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The rainbow k-connectivity of the non-commutative graph of a finite group

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Abstract

The non-commuting graph $\Gamma(G)$ of a non–abelian group G is defined as follows. The vertex set $V(\Gamma(G))$ of $\Gamma(G)$ is $G \setminus Z(G)$ where $Z(G)$ denotes the center of G and two vertices x and y are adjacent if and only if $xy \neq yx$. We prove that the rainbow k-connectivity of $\Gamma(G)$ is equal to $\left\lceil \frac{k}{2} \right\rceil + 2$, for $3 \leq k \leq |Z(G)|$.

Keywords: non-commuting graph, non-abelian group, rainbow connectivity, rainbow path
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1. Introduction

Let G be a group and $Z(G)$ be the center of G. The *non-commuting graph* $\Gamma(G)$ associated to G is the graph with vertex set $G \setminus Z(G)$ and such that two vertices x and y are adjacent whenever $xy \neq yx$. The non-commuting graph of a group was first considered by Paul Erdös in 1975, [6]. Subsequently, it was strongly developed in [1].

Let Γ be a connected graph with the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$. Define a coloring $\varphi : E(\Gamma) \to \{1, 2, \ldots, t\}$, $t \in \mathbb{N}$, where adjacent edges may be colored the same. Given an edge coloring of Γ, a path P is *rainbow* if no two edges of P are colored the same. An edge-colored

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graph Γ is *rainbow connected* if every pair of vertices of Γ are connected by a rainbow. The *rainbow connection number* $\mathrm{rc}_1(\Gamma)$ of Γ is defined to be the minimum integer t such that there exists an edge-coloring of Γ with t colors that makes Γ rainbow connected.

From a generalization given by Chartrand, Johns, McKeon and Zhang in 2009 [2], an edge-colored graph Γ is called *rainbow k–connected* if any two distinct vertices of Γ are connected by at least k internally disjoint rainbow paths. The *rainbow k–connectivity* of Γ, denoted by $\mathrm{rc}_\mathrm{k}(\Gamma)$, is the minimum number of colors required to color the edges of Γ to make it rainbow k–connected, and φ is called a *rainbow k–coloring* of Γ. We usually denote $\mathrm{rc}_1(\Gamma)$ by $\mathrm{rc}(\Gamma)$.

The importance of rainbow connection number emerge from applications to the secure transfer of classified information between agencies [2]. Recently, Septyanto in [8], showed another form to see the application.

The *commutator* of an ordered pair g_1, g_2 of elements of G is the element

$$[g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2 \in G$$

G is abelian if and only if $[g_1, g_2] = 1$

Let $G(V, E)$, and let $a = (e_1, ..., e_j)$ be a path with $e_i \in E$. Then $l(a) := j$ is called the *length* of a.

We denote by $P(x, y)$ the set of all x, y paths in G. Then $d(x, y) := min\{l(a)|a \in P(x, y)\}$ is called the *distance* from x to y.

We call $diam(G) := max\{d(x, y)|x, y \in G\}$ the *diameter* of G. The length of a shortest cycle of G is called the *girth* of G.

When a pair of vertices g_i, g_j are joined, we denoted by $g_i \sim g_j$. In otherwise we denoted by $g_i \nsim g_j$.

A non–commutative graph $\Gamma(G)$ is connected and the diameter of $\Gamma(G)$ is 2, $diam(\Gamma(G)) = 2$.

Theorem 1.1. [1] *For any non–abelian group G, $diam(\Gamma(G)) = 2$. In particular, $\Gamma(G)$ is connected.*

In [9], it is shown that $\mathrm{rc}(\Gamma(G)) = \mathrm{rc}_2(\Gamma(G)) = 2$.

Theorem 1.2. [9] *Let G be a finite non-abelian group. Then $\mathrm{rc}(\Gamma(G)) = \mathrm{rc}_2(\Gamma(G)) = 2$.*

In the present article, we estimate $\mathrm{rc}_\mathrm{k}(\Gamma(G))$ for $3 \le k \le |Z(G)|$. Our main result is the following theorem.

