



Two classes of non-Leech trees

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Abstract

Let T be a tree of order n . For any edge labeling $f : E \rightarrow \{1, 2, 3, \dots\}$ the weight of a path P is the sum of the labels of the edges of P and is denoted by $w(P)$. If the weights of the ${}^n C_2$ paths in T are exactly $1, 2, \dots, {}^n C_2$, then f is called a Leech labeling and a tree which admits a Leech labeling is called a Leech tree. In this paper we determine all Leech trees having diameter three. We also prove that the tree obtained from the path $P_n = (v_1, v_2, \dots, v_n)$ by attaching a pendent vertex at v_{n-1} is not a Leech tree for all $n \geq 4$.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph with neither loops nor multiple edges. The order $|V|$ and the size $|E|$ are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [2].

Let $f : E \rightarrow \{1, 2, 3, \dots\}$ be an edge labeling of G . The weight of a path P in G is the sum of the labels of the edges of P and is denoted by $w(P)$. Leech [5] introduced the concept of a Leech tree, while considering a problem in electrical engineering, where edge labels represent electrical

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resistance. Let T be a tree of order n . An edge labeling $f : E \rightarrow \{1, 2, 3, \dots\}$ is called a Leech labeling if the weights of the ${}^n C_2$ paths in T are exactly $1, 2, \dots, {}^n C_2$. A tree which admits a Leech labeling is called a Leech tree. Since each edge label is the weight of a path of length one, it follows that f is an injection and 1,2 are edge labels for all $n \geq 3$. Leech found five Leech trees which are given in Figure 1 and these are the only known Leech trees.

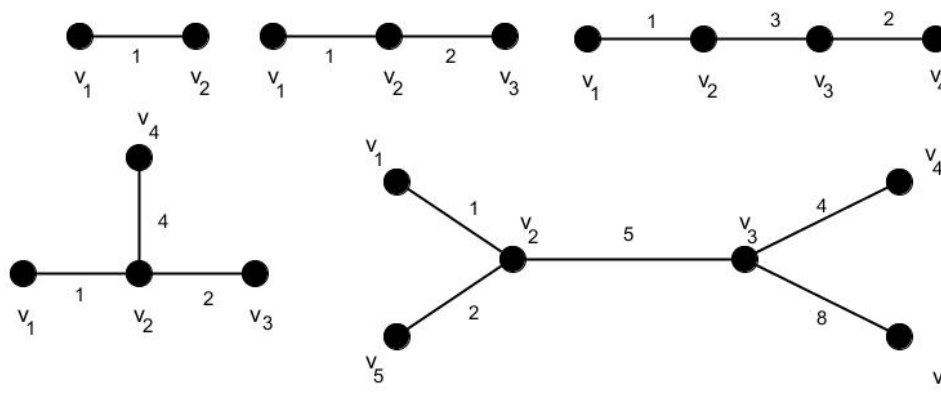


Figure 1. Leech trees.

Taylor [8] proved that if T is a Leech tree of order n , then $n = k^2$ or $k^2 + 2$ for some integer k . Since then it has been proved by several authors ([1],[7],[9]) that no Leech trees of order 9, 11 or 16 exist, leaving $n = 18$ as the smallest open case. Some variations of Leech trees such as modular Leech trees ([3], [4]), minimal distinct distance trees [1] and leaf-Leech trees [6] have been investigated by several authors.

Szekely et al. [7] have proved the following theorem.

Theorem 1.1. [7] *If there is a Leech tree on n vertices, then it has no paths larger than $\frac{n}{\sqrt{2}}(1 + o(1))$.*

A double star $B_{r,s}$ is a tree with vertex set $\{u, v, u_1, u_2, \dots, u_r, v_1, v_2, \dots, v_s\}$ in which uv is an edge, each u_i is a pendent vertex adjacent to u and each v_j is pendent vertex adjacent to v . If both r and s are not zero, the double star $B_{r,s}$ are precisely the trees having diameter three. In this paper we determine all Leech trees having diameter three. We also prove that the tree obtained from the path $P_n = (v_1, v_2, \dots, v_n)$ by attaching a pendent vertex at v_{n-1} is not a Leech tree for all $n \geq 4$.

2. Leech trees with diameter three

For any positive integer n , we denote the set $\{1, 2, \dots, n\}$ by $[n]$. We proceed to prove that the double stars $B_{1,1}$ and $B_{2,2}$ are the only Leech trees of diameter three.

