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Squared distance matrix of a weighted tree

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Abstract

Let T be a tree with vertex set $\{1,\dots,n\}$ such that each edge is assigned a nonzero weight. The squared distance matrix of T, denoted by Δ, is the $n\times n$ matrix with (i,j)-element $d(i,j)^2$, where $d(i,j)$ is the sum of the weights of the edges on the (ij)-path. We obtain a formula for the determinant of Δ. A formula for Δ^{-1} is also obtained, under certain conditions. The results generalize known formulas for the unweighted case.

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1. Introduction

Let G be a connected graph with vertex set $V(G)=\{1,\dots,n\}$. The distance between vertices $i,j\in V(G)$, denoted $d(i,j)$, is the minimum length (the number of edges) of a path from i to j (or an ij-path). We set $d(i,i)=0, i=1,\dots,n$. The distance matrix $D(G)$, or simply D, is the $n\times n$ matrix with (i,j)-element $d_{ij}=d(i,j)$.

A classical result of Graham and Pollak [7] asserts that if T is a tree with n vertices, then the determinant of the distance matrix D of T is $(-1)^{n-1}(n-1)2^{n-2}$. Thus the determinant depends only on the number of vertices in the tree and not on the tree itself. A formula for the inverse of the distance matrix of a tree was given by Graham and Lovász [6]. Several extensions and generalizations of these results have been proved (see, for example [1], [2], [5], [8], [9] and the references contained therein).

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Let T be a tree with vertex set $\{1, \ldots, n\}$ and let D be the distance matrix of T. The squared distance matrix Δ is defined to be the Hadamard product $D \circ D$, and thus has the (i,j)-element $d(i,j)^2$. A formula for the determinant of Δ was proved in [3], while the inverse and the inertia of Δ were considered in [4].

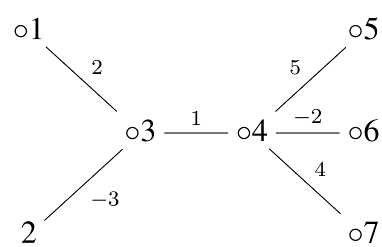
In this paper we consider weighted trees. Let T be a tree with vertex set $V(T) = \{1, \ldots, n\}$ and edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$. We assume that each edge is assigned a weight and let the weight assigned to e_i be denoted w_i, which is a nonzero real number (not necessarily positive).

For $i, j \in V(T), i \neq j$, the distance $d(i,j)$ is defined to be the sum of the weights of the edges on the (unique) ij-path. We set $d(i,i) = 0, i = 1, \ldots, n$. Let D be the $n \times n$ distance matrix with $d_{ij} = d(i,j)$.

The Laplacian of T is the $n \times n$ matrix defined as follows. The rows and the columns of L are indexed by $V(T)$. For $i \neq j$, the (i,j)-element is 0 if i and j are not adjacent. If i and j are adjacent, and if the edge joining them is e_k, then the (i,j)-element of L is set equal to $-1/w_k$. The diagonal elements of L are defined so that L has zero row (and column) sums.

The paper is organized as follows. In this section we review some basic properties of the distance matrix of a tree such as formulas for its determinant and inverse. Some preliminary results are obtained in Section 2. Sections 3 and 4 are devoted to the determinant and the inverse of Δ, respectively.

Example. Consider the tree



The Laplacian of the tree is given by

$$\begin{bmatrix} 1/2 & 0 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & -1/3 & 1/3 & 0 & 0 & 0 & 0 \\ -1/2 & 1/3 & 7/6 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 19/20 & -1/5 & 1/2 & -1/4 \\ 0 & 0 & 0 & -1/5 & 1/5 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -1/4 & 0 & 0 & 1/4 \end{bmatrix}.$$

We let Q be the $n \times (n-1)$ vertex-edge incidence matrix of the underlying unweighted tree, with an orientation assigned to each edge. Thus the rows and the columns of Q are indexed by $V(T)$ and $E(T)$ respectively. If $i \in V(T), e_j \in E(T)$, the (i,j)-element of Q is 0 if i and e_j are not incident, it is $1(-1)$ if i and e_j are incident and i is the initial (terminal) vertex of e_j. It is well-known [1] that Q has rank $n-1$ and any minor of Q is either 0 or ± 1 (thus Q is totally unimodular).