Theorem 1.3. *Let G be a finite non-abelian group. Then $\mathrm{rc}_\mathrm{k}(\Gamma(G)) \le k$, for $3 \le k \le |Z(G)|$ with $|Z(G)| \ge 3$. Specifically $rc_k(\Gamma(G)) = \lceil \frac{k}{2} \rceil + 2$.*

2. $\mathrm{rc_k}(\Gamma(G))$ with $1 \leq k \leq |Z(G)|$

Let G be a finite non-abelian group, from now on we write the vertices of $\Gamma(G)$ as the partition

$$V(\Gamma(G)) = g_1 Z \,\dot\cup\, g_2 Z \,\dot\cup \cdots \dot\cup\, g_m Z,$$

with $Z = Z(G)$, $g_i Z \neq Z$, $m = [G : Z(G)] - 1$ and where $g_i Z$ is an independent subset of $\Gamma(G)$.

Proposition 2.1. *Let G be a finite non-abelian group. Then the m–partite graph $\Gamma(G)$ with partition $V(\Gamma(G)) = g_1 Z \,\dot\cup\, g_2 Z \,\dot\cup \cdots \dot\cup\, g_m Z$, provides an adjacency by blocks.*

Proof. Observe that every pair of vertices $g_i \sim g_j$, if and only if for all $x, y \in Z$ $g_i x \sim g_j y$. In addition, for each i, the vertex $g \in V(\Gamma(G))$ is adjacent to g_i if and only if it is adjacent to every element of the set $g_i Z$. In other words, it is an adjacency by blocks. □

Definition 2.2. Let G be a non-commutative finite group, with m–partition

$$V(\Gamma(G)) = g_1 Z \,\dot\cup\, g_2 Z \,\dot\cup \cdots \dot\cup\, g_m Z$$

adjacency by blocks. We define the *skeleton* of the m–partition as the subgraph induced by $M = \{g_1, g_2, \ldots, g_m\}$. The skeleton is denoted by $S^M_{\Gamma(G)}$.

Remark 2.3. The graph $\Gamma(G)$ is not complete , however $S^M_{\Gamma(G)}$ can be complete, we can see this in the follow example: Let $G = D_{2\times 4} := \langle a, x : a^4 = x^2 = 1, xax = a^{-1}\rangle$, the dihedral group of order 8. Then $Z := Z(G) = \{1, a^2\}$, and we have

$$V(\Gamma(G)) = aZ \dot\cup xZ \dot\cup axZ.$$

Since each pair of $\{a, x, ax\}$ do not commute, we have $S^M_{\Gamma(D_{2\times 4})}$ is complete.

By Theorem 1.2, there is a coloration

$$\varphi : E(\Gamma(G)) \to \{1, 2\}$$

such that $\mathrm{rc}(\Gamma) = \mathrm{rc}_2(\Gamma) = 2$. Thus, the graph $\Gamma(G)$ is not complete, implies that $\varphi(E(S^M_{\Gamma(G)})) = \{1, 2\}$. Therefore, the coloration

$$\phi := \varphi|_{E(S^M_{\Gamma(G)})} : E(S^M_{\Gamma(G)}) \to \{1, 2\}$$

meets the 2–connectivity, that is to say, $\mathrm{rc}(S^M_{\Gamma(G)}) \leq 2$. Consider $Z(G) = \{e = z_1, z_2, z_3, \ldots, z_s\}$ and define the following coloring of $\Gamma(G)$:

$$\psi : E(\Gamma(G)) \to \{1, 2\} \text{ given by}$$

$$\psi(\{g_i z_p, g_j z_p\}) = \phi(\{g_i, g_j\}) \text{ for } 1 \leq i, j, p \leq m; i \neq j;$$

$$\psi(\{g_i z_p, g_j z_q\}) \neq \phi(\{g_i, g_j\}) \text{ for } 1 \leq i, j, p, q \leq m; i \neq j; p \neq q.$$

In the next section we give a coloring for $3 \leq k \leq s$ with $p \neq q$. Moreover in section 6 we will proof that this coloring works.