Theorem 2.1. *The double star $T = B_{r,s}$, where both r and s are non-zero, is a Leech tree if and only if $T = B_{1,1}$ or $B_{2,2}$.*

Proof. Let f be a Leech labeling of $B_{r,s}$ where $r \geq s \geq 1$. Let S denote the set of all path weights in the labeling process. Since uv is the unique nonpendent edge of T , the label 1 is assigned to either uv or to a pendent edge. We consider two cases.

Case 1: $f(uv) = 1$.

Without loss of generality, let $f(uu_1) = 2$. Then $S = [3]$. The next available edge label is 4, which has to be used to get path of weight 4. If $f(vv_1) = 4$, then $S = [7] - \{6\}$. To get path weight 6, one of the pendent edges must be assigned the label 6 and this gives two paths of weight 7, which is a contradiction. Hence $f(uu_2) = 4$ and $S = [6]$. To get path weight 7 an edge has to be assigned label 7. If $f(uu_3) = 7$, then $S = [11] - \{10\}$ and a path of weight 10 cannot be obtained in any subsequent labeling. If $f(vv_1) = 7$, then $S = [11] - \{9\}$ and a path of weight 9 cannot be obtained in any subsequent labeling.

Case 2: $f(uu_1) = 1$.

We consider three subcases.

Subcase a: $f(uv) = 2$.

Then $S = \{1, 2, 3\}$. If $f(vv_1) = 4$, then $S = [7] - \{5\}$. If an edge incident at v is given the label 5, then path weight 7 is repeated. If an edge incident at u is given the label 5, then path weight 6 is repeated. Hence $f(uu_2) = 4$ and $S = [6]$. The next available edge label is 7 which has to be used to get path weight 7. If $f(vv_1) = 7$, then $S = [13] - \{8, 11, 12\}$. Now if an edge is assigned label 8, then we get two paths with weight 10, a contradiction. Hence $f(uu_3) = 7$ and $S = [11] - \{10\}$. Now if $f(uu_4) = 10$, then path weight 11 is repeated. If $f(vv_1) = 10$, then $S = [19] - \{14, 15, 17, 18\}$ and if any edge is assigned label 14, then path weight 16 is repeated.

Subcase b: $f(uu_2) = 2$.

Then $S = [3]$. If $f(uv) = 4$, then proceeding as in subcase a, we get a contradiction. Suppose $f(uu_3) = 4$. Then $S = [6]$. Now the label 7 can be assigned to one of the edges uu_4, uv or vv_1 . Suppose $f(uu_4) = 7$. Then $S = [11] - \{10\}$. If 10 is assigned to uu_5 or uv , then path weight 11 is repeated. Hence let $f(vv_1) = 10$. Then $S = [11]$. Now if $f(uu_5) = 12$, then the paths (u_5, u, v) and (v_1, v, u, u_2) both have weight $12 + f(uv)$. A similar contradiction arises if $f(vv_j) = 12$. Hence $f(uv) = 12$ and $S = [29] - \{15, 17, 18, 20, 21, 25, 27, 28\}$. Now no edge can be assigned the label 15, which is a contradiction. Next, we take $f(uv) = 7$. Then $S = [11] - \{10\}$. Now any edge incident at u cannot be assigned label 10, since this gives two paths of weight 11. Suppose $f(vv_1) = 10$. Then $12 \notin S$ and further 12 cannot be assigned as a label to any edge, which is a contradiction.

Now, suppose $f(vv_1) = 7$. Then 8 cannot be assigned to any pendent edge since this gives two paths of weight $8 + f(uv)$. Hence $f(uv) = 8$. Then $11 \notin S$ and if the label 11 is to be assigned to any edge, we get two paths of weight $11 + f(uv)$ which is again a contradiction.

Finally, suppose $f(vv_1) = 4$. Then $S = [4]$. If 5 is assigned to a pendent edge, then we get two paths of weight $5 + f(uv)$. Hence $f(uv) = 5$ and $S = [11] - \{8\}$. If the label 8 is assigned to uu_3 , we get two paths of weight 9. Hence $f(vv_2) = 8$ and $S = [15]$. This gives a Leech labeling of $B_{2,2}$. If there are more edges, we continue the labeling process. If $f(uu_3) = 16$, then $19 \notin S$ and assigning the label 19 to any edge gives two paths of weight 20 or 21. If $f(vv_3) = 16$, then $17 \notin S$ and label 17 cannot be assigned to any edge.

Subcase c: $f(vv_1) = 2$.

Then $S = \{1, 2\}$. If 3 is assigned to a pendent edge, we get two paths of weight $3 + f(uv)$.