Let F be the $n \times n$ diagonal matrix with diagonal elements $w_1, \ldots, w_{n-1}$. It can be verified that $L = QF^{-1}Q'$.

Lemma 1.1. *The following assertions are true:*

(i) $Q'DQ = -2F$.

(ii) $LDL = -2L$.

Proof. (i). The result follows from the following observation which is easily verified: If $e_p = \{i, j\}$ and $e_q = \{k, \ell\}$ are edges of T, then

$$d(i,k) + d(j,\ell) - d(i,\ell) - d(j,k)$$

equals 0 if e_p and e_q are distinct, and equals $-2w_p$, if $e_p = e_q$.

(ii). We have

$$\begin{aligned} LDL &= QF^{-1}Q'DQF^{-1}Q' \\ &= QF^{-1}(-2F)F^{-1}Q' \text{ by (i)} \\ &= -2QF^{-1}Q' \\ &= -2L, \end{aligned}$$

and the proof is complete. ■

Let δ_i denote the degree of the vertex $i, i = 1, \ldots, n$, and let δ be the $n \times 1$ vector with components $\delta_1, \ldots, \delta_n$. We set $\tau_i = 2 - \delta_i, i = 1, \ldots, n$, and let τ be the $n \times 1$ vector with components $\tau_1, \ldots, \tau_n$.

Theorem 1.1. *The following assertions are true:*

(i) $\det D = (-1)^{n-1}2^{n-2}(\sum_i w_i)(\prod_i w_i)$.

(ii) If $\sum_i w_i \neq 0$, *then* D *is nonsingular and*

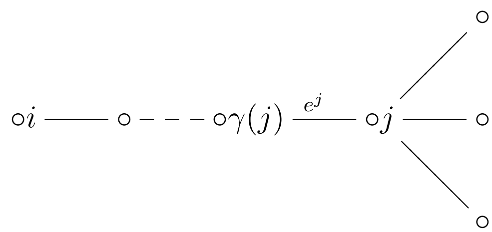
$$D^{-1} = -\frac{1}{2}L + \frac{1}{2\sum_i w_i}\tau\tau'.$$

(iii) $D\tau = (\sum_i w_i)\mathbf{1}$.

Proof. Parts (i) and (ii) are well-known, see for example, [2]. To prove (iii), note that from (ii),

$$D^{-1}\mathbf{1} = \frac{1}{2\sum_i w_i}\tau\tau'\mathbf{1} = \frac{1}{\sum_i w_i}\tau,$$

since $\mathbf{1}'\tau = 2$. It follows that $D\tau = (\sum_i w_i)\mathbf{1}$ and the proof is complete. ■



2. Preliminary results

We now turn to the main results for the case of a weighted tree. Let T be a tree with vertex set $V(T) = \{1, \ldots, n\}$ and edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$. Let $w_1, \ldots, w_{n-1}$ be the edge-weights. Recall that δ_i is the degree of vertex i and $\tau_i = 2 - \delta_i$. We write $j \sim i$ if vertex j is adjacent to vertex i. We let $\hat{\delta}_i$ be the weighted degree of i, which is defined as

$$\hat{\delta}_i = \sum_{j:j\sim i} w(\{i,j\}), i = 1, \ldots, n.$$

Let $\hat{\delta}$ be the $n \times 1$ vector with components $\hat{\delta}_1, \ldots, \hat{\delta}_n$.

Let Δ be the squared distance matrix of T, which is the $n \times n$ matrix with its (i,j)-element equal to d_{ij}^2 or equivalently, $d(i,j)^2$. The next result was obtained in [4] for the unweighted case,

Lemma 2.1. $\Delta\tau = D\hat{\delta}$.