3. About edge-connectivity

We need to find k-rainbow paths between any two vertices for $\Gamma(G)$, with $k \geq 3$. We may ask for the maximum number of paths from v_1 to v_2 vertices, no two of which have an edge in common (such paths are called *edge-disjoint paths*). As a consequence of Menger's theorem about max-flow and min-cut, Witney [10] presented that a graph is k-connected if and only if any two vertices are connected by k internally disjoint paths. With Whitney's result we can answer how many edge-disjoint paths are connecting a given pair of vertices on $\Gamma(G)$.

Definition 3.1. The *edge-connectivity* is the minimum size of a subset $C \subset E(G)$ for which $G - C$ is not connected for a graph G. The edge-connectivity of G is denoted by $\lambda(G)$. If $\lambda(G) \geq k$ then G es called k-edge connected.

The next theorem is a result implied by Menger's theorem. This form can be found in [7, Chapter 15].

Theorem 3.2. *An undirected graph $G = (V, E)$ is k-edge-connected if and only if there exist k edge-disjoint paths between any two vertices s and t.*

As we can obtain the rainbow-connectivity number of $\Gamma(G)$ and this graph is connected by blocks with $s = |Z(G)|$ as size of each block, we have that the graph $\Gamma(G)$ is s-edge-connected and there exist s edge-disjoint paths in $\Gamma(G)$. Then, our problem now is coloring the s edge-disjoint paths of $\Gamma(G)$.

Remark 3.3. By 1.1 we note that there exist two cases that we need analyze, for $g_i, g_j, g_k, g_l \in S^M_{\Gamma(G)}$ and $z_r, z_t, z_w, z_p \in Z(G)$. The first case is when $g_i z_r \sim g_j z_t$ which give us a bipartite complete graph in $\Gamma(G)$. The second case is when we have $g_i z_r \sim g_j z_t \sim g_k z_w$, but $g_i z_r \nsim g_k z_w$.

Remark 3.4. We note that $\lambda(G) \geq s$. Then, if we want a path between end vertices $g_i z_r$ and $g_j z_t$, without loss of generality we start with $g_i z_r$, necessarily, from 3.2, the edges $g_i z_r \sim g_j z_{t_b}$ with $t_b \in \{1, ..., s\}$, are in the set of edge-disjoint paths. The same happens for the edges $g_i z_{r_a} \sim g_j z_t$ with $r_a \in \{1, ..., s\}$ because we have s disjoint paths, therefore we need all out-edge from $g_i z_r$, and all in-edge to $g_j z_t$, thus all our edge-disjoint paths have the following form: $(g_i z_r, g_j z_{t_b}, ..., g_i z_{r_a}, g_j z_t)$, with $t_a, r_b \in \{1, ..., s\}$.

4. Rainbow *k–connectivity*

4.1. Case when $g_i \sim g_j \in V(S^M_{\Gamma(G)})$

Let $s = |Z(G)|$ and let $\bar{r} \equiv r \bmod s$ with $1 \leq r \leq s$. If $g_i \sim g_j \in V(S^M_{\Gamma(G)})$, then the set of edges is given by

$$\begin{array}{rcl}
E_1 & = & \{e \in E(\Gamma(G)) | g_i z_r \sim g_j z_r \text{ such that } \psi(\{g_i z_p, g_j z_p\}) = 1\} \bigcup \\
 & & \{e \in E(\Gamma(G)) | \text{ for } g_i z_r \sim g_j z_{\overline{r+1}} \text{ such that } \psi(\{g_i z_p, g_j z_p\}) = 2 \text{ with} \\
 & & 1 \leq i, j, p \leq m; i \neq j\} \\
E_2 & = & \{e \in E(\Gamma(G)) | g_i z_r \sim g_j z_r \text{ such that } \psi(\{g_i z_p, g_j z_p\}) = 2\} \bigcup \\
 & & \{e \in E(\Gamma(G)) | \text{ for } g_i z_r \sim g_j z_{\overline{r+1}} \text{ such that } \psi(\{g_i z_p, g_j z_p\}) = 1 \text{ with} \\
 & & 1 \leq i, j, p \leq m; i \neq j\} \\
E_3 & = & \{e \in E(\Gamma(G)) | g_i z_r \sim g_j z_{\overline{r+2}}\} \\
\vdots & \vdots & \vdots \\
E_n & = & \{e \in E(\Gamma(G)) | g_i z_r \sim g_j z_{\overline{r+n-1}}\} \\
E_{n+1} & = & \{e \in E(\Gamma(G)) | g_i z_r \sim g_j z_{\overline{r+n}}\} \\
E_{n+2} & = & E(\Gamma(G)) \setminus \big(E_1 \cup \cdots \cup E_{n+1}\big)
\end{array}$$

