_{G\circ H} \cap V(H_i)| \leq 1$, $i = 1, 2, ..., |V(G)|$. This implies that there exists u_i in H_i of degree $(|V(H)| - 1)$ and so $\triangle(H) = |V(H)| - 1$.

Case 2. All elements of $C_{G\circ H}$ are in G'. In this case, if $|C_{G\circ H}| > 1$, it is clear that $G \circ H$ doesn't have any 1-perfect code. Now, suppose $|C_{G\circ H}| = |\{v\}| = 1$. If G is nontrivial, then there exists a neighbour vertex x_i of a vertex in $C_{G\circ H}$. Thus, $d(H_i, C_{G\circ H}) = 2$, a contradiction. Therefore, G cannot be nontrivial.

□

Proposition 4.4. *If $G \circ H$ has an r-perfect code $C_{G\circ H}$ with $r > 1$ and $C_{G\circ H} \nsubseteq V(G')$, then* $|C_{G\circ H}| = 1$.

Proof. Suppose $C_{G\circ H}$ is an r-perfect code in $G \circ H$ and $v \in C_{G\circ H} \cap V(H_i)$. To the contrary, assume that $|C_{G\circ H}| > 1$ and $v \neq u \in C_{G\circ H}$. By Corollary 4.1, $u \notin V(G')$ and by Proposition 4.2, $u \in V(H_j)$ with $i \neq j$. Hence, there is a vertex $x_l \in G'$ that $d_{G\circ H}(u, x_l) = r$ and so $d(w, u) = r+1$ for each $w \in V(H_l)$, contrary to the assumption that $C_{G\circ H}$ is an r-perfect code. □

Theorem 4.1. *Let G and H be two graphs and $r \geq 2$. Then $G \circ H$ is an r-perfect graph iff* $\mathrm{rad}(G) \leq r - 1$.

Proof. We first assume that $\mathrm{rad}(G) \leq r - 1$ and x_i is the central vertex of G'. Then $\{x_i\}$ is the r-perfect code in $G \circ H$.

Conversely, suppose that $G \circ H$ has an r-perfect code $C_{G\circ H}$. There are two possible cases for the elements of $C_{G\circ H}$ as follows:

Case 1. $C_{G\circ H} \cap G' = \emptyset$. Thus, by Proposition 4.2, $|C_{G\circ H} \cap V(H_i)| \leq 1$, $i = 1, 2, \ldots, |V(G)|$. To the contrary, assume that $\mathrm{rad}(G) > r - 1$. Let u be an element of $C_{G\circ H}$ which $u \in V(H_i)$. Since $\mathrm{rad}(G) > r - 1$, then there exists $x_j x_l$ in the edge set of G' that $d_{G\circ H}(u, x_j) = r$ and $d_{G\circ H}(u, x_l) = r + 1$. Therefore, $d_{G\circ H}(u, v) = r + 1$ for each $v \in V(H_j)$, contrary to the assumption that $C_{G\circ H}$ is an r-perfect code.

Case 2. All elements of $C_{G\circ H}$ are in G'. By proposition 4.1, $|C_{G\circ H}| = 1$ and so $\mathrm{rad}(G) \leq r - 1$.

□

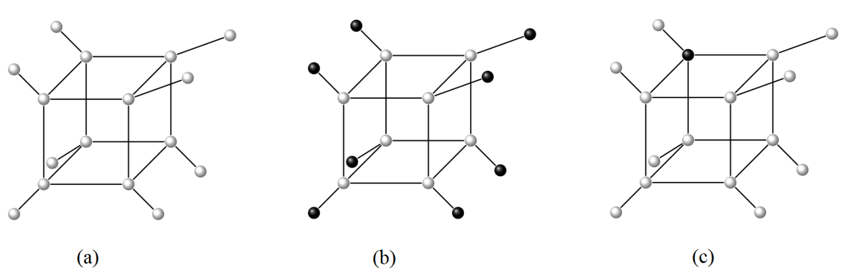


Figure 4. (a) G; (b) the 1-perfect code in G; (c) the 4-perfect code in G.

Example 4. *Octanitrocubane is the most powerful chemical explosive with formula $C_8(NO2)_8$. Let G be the graph of this molecule, see part (a) of Figure 4. So, It is clear that G is formed by corona product Q_3 with P_1. That is $G \cong Q_3 \circ P_1$. According to Proposition 4.3, G has the 1-perfect code as shown in part (b) of Figure 4. On the other hand, Since Q_3 and P_1 don't have conditions of Theorem 4.1, then G doesn't have any 2-perfect code. Moreover, applying Theorem 4.1 and this fact that $\mathrm{rad}(Q_3) = 3$, G is a 4-perfect graph as shown in part (c) of Figure 4.*

A caterpillar is a tree in which all the vertices are within distance 1 of a central path.

Example 5. *Consider a caterpillar $H = P_n \circ \bar{K}_m$. By previous results, if $n = 1$, H is a 1-perfect graph and if $n > 1$, then H is a $([\frac{n}{2}] + 1)$-perfect graph.*

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