Proof. Let $i \in \{1, \ldots, n\}$ be fixed. For $j \neq i$, let $\gamma(j)$ be the predecessor of j on the ij-path (in the underlying unoriented tree). Let e^j be the edge $\{\gamma(j), j\}$ and set $\theta^j = \hat{\delta}_j - w(e^j)$. We have

$$\begin{aligned} & 2\sum_{j=1}^{n} d(i,j)^2 \\ = & \sum_{j=1}^{n} d(i,j)^2 + \sum_{j\neq i}(d(i,\gamma(j)) + w(e^j))^2 \\ = & \sum_{j=1}^{n} d(i,j)^2 + \sum_{j\neq i} d(i,\gamma(j))^2 + 2\sum_{j\neq i} d(i,\gamma(j))w(e^j) + \sum_{j\neq i} w(e^j)^2. \end{aligned} \tag{1}$$

Note that

$$\sum_{j\neq i} d(i,\gamma(j))^2 = \sum_{j=1}^{n}(\delta_j - 1)d(i,j)^2, \tag{2}$$

since vertex j serves as a predecessor of $\delta_j - 1$ vertices in paths from i. Also note that

$$\sum_{j\neq i} w(e^j)^2 = \sum_{k=1}^{n-1} w(e_k)^2. \tag{3}$$

We have

$$\begin{aligned} & \sum_{j=1}^{n} d(i,j)\hat{\delta}_j \\ = \; & \sum_{j\neq i}(d(i,\gamma(j)+w(e^j))(w(e^j)+\theta^j) \\ = \; & \sum_{j\neq i} d(i,\gamma(j))w(e^j)+\sum_{j\neq i} w(e^j)^2+\sum_{j\neq i}(d(i,\gamma(j))+w(e^j))\theta^j. \end{aligned} \tag{4}$$

Observe that θ^j is the sum of the weights of all the edges incident to j, except the edge e^j, which is on the ij-path. Thus $(d(i,\gamma(j))+w(e^j))\theta^j$ equals $\sum d(i,\gamma(\ell))w(e^\ell)$, where the summation is over all vertices adjacent to j, except i. Therefore it follows that

$$\sum_{j\neq i} d(i,\gamma(j))w(e^j)=\sum_{j\neq i}(d(i,\gamma(j))+w(e^j))\theta^j. \tag{5}$$

From (1)-(5) we get

$$2\sum_{i=1}^{n} d(i,j)^2=\sum_{j=1}^{n} d(i,j)^2\delta_j+\sum_{j=1}^{n} d(i,j)\hat{\delta}_j,$$

which is equivalent to

$$\sum_{i=1}^{n} d(i,j)^2\tau_j=\sum_{j=1}^{n} d(i,j)\hat{\delta}_j,$$

and the proof is complete. ■

Next we define the edge orientation matrix of T. We assign an orientation to each edge of T. Let $e_i=(p,q); e_j=(r,s)$ be edges of T. We say that e_i and e_j are similarly oriented, denoted by $e_i \Rightarrow e_j$, if $d(p,r)=d(q,s)$. Otherwise e_i and e_j are said to be oppositely oriented, denoted by $e_i \rightleftharpoons e_j$. For example, in the following diagram e_i and e_j are similarly oriented.

$$\circ p \longrightarrow \circ q - - - \circ r \longrightarrow \circ s$$

The edge orientation matrix of T is the $(n-1)\times(n-1)$ matrix H having the rows and the columns indexed by the edges of T. The (i,j)-element of H, denoted by $h(i,j)$ is defined to be $1(-1)$ if the corresponding edges e_i, e_j of T are similarly (oppositely) oriented. The diagonal elements of H are set to be 1. We assume that the same orientation is used while defining the matrix H and the incidence matrix Q.

If the tree T has no vertex of degree 2, then we let $\hat{\tau}$ be the diagonal matrix with diagonal elements $1/\tau_1,\ldots,1/\tau_n$. We state some basic properties of H next, see [3].