with $n = \left\lfloor \frac{k}{2} \right\rfloor$. The coloring given by:

$$\begin{array}{rcl}
\psi : E(\Gamma(G)) & \longrightarrow & \{1, ..., n+2\} \\
f & \mapsto & i \qquad \text{if } f \in E_i
\end{array}$$

For an easier study of this kind of graph we use a table called *rainbow table*, whose entries (r_a, t_b) are the color from edge $(g_i z_{r_a}, g_j z_{t_b})$. This table is the following form:

	g_jz_1	g_jz_2	g_jz_3	$\cdots$	g_jz_n	g_jz_{n+1}	g_jz_{n+2}	$\cdots$	g_jz_s
g_iz_1	1	2	3	$\cdots$	n	$n+1$			
g_iz_2		1	2	$\cdots$	$n-1$	n	$n+1$		
g_iz_3			1	$\cdots$	$n-2$	$n-1$	n	$\cdots$	
$\vdots$					$\vdots$	$\vdots$	$\vdots$		
g_iz_n					1	2	3	$\cdots$	$n+1$
g_iz_{n+1}	$n+1$					1	2	$\cdots$	n
$\vdots$	$\vdots$								$\vdots$
g_iz_s	2	3	4	$\cdots$	$n+1$				1

Case $g_i \sim g_j$ in $S^M_{\Gamma(G)}$, $s = |Z(G)|$ and $n = \left\lfloor \frac{k}{2} \right\rfloor$.

The $(n+2)$-color in the table is given by white space.

4.2. *Case when $g_i \sim g_j \sim g_l$ but $g_i \nsim g_l$ in $S^M_{\Gamma(G)}$*

Let $s = |Z(G)|$ and let $\bar{r} \equiv r \bmod s$ with $1 \leq r \leq s$. If $g_i \sim g_j \in V(S^M_{\Gamma(G)})$, then the set of edges is given by

$$
\begin{aligned}
E_1 &= \{e \in E(\Gamma(G)) | g_i z_r \sim g_j z_r \text{ such that } \psi(\{g_i z_p, g_j z_p\}) = 1\} \bigcup \\
&\quad \{e \in E(\Gamma(G)) | \text{ for } g_i z_r \sim g_j z_{\overline{r+1}} \text{ such that } \psi(\{g_i z_p, g_j z_p\}) = 2 \text{ with} \\
&\quad 1 \leq i, j, p \leq m; i \neq j\} \\
E_2 &= \{e \in E(\Gamma(G)) | g_i z_r \sim g_j z_r \text{ such that } \psi(\{g_i z_p, g_j z_p\}) = 2\} \bigcup \\
&\quad \{e \in E(\Gamma(G)) | \text{ for } g_i z_r \sim g_j z_{\overline{r+1}} \text{ such that } \psi(\{g_i z_p, g_j z_p\}) = 1 \text{ with} \\
&\quad 1 \leq i, j, p \leq m; i \neq j\} \\
E_3 &= \{e \in E(\Gamma(G)) | g_i z_r \sim g_j z_{\overline{r+2}}\} \\
\vdots &\quad \vdots \quad \vdots \\
E_n &= \{e \in E(\Gamma(G)) | g_i z_r \sim g_j z_{\overline{r+n-1}}\} \\
E_{n+1} &= \{e \in E(\Gamma(G)) | g_i z_r \sim g_j z_{\overline{r+n}}\} \\
E_{n+2} &= E(\Gamma(G)) \setminus \left(E_1 \cup \cdots \cup E_{n+1}\right)
\end{aligned}
$$

with $n = \lceil \frac{k}{2} \rceil$. The coloring given by:

$$
\begin{aligned}
\psi : E(\Gamma(G)) &\longrightarrow \{1, ..., n+2\} \\
f &\mapsto i \qquad \text{if } f \in E_i
\end{aligned}
$$