Hence $f(uv) = 3$ and $S = [6]$. This gives a Leech labeling of $B_{1,1}$. Suppose there are more edges. If $f(uu_2) = 7$, then $9 \notin S$ and 9 cannot be assigned to any edge. If $f(vv_2) = 7$, then $8 \notin S$ and the label 8 cannot be assigned to any edge. Thus $B_{2,2}$ and $B_{1,1}$ are the only double stars which are Leech Trees. \square

3. A family of non-Leech trees

All the known Leech trees given in Figure 1 have diameter at most three. Theorem 2.1 implies that there is no other Leech tree with diameter at most three. The path P_n are trees with diameter $n - 1$ and P_n is a Leech tree if and only if $n = 2, 3$ or 4. Thus all trees of diameter $n - 1$ have been characterized. Hence the most natural question in this direction is to determine all Leech trees of diameter $n - 2$. We observe that $K_{1,3}$ is a Leech tree with diameter $n - 2$. Now let $P_n = (v_1, v_2, \dots, v_n)$ be a path on n vertices with $n \geq 3$. Let G_n be the graph obtained from P_n by attaching a pendant vertex v'_n adjacent to v_{n-1} . Clearly G_n is a graph of order $n + 1$ and has diameter $n - 1$. In this section we prove that G_n is not a Leech tree for $n \geq 4$. This shows $K_{1,3}$ is the only Leech tree with diameter $n - 2$, thus giving a characterization of all Leech trees with diameter $n - 2$. Though most of the cases in the proof of this theorem follow from Theorem 1.1, we have given an independent proof and we believe that this proof technique along with Theorem 1.1 will be of help in moving towards the proof of the well known basic conjecture that the number of Leech trees is finite.

Since any tree of order $n < 18$ other than the five Leech trees given in Figure 1 is not a Leech tree, it is enough to prove that G_n is not a Leech tree for $n \geq 17$. We need the following Lemma.

Lemma 3.1. *Suppose f is a Leech labeling of G_n where $n \geq 17$. Let $P = (v'_n, v_{n-1}, v_n)$. Then $w(P) < {}^{n+1}C_2 - 85$.*

Proof. Suppose $w(P) \geq {}^{n+1}C_2 - 85 = \frac{n^2+n-170}{2}$. Then one of the edges $v'_n v_{n-1}$ or $v_{n-1} v_n$ must have label greater than or equal to $\frac{w(P)}{2}$.

$$\text{Let } f(v_{n-1}v_n) \geq \frac{w(P)}{2} \geq \frac{n^2+n-170}{4}.$$

The ${}^{n-1}C_2$ subpaths of the path $(v_1, v_2, \dots, v_{n-1})$ have distinct weights and hence $w((v_1, v_2, \dots, v_{n-1})) \geq {}^{n-1}C_2$.

Hence $w(P_n) \geq {}^{n-1}C_2 + f(v_{n-1}v_n) \geq \frac{(n-1)(n-2)}{2} + \frac{n^2+n-170}{4} = \frac{3n^2-5n-166}{4}$. Since $w(P_n) \leq {}^{n+1}C_2$, it follows that $\frac{3n^2-5n-166}{4} \leq {}^{n+1}C_2$. This gives $n^2 - 7n - 166 \leq 0$. However $n^2 - 7n - 166 > 0$ for $n \geq 17$ which is a contradiction. Hence $w(P) < {}^{n+1}C_2 - 85$. \square

Theorem 3.1. *The tree G_n is not a Leech tree for $n \geq 17$.*

Proof. Suppose there exists a Leech labeling f for G_n . There are three maximal paths $P_n = (v_1, v_2, \dots, v_n)$, $P'_n = (v_1, v_2, \dots, v_{n-1}, v'_n)$ and $P = (v'_n, v_{n-1}, v_n)$ and one of these paths must have weight ${}^{n+1}C_2$. It follows from Lemma 3.1 that either P_n or P'_n has weight ${}^{n+1}C_2$. We may assume without loss of generality that $w(P_n) = {}^{n+1}C_2$. Now the path of weight ${}^{n+1}C_2 - 1$ is either P'_n or $P_n - v_1$.

Case 1: $w(P'_n) = {}^{n+1}C_2 - 1$.