Theorem 2.1. *Let T be a directed tree on n vertices, let H and Q be the edge orientation matrix and the vertex-edge incidence matrix of T, respectively. Then* $\det H = 2^{n-2}\prod_{i=1}^{n}\tau_i$. *Furthermore, if T has no vertex of degree 2, then H is nonsingular and $H^{-1}=\frac{1}{2}Q'\hat{\tau}Q$.*

Let $w_1, \ldots, w_{n-1}$ be the edge-weights. Recall that F be the diagonal matrix with diagonal elements $w_1, \ldots, w_{n-1}$.

Also note that,

$$(FHF)_{ij} = \begin{cases} w_i w_j, & \text{if } e_i \Rightarrow e_j; \\ -w_i w_j, & \text{if } e_i \rightleftharpoons e_j. \end{cases}$$

Lemma 2.2. $Q'\Delta Q = -2FHF$.

Proof. For $i, j \in \{1, \ldots, n-1\}$, let the edge e_i be from p to q and the edge e_j be from r to s. Then

$$(Q'\Delta Q)_{ij} = \begin{cases} d(p,r)^2 + d(q,s)^2 - d(p,s)^2 - d(q,r)^2, & \text{if } e_i \Rightarrow e_j; \\ d(p,s)^2 + d(q,r)^2 - d(p,r)^2 - d(q,s)^2, & \text{if } e_i \rightleftharpoons e_j. \end{cases} \quad (6)$$

Let $d(r,s) = \alpha$. It follows from (6) that

$$\begin{aligned} (Q'\Delta Q)_{ij} &= \begin{cases} (w_i+\alpha)^2 + (w_j+\alpha)^2 - (w_i+w_j+\alpha)^2 - \alpha^2 = -2w_i w_j, & \text{if } e_i \Rightarrow e_j; \\ (w_i+w_j+\alpha)^2 + \alpha^2 - (w_i+\alpha)^2 - (w_j+\alpha)^2 = 2w_i w_j, & \text{if } e_i \rightleftharpoons e_j. \end{cases} \\ &= -2(FHF)_{ij}, \end{aligned}$$

and the proof is complete. ∎

Let $\tilde{\tau}$ be the diagonal matrix with diagonal elements $\tau_1, \ldots, \tau_n$.

Lemma 2.3. $\Delta L = 2D\tilde{\tau} - \mathbf{1}\hat{\delta}'$.

Proof. Let $i, j \in \{1, \ldots, n\}$ be fixed. Let vertex j have degree p. Suppose j is adjacent to vertices $u_1, \ldots, u_p$ and let $e_{\ell_1}, \ldots, e_{\ell_p}$ be the corresponding edges with weights $w_{\ell_1}, \ldots, w_{\ell_p}$, respectively. We consider two cases.

Case 1. $i = j$. We have

$$\begin{aligned} (\Delta L)_{jj} &= \sum_{k=1}^{n} d(j,k)^2 \ell_{kj} \\ &= w_{\ell_1}^2(-w_{\ell_1})^{-1} + \cdots + w_{\ell_p}^2(-w_{\ell_p})^{-1} \\ &= -(w_{\ell_1} + \cdots + w_{\ell_p}) \\ &= -\hat{\delta}_j. \end{aligned}$$

Since the (j,j)-element of $2D\tilde{\tau} - \mathbf{1}\hat{\delta}'$ is $-\hat{\delta}_j$, the proof is complete in this case.

Case 2. $i \neq j$. We assume, without loss of generality, that the ij-path passes through u_1 (it is possible that $i = u_1$). Let $d(i,j) = \alpha$. Then $d(i,u_1) = \alpha - w_{\ell_1}, d(i,u_2) = \alpha + w_{\ell_2}, \ldots, d(i,u_p) =$