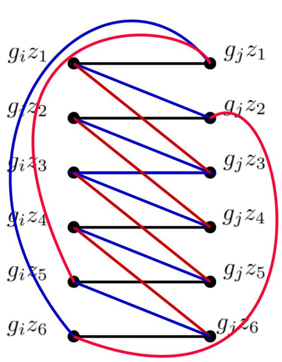
This give us a table as:

	g_iz_1	g_iz_2	$\cdots$	g_iz_n	g_iz_{n+1}	$\cdots$	g_iz_s	g_lz_1	g_lz_2	$\cdots$	g_lz_{n-1}	g_lz_n	g_lz_{n+1}	$\cdots$	g_lz_s
g_jz_1	1			$n+1$	n	$\cdots$	2	2	1	$\cdots$	$n-1$	n	$n+1$	$\cdots$	
g_jz_2	2	1			$n+1$	$\cdots$	3		2	$\cdots$	$n-2$	$n-1$	n	$\cdots$	
$\vdots$	$\vdots$	$\vdots$					$\vdots$			$\ddots$	$\vdots$	$\vdots$			
g_jz_{n-1}	$n-1$	$n-2$	$\ddots$				n				2	1	3	$\cdots$	$n+1$
g_jz_n	n	$n-1$	$\cdots$	1			$n+1$	$n+1$				2	1	$\cdots$	n
g_jz_{n+1}	$n+1$	n	$\cdots$	$\vdots$	1			n	$n+1$				2	$\cdots$	$n-1$
$\vdots$				$\vdots$	$\vdots$			$\vdots$	$\vdots$					$\ddots$	$\vdots$
g_jz_s				n	$n-1$	$\cdots$	1	1	3	$\cdots$	n	$n+1$			2

Case when $g_i \sim g_j \sim g_l$ but $g_i \nsim g_l$ in $S^M_{\Gamma(G)}$ with $n = \lceil \frac{k}{2} \rceil$ and $(n+2)$-color with white spaces.

5. How to build the rainbow table

Example 5.1. We give the case when $s = 6$ and $g_1 \sim g_2$ in $S^M_{\Gamma(G)}$ with the coloring assigned before. Without loss of generality suppose that $\psi(\{g_1 z_p, g_2 z_p\}) = 1$, then the rainbow table is given by:



	g_2z_1	g_2z_2	g_2z_3	g_2z_4	g_2z_5	g_2z_6
g_1z_1	1	2	3			
g_1z_2		1	2	3		
g_1z_3			1	2	3	
g_1z_4				1	2	3
g_1z_5	3				1	2
g_1z_6	2	3				1

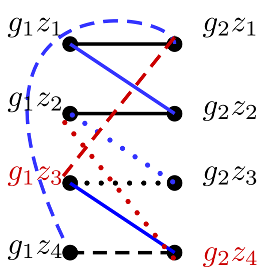
We can see that there is not exist a rainbow k-connectivity with 4 colors. To give s edge-disjoint paths with ends vertices g_1z_2 and g_2z_4, the first path cross above g_2z_1, then we start the path with $g_1z_2 \overset{4}{\sim} g_2z_1$. Now, we need move from g_2z_1 but our only options are $g_2z_1 \overset{1}{\sim} g_1z_1$, $g_2z_1 \overset{3}{\sim} g_1z_5$ and $g_2z_1 \overset{2}{\sim} g_1z_6$ and these edges can not arrive to g_2z_4 because all the in-edge repeat color 4. For this reason we need to ensure that there exist enough in-edge that cover complete the out-edge in the set edges with majority color. For the existence of all edge-disjoint paths for any vertex we need to add one color more, and the table is given by

	g_2z_1	g_2z_2	g_2z_3	g_2z_4	g_2z_5	g_2z_6
g_1z_1	1	2	3	4		
g_1z_2		1	2	3	4	
g_1z_3			1	2	3	4
g_1z_4	4			1	2	3
g_1z_5	3	4			1	2
g_1z_6	2	3	4			1

Example 5.2. We will do an example step-by-step about how we found all the edge-disjoint paths with our table. Let $g_1 \sim g_2$ in $S^M_{\Gamma(G)}$ and $|Z(G)| = 4$. Then, we will build our rainbow table with 3 colors the following form.