In this case $f(v_{n-1}v'_n) = f(v_{n-1}v_n) - 1$ and $w((v_i, v_{i+1}, \dots, v_{n-1}, v'_n)) = w((v_i, v_{i+1}, \dots, v_{n-1}, v_n)) - 1$. Hence the path of weight $^{n+1}C_2 - 2$ is either $P_n - v_1$ or $P_n - v_n$. Suppose $w(P_n - v_1) = ^{n+1}C_2 - 2$. Then $w(P'_n - v_1) = ^{n+1}C_2 - 3$. Hence $f(v_1v_2) = 2$. Since $f(v_2v_3) \neq f(v_1v_2) = 2$, the path $P_n - \{v_1, v_2\}$ cannot have weight $^{n+1}C_2 - 4$. Hence the path of weight $^{n+1}C_2 - 4$ is $P_n - v_n$ and so $f(v_{n-1}v_n) = 4$ and $f(v_{n-1}v'_n) = 3$. Also $w(P_n - \{v_n, v_1\}) = ^{n+1}C_2 - 6$. Now the path of weight $^{n+1}C_2 - 5$ is either $P_n - \{v_1, v_2\}$ or $P_n - \{v_n, v_{n-1}\}$. If $w(P_n - \{v_1, v_2\}) = ^{n+1}C_2 - 5$, then $f(v_2v_3) = 3 = f(v_{n-1}v'_n)$. If $w(P_n - \{v_n, v_{n-1}\}) = ^{n+1}C_2 - 5$, then $f(v_{n-2}v_{n-1}) = 1$ and hence $w(v_{n-2}v_{n-1}v'_n) = f(v_{n-1}v_n)$, giving a contradiction.

Now, suppose $w(P_n - v_n) = ^{n+1}C_2 - 2$. Then $f(v_{n-1}v_n) = 2$ and $f(v_{n-1}v'_n) = 1$. Now $f(v_i v_{i+1}) \geq 3$ for all i , $1 \leq i \leq n - 2$ and hence there is no path of weight $^{n+1}C_2 - 3$, which is a contradiction.

Case 2: $w(P_n - v_1) = ^{n+1}C_2 - 1$.

In this case $f(v_1v_2) = 1$. Since $f(v_2v_3) \geq 2$, we have $w(P_n - \{v_1, v_2\}) \leq ^{n+1}C_2 - 3$. Hence $w(P'_n) = ^{n+1}C_2 - 2$ and $w(P'_n - \{v_1\}) = ^{n+1}C_2 - 3$. Also $f(v_{n-1}v'_n) = f(v_{n-1}v_n) - 2$ and no subpath of P'_n can have weight $^{n+1}C_2 - 4$. Hence one of the paths $P_n - \{v_1, v_2\}$ or $P_n - v_n$ must have weight $^{n+1}C_2 - 4$. Suppose $w(P_n - \{v_1, v_2\}) = ^{n+1}C_2 - 4$. Then $f(v_2v_3) = 3$ and $w(P'_n - \{v_1, v_2\}) = ^{n+1}C_2 - 6$. Now let $f(v_{n-1}v_n) = k$. Then $f(v_{n-1}v'_n) = k - 2$. Since 1, 3 and 4 are already path weights, it follows that $k \geq 7$ and there is no path of weight $^{n+1}C_2 - 5$, which is a contradiction. Hence it follows that $w(P_n - v_n) = ^{n+1}C_2 - 4$. In this case $f(v_{n-1}v_n) = 4$, $f(v_{n-1}v'_n) = 2$ and $w(P_n - \{v_1, v_2\}) = ^{n+1}C_2 - 5$. Since the pendent edges of P_n have labels 1 and 4 and $w((v'_n, v_{n-1}, v_n)) = 6$, it follows that no subpath of P_n has weight $^{n+1}C_2 - 6$. Also since $w(P'_n) = ^{n+1}C_2 - 2$ and the pendent edges of P'_n have labels 1 and 2, no subpath of P'_n has weight $^{n+1}C_2 - 6$. Thus there is no path of weight $^{n+1}C_2 - 6$, which is a contradiction. Hence there is no Leech labeling for G_n . \square

Observation 3.1. In Lemma 3.1 we have assumed that $n \geq 17$, which gives $w((v'_n, v_{n-1}, v_n)) < ^{n+1}C_2 - 85$. However in the proof of Theorem 3.1 we only require $w((v'_n, v_{n-1}, v_n)) < ^{n+1}C_2 - 6$. Hence the proof of Theorem 3.1 holds for $n \geq 9$ which shows that the proof is independent of the known results of non existence of Leech trees of order 9, 11 and 16.

4. Conclusion and Scope

All the five known Leech trees given in Figure 1 are double stars and we have proved that no other double star is a Leech tree. Hence the basic conjecture on Leech trees can be restated as follows.

Leech Tree Conjecture: The only Leech trees are the double stars $B_{0,0}, B_{1,1}, B_{2,2}, B_{1,0} = B_{0,1}$ and $B_{2,0} = B_{0,2}$.

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