$\alpha + w_{\ell_p}$. We have

$$
\begin{aligned}
(\Delta L)_{ij} &= \sum_{k=1}^{n} d(i,k)^2 \ell_{kj} \\
&= d(i,u_1)^2(-w_{\ell_1})^{-1} + \cdots + d(i,u_p)^2(-w_{\ell_p})^{-1} + d(i,j)^2\ell_{jj} \\
&= (\alpha - w_{\ell_1})^2(-w_{\ell_1})^{-1} + (\alpha + w_{\ell_2})^2(-w_{\ell_2})^{-1} + \cdots + (\alpha + w_{\ell_p})^2(-w_{\ell_p})^{-1} \\
&+ \alpha^2((w_{\ell_1})^{-1} + \cdots + (w_{\ell_p})^{-1}) \\
&= (-2\alpha w_{\ell_1} + w_{\ell_1}^2)(-w_{\ell_1})^{-1} + (2\alpha w_{\ell_2} + w_{\ell_2}^2)(-w_{\ell_2})^{-1} + \cdots \\
&+ (2\alpha w_{\ell_p} + w_{\ell_p}^2)(-w_{\ell_p})^{-1} \\
&= 2\alpha - 2\alpha(p-1) - (w_{\ell_1} + \cdots + w_{\ell_p}) \\
&= 2\alpha\tau_j - (w_{\ell_1} + \cdots + w_{\ell_p}),
\end{aligned}
$$

which is the (i,j)-element of $2D\tilde{\tau} - \mathbf{1}\hat{\delta}'$ and the proof is complete. ■

3. Determinant

Our next objective is to obtain a formula for the determinant of the squared distance matrix. We first consider the case when the tree has no vertex of degree 2.

Theorem 3.1. *Let T be a tree with vertex set $V(T) = \{1, \ldots, n\}$, edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$, and edge weights $w_1, \ldots, w_{n-1}$. Suppose T has no vertex of degree 2. Then*

$$\det \Delta = (-1)^{n-1} \frac{4^{n-2}}{2} \prod_{i=1}^{n} \tau_i \prod_{i=1}^{n-1} w_i^2 \sum_{i=1}^{n} \frac{\hat{\delta}_i^2}{\tau_i}. \tag{7}$$

Proof. We assign an orientation to the edges of the tree and let H and Q be, respectively, edge orientation matrix and the vertex-edge incidence matrix of T.

Let Δ_i denote the i-th column of Δ, and let t_i be the column vector with 1 at the i-th place and zeros elsewhere, $i = 1, \ldots, n$. Then

$$\begin{bmatrix} Q' \\ t_1' \end{bmatrix} \Delta \begin{bmatrix} Q & t_1 \end{bmatrix} = \begin{bmatrix} Q'\Delta Q & Q'\Delta_1 \\ \Delta_1' Q & 0 \end{bmatrix}. \tag{8}$$

Since $\det \begin{bmatrix} Q' \\ t_1' \end{bmatrix} = \pm 1$, it follows from (8) that

$$\begin{aligned}
\det \Delta &= \begin{bmatrix} Q'\Delta Q & Q'\Delta_1 \\ \Delta_1' Q & 0 \end{bmatrix} \\
&= \begin{bmatrix} -2FHF & Q'\Delta_1 \\ \Delta_1' Q & 0 \end{bmatrix} \text{ by Lemma 2.1} \\
&= (\det(-2FHF))(-\Delta_1' Q(-2FHF)^{-1}Q'\Delta_1) \\
&= (-2)^{n-1}\prod_{i=1}^{n-1} w_i^2 (\det H) 2\Delta_1' Q F^{-1} H^{-1} F^{-1} Q' \Delta_1 \\
&= (-1)^{n-1} 2^n \prod_{i=1}^{n-1} w_i^2 (\det H) \Delta_1' Q F^{-1} Q' \hat{\tau} Q F^{-1} Q' \Delta_1, \qquad (9)
\end{aligned}$$

in view of Theorem 2.1.

By Lemma 2.2 we have

$$\begin{aligned}
\Delta_1' Q F^{-1} Q' \hat{\tau} Q F^{-1} Q' \Delta_1 &= \sum_i (2d_{1i}\tau_i - \hat{\delta}_i)^2 \frac{1}{\tau_i} \\
&= \sum_i (4d_{1i}^2\tau_i^2 + \hat{\delta}_i^2 - 4d_{1i}\tau_i\hat{\delta}_i)\frac{1}{\tau_i} \\
&= \sum_i 4d_{1i}^2\tau_i + \sum_i \frac{\hat{\delta}_i^2}{\tau_i} - 4\sum_i d_{1i}\hat{\delta}_i \qquad (10)
\end{aligned}$$

It follows from (10) and Lemma 2.1 that

$$\Delta_1' Q F^{-1} Q' \hat{\tau} Q F^{-1} Q' \Delta_1 = \sum_i \frac{\hat{\delta}_i^2}{\tau_i}. \qquad (11)$$

Also by Theorem 2.1,

$$\det H = 2^{n-2} \prod_{i=1}^{n} \tau_i. \qquad (12)$$

The proof is complete by substituting (11) and (12) in (9). ■

Corollary 3.1. *[3] Let T be an unweighted tree with vertex set $V(T) = \{1, \dots, n\}$. Suppose T has no vertex of degree 2. Then*

$$\det \Delta = (-1)^n 4^{n-2} \left(2n - 1 - 2\sum_i \frac{1}{\tau_i}\right) \prod_{i=1}^{n} \tau_i. \qquad (13)$$

Proof. We set $w_i = 1, i = 1, \ldots, n-1$ in Theorem 3.1. Then $\hat{\delta}_i = \delta_i = 2 - \tau_i, i = 1, \ldots, n$. We have

$$\begin{aligned}\sum_i \frac{{\delta_i}^2}{\tau_i} &= \sum_i \frac{(2-\tau_i)^2}{\tau_i} \\ &= \sum_i \frac{4+\tau_i^2-4\tau_i}{\tau_i} \\ &= 4\sum_i \frac{1}{\tau_i} + \sum_i \tau_i - 4n \\ &= 4\sum_i \frac{1}{\tau_i} + 2 - 4n \\ &= -2\left(2n-1-2\sum_i \frac{1}{\tau_i}\right). \qquad (14)\end{aligned}$$

The proof is complete by substituting (14) in (7). ■

We turn to the case when there is a vertex of degree 2.

Theorem 3.2. *Let T be a tree with vertex set $V(T) = \{1, \ldots, n\}$, edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$, and edge weights $w_1, \ldots, w_{n-1}$. Let q be a vertex of degree 2 and let p and r be neighbors of q. Let $e_i = (pq), e_j = (qr)$. Then*

$$\det \Delta = (-1)^{n-1} 2^{2n-5} (w_i + w_j)^2 \prod_{s=1}^{n-1} w_s^2 \prod_{k \neq q} \tau_k. \qquad (15)$$

Proof. We assume, without loss of generality, that e_i is directed from p to q and e_j is directed from q to r.

$$\circ p \xrightarrow{e_i} \circ q \xrightarrow{e_j} \circ r$$

Let z_q be the $n \times 1$ unit vector with 1 at the q-th place and zeros elsewhere. Let Δ_q be the q-th column of Δ. We have

$$\begin{bmatrix} Q' \\ z_q' \end{bmatrix} \Delta \begin{bmatrix} Q & z_q \end{bmatrix} = \begin{bmatrix} Q'\Delta Q & Q'\Delta_q \\ \Delta_q' Q & 0 \end{bmatrix} = \begin{bmatrix} -2FHF & Q'\Delta_q \\ \Delta_q' Q & 0 \end{bmatrix}, \qquad (16)$$

in view of Lemma 2.2. It follows from (16) that

$$\begin{bmatrix} F^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Q' \\ z_q' \end{bmatrix} \Delta \begin{bmatrix} Q & z_q \end{bmatrix} \begin{bmatrix} F^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2H & F^{-1}Q'\Delta_q \\ \Delta_q' Q F^{-1} & 0 \end{bmatrix}. \qquad (17)$$

Taking determinants of matrices in (17) we get

$$(\det F^{-1})^2 \det \Delta = \det \begin{bmatrix} -2H & F^{-1}Q'\Delta_q \\ \Delta_q' Q F^{-1} & 0 \end{bmatrix}. \qquad (18)$$

Note that the i-th and the j-th columns of H are identical.

Let $H(j|j)$ denote the submatrix obtained by deleting row j and column j from H. In $\begin{bmatrix} -2H & F^{-1}Q'\Delta_q \\ \Delta_q' QF^{-1} & 0 \end{bmatrix}$, subtract column i from column j, row i from row j, and then expand the determinant along column j. Then we get

$$\begin{aligned} \det \begin{bmatrix} -2H & F^{-1}Q'\Delta_q \\ \Delta_q' QF^{-1} & 0 \end{bmatrix} &= -((\Delta_q' QF^{-1}))_j - (\Delta_q' QF^{-1})_j)^2 \det(-2H(j|j)) \\ &= -(-2)^{n-2} \det H(j|j)(-w_j - w_i)^2, \end{aligned} \tag{19}$$

Note that $H(j|j)$ is the edge orientation matrix of the tree obtained by deleting vertex q and replacing edges e_i and e_j by a single edge directed from p to r in the tree. Hence by Theorem 2.1,

$$\det H(j|j) = 2^{n-3} \prod_{k \neq q} \tau_k. \tag{20}$$

It follows from (17),(18) and (19) that

$$\begin{aligned} \det \Delta &= -(\det F)^2 (-1)^n 2^{n-2} 2^{n-3} (\prod_{k \neq q} \tau_k)(w_i + w_j)^2 \\ &= (-1)^{n-1} 2^{2n-5} (w_i + w_j)^2 \prod_{s=1}^{n-1} w_s^2 \prod_{k \neq q} \tau_k, \end{aligned} \tag{21}$$

and the proof is complete. ■

Corollary 3.2. *Let T be a tree with vertex set $V(T) = \{1, \ldots, n\}$, edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$, and edge weights $w_1, \ldots, w_{n-1}$. Suppose T has at least two vertices of degree 2. Then $\det \Delta = 0$.*

Proof. The result follows from Theorem 3.2 since $\tau_i = 0$ for at least two values of i. ■

4. Inverse

We now turn to the inverse of Δ, when it exists. When the tree has no vertex of degree 2, we can give a concise formula for the inverse. We first prove some preliminary results.

Lemma 4.1. *Let the tree have no vertex of degree 2. Then*

$$\Delta(2\tau - L\hat{\tau}\hat{\delta}) = (\hat{\delta}'\hat{\tau}\hat{\delta})\mathbf{1}. \tag{22}$$

Proof. By Lemma 2.3, $\Delta L = 2D\tilde{\tau} - \mathbf{1}\hat{\delta}'$. Hence

$$\Delta L\hat{\tau}\hat{\delta} = 2D\hat{\delta} - (\hat{\delta}'\hat{\tau}\hat{\delta})\mathbf{1}. \tag{23}$$

Since by Lemma 2.1, $\Delta\tau = D\hat{\delta}$, we obtain the result from (23). ■

For a square matrix A, we denote by $\operatorname{cof} A$, the sum of the cofactors of A.

Lemma 4.2. *Let T be a tree with vertex set $V(T) = \{1, \dots, n\}$, edge set $E(T) = \{e_1, \dots, e_{n-1}\}$, and edge weights $w_1, \dots, w_{n-1}$. Suppose T has no vertex of degree 2. Then*

$$\operatorname{cof} \Delta = (-1)^{n-1} 2^{2n-3} \prod_{k=1}^{n-1} w_k^2 \prod_{i=1}^{n} \tau_i. \tag{24}$$

Proof. By Lemma 2.2, $Q'\Delta Q = -2FHF$. Taking determinant of both sides and using Cauchy-Binet formula, we get

$$\begin{aligned} \operatorname{cof} \Delta &= (-2)^{n-1} (\det F)^2 \det H \\ &= (-2)^{n-1} \prod_{k=1}^{n-1} w_k^2 2^{n-2} \prod_{i=1}^{n} \tau_i \text{ by Theorem 2.1} \\ &= (-1)^{n-1} 2^{2n-3} \prod_{k=1}^{n-1} w_k^2 \prod_{i=1}^{n} \tau_i, \end{aligned} \tag{25}$$

and the proof is complete. ■

Corollary 4.1. *Let the tree have no vertex of degree 2 and let $\beta = \hat{\delta}'\hat{\tau}\hat{\delta}$. If $\beta \neq 0$, then Δ is nonsingular and*

$$\mathbf{1}'\Delta^{-1}\mathbf{1} = \frac{4}{\beta}. \tag{26}$$

Proof. Observe that $\beta = \sum_{i=1}^{n} \frac{\hat{\delta}_i^2}{\tau_i}$. By Theorem 3.1,

$$\det \Delta = (-1)^{n-1} \frac{4^{n-2}}{2} \prod_{i=1}^{n} \tau_i \prod_{i=1}^{n-1} w_i^2 \sum_{i=1}^{n} \frac{\hat{\delta}_i^2}{\tau_i}. \tag{27}$$

If $\beta \neq 0$, then Δ is nonsingular by (27). Note that $\mathbf{1}'\Delta^{-1}\mathbf{1} = \frac{\operatorname{cof} \Delta}{\det \Delta}$. The proof is complete using Lemma 4.2 and (27). ■

Theorem 4.1. *Let the tree have no vertex of degree 2 and let $\beta = \hat{\delta}'\hat{\tau}\hat{\delta}$. Let $\eta = 2\tau - L\hat{\tau}\hat{\delta}$. If $\beta \neq 0$, then Δ is nonsingular and*

$$\Delta^{-1} = -\frac{1}{4} L\hat{\tau}L + \frac{1}{4\beta}\eta\eta'. \tag{28}$$

Proof. Let $X = -\frac{1}{4}L\hat{\tau}L + \frac{1}{4\beta}\eta\eta'$. Then

$$\Delta X = -\frac{1}{4}\Delta L\hat{\tau}L + \frac{1}{4\beta}\Delta\eta\eta'. \tag{29}$$

By Lemma 2.3, $\Delta L = 2D\tilde{\tau} - \mathbf{1}\hat{\delta}'$. Hence

$$\Delta L\hat{\tau}L = 2DL - \mathbf{1}\hat{\delta}'\hat{\tau}L. \tag{30}$$

Using Theorem 1.1, we can see that

$$DL = -2I + \mathbf{1}\tau'. \tag{31}$$

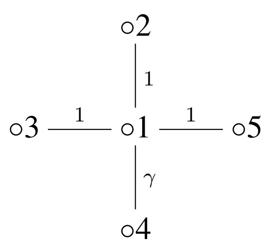
Finally, by Lemma 4.1, $\Delta\eta = \beta$. This fact and (29), (30) and (31) lead to

$$\Delta X = I - \frac{1}{2}\mathbf{1}\tau' + \frac{1}{4}\hat{\delta}'\hat{\tau}L + \frac{1}{4\beta}\mathbf{1}\eta'. \tag{32}$$

Since $\eta = 2\tau - L\hat{\tau}\hat{\delta}$, it follows from (32) that $\Delta X = I$ and the proof is complete. ■

We conclude with an example to show that the condition $\beta \neq 0$ is necessary in Theorem 4.1.

Example Consider the tree



The distance matrix of the tree is given by

$$D = \begin{bmatrix} 0 & 1 & 1 & \gamma & 1 \\ 1 & 0 & 2 & 1+\gamma & 2 \\ 1 & 2 & 0 & 1+\gamma & 2 \\ \gamma & 1+\gamma & 1+\gamma & 0 & 1+\gamma \\ 1 & 2 & 2 & 1+\gamma & 0 \end{bmatrix}.$$

It can be checked that $\det \Delta = -32\gamma^2(\gamma^2 - 6\gamma - 3)$. Thus Δ is singular if $\gamma = 3 + 2\sqrt{3}$. Note that $\hat{\delta}' = [\gamma + 3, 1, 1, \gamma, 1]$, $\tau' = [-2, 1, 1, 1, 1]$ and hence, if $\gamma = 3 + 2\sqrt{3}$, then $\sum_{i=1}^{4} \frac{\hat{\delta}^2}{\tau_i} = 0$.

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