	g_2z_1	g_2z_2	g_2z_3	g_2z_4
g_1z_1	1	2		
g_1z_2		1	2	
g_1z_3			1	2
g_1z_4	2			1

From this table we can found $\mathrm{rc}_3(\Gamma(G)) = 3$ for any vertices. For example, for end vertices g_1z_3, g_2z_4



1-path: $g_1z_3 \overset{2}{\sim} g_2z_4$

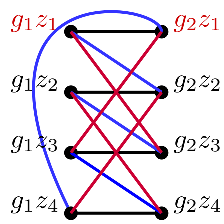
2-path: $g_1z_3 \overset{3}{\sim} g_2z_1 \overset{2}{\sim} g_1z_4 \overset{1}{\sim} g_2z_4$

3-path: $g_1z_3 \overset{1}{\sim} g_2z_3 \overset{2}{\sim} g_1z_2 \overset{3}{\sim} g_2z_4$

If we note, we can not find 4 edge-disjoint paths with 3 colors, because g_1z_1 to g_2z_1 passes through g_2z_3, the paths are the followings: $g_1z_1 \overset{3}{\sim} g_2z_3 \overset{2}{\sim} g_1z_2 \overset{3}{\sim} g_2z_1$ or $g_1z_1 \overset{3}{\sim} g_2z_3 \overset{1}{\sim} g_1z_3 \overset{3}{\sim} g_2z_1$. Then, we need add another color, then the table is 4 colors the following form:

	g_2z_1	g_2z_2	g_2z_3	g_2z_4
g_1z_1	1	2	3	
g_1z_2		1	2	3
g_1z_3	3		1	2
g_1z_4	2	3		1

Then, with all this 4 colors we found all 4 edge-disjoint paths from g_1z_1 to g_2z_1, and they are the followings:



1-path: $g_1z_1 \overset{1}{\sim} g_2z_1$

2-path: $g_1z_1 \overset{2}{\sim} g_2z_2 \overset{1}{\sim} g_1z_2 \overset{4}{\sim} g_2z_1$

3-path: $g_1z_1 \overset{3}{\sim} g_2z_3 \overset{4}{\sim} g_1z_4 \overset{2}{\sim} g_2z_1$

4-path: $g_1z_1 \overset{4}{\sim} g_2z_4 \overset{2}{\sim} g_1z_3 \overset{3}{\sim} g_2z_1$

and the same is true for any pair of vertices.

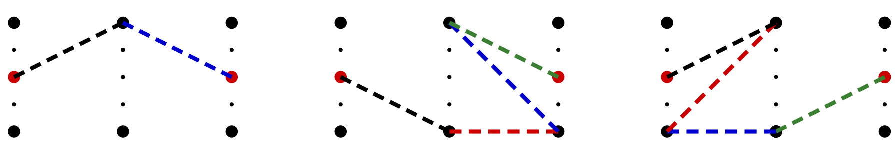
6. Proofs

6.1. Case 3-partite with $|Z(G)| = 3$

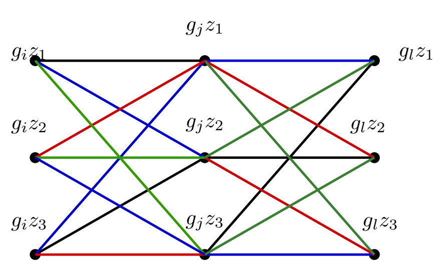
The coloring given before can not help us to find all the disjoint-edge paths for the case when $g_i \sim g_j \sim g_l$ but $g_i \nsim g_l$ in $S^M_{\Gamma(G)}$, for example, the rainbow table for this case is the next

	g_iz_1	g_iz_2	g_iz_3	g_lz_1	g_lz_2	g_lz_3
g_jz_1	1		2	2	1	
g_jz_2	2	1			2	1
g_jz_3		2	1	1		2

But, we can see that for go from g_iz_1 to g_lz_2 we have same colors then, we need to do paths with length at least 4 like the following picture:



The coloring given for this specifical case is the following: The rainbow tables for each case



are the following:

$$\begin{array}{c} \\ g_jz_1 \\ g_jz_2 \\ g_jz_3 \end{array}\begin{array}{c} \begin{array}{cccccc} g_iz_1 & g_iz_2 & g_iz_3 & g_lz_1 & g_lz_2 & g_lz_3 \end{array} \\ \left[\begin{array}{cccccc} 1 & 3 & 2 & 2 & 3 & 4 \\ 2 & 4 & 1 & 4 & 1 & 3 \\ 4 & 2 & 3 & 1 & 4 & 2 \end{array}\right] \end{array}$$

With $\psi(\{g_i, g_j\}) = 1$ in $S^M_{\Gamma(G)}$.

$$\begin{array}{c} \\ g_jz_1 \\ g_jz_2 \\ g_jz_3 \end{array}\begin{array}{c} \begin{array}{cccccc} g_iz_1 & g_iz_2 & g_iz_3 & g_lz_1 & g_lz_2 & g_lz_3 \end{array} \\ \left[\begin{array}{cccccc} 2 & 3 & 4 & 1 & 3 & 2 \\ 4 & 1 & 3 & 2 & 4 & 1 \\ 1 & 4 & 2 & 4 & 2 & 3 \end{array}\right] \end{array}$$

With $\psi(\{g_j, g_l\}) = 1$ in $S^M_{\Gamma(G)}$.

Theorem 6.1. *Let G be a non–abelian group with $|Z(G)| = 3$ and $\Gamma(G)$ be the non-commutative graph associated to G, then $\mathrm{rc}_3(\Gamma(G)) = 4$.*

Proof. Let the set of edges be the following form:

$$\begin{array}{rcl} E_1 & = & \{e \in E(\Gamma(G)) | g_iz_{k_r} \sim g_jz_1 \text{ such that } \psi(\{g_i, g_j\}) = 1 \text{ for } g_i, g_j \in S^M_{\Gamma(G)} \text{ and } k_r = 1, 2, 3\} \\ & & \bigcup\{e \in E(\Gamma(G)) | g_jz_2 \sim g_lz_2, g_jz_3 \sim g_lz_1 \text{ such that } \psi(\{g_j, g_l\}) = 2 \text{ for } g_j, g_l \in S^M_{\Gamma(S)}\} \\ E_2 & = & \{e \in E(\Gamma(G)) | g_iz_{k_r} \sim g_jz_2 \text{ such that } \psi(\{g_i, g_j\}) = 1 \text{ for } g_i, g_j \in S^M_{\Gamma(G)} \text{ and } k_r = 1, 2, 3\} \\ & & \bigcup\{e \in E(\Gamma(G)) | g_jz_{j_a} \sim g_lz_{j_a} \text{ such that } \psi(\{g_j, g_l\}) = 2 \text{ for } g_j, g_l \in S^M_{\Gamma(S)} \text{ and } j_a = 1, 3\} \\ E_3 & = & \{e \in E(\Gamma(G)) | g_iz_{k_r} \sim g_jz_3 \text{ such that } \psi(\{g_i, g_j\}) = 1 \text{ for } g_i, g_j \in S^M_{\Gamma(G)} \text{ and } k_r = 1, 2, 3\} \\ & & \bigcup\{e \in E(\Gamma(G)) | g_jz_1 \sim g_lz_2, g_jz_2 \sim g_lz_3 \text{ such that } \psi(\{g_j, g_l\}) = 2 \text{ for } g_j, g_l \in S^M_{\Gamma(S)}\} \\ E_4 & = & E \setminus (E_1 \cup E_2 \cup E_3) \end{array}$$

And the coloring is given by

$$\begin{array}{rcll} \psi : E(\Gamma(G)) & \longrightarrow & \{1, 2, 3, 4\} & \\ f & \mapsto & i & \text{if } i \in E_i. \end{array}$$

The following are all the 3 edge-disjoint paths for each pair of vertices when $\phi(\{g_j, g_l\}) = 2$

$g_jz_1 \overset{2}{\sim} g_lz_1$ $g_jz_1 \overset{4}{\sim} g_lz_3 \overset{2}{\sim} g_jz_3 \overset{1}{\sim} g_lz_1$ $g_jz_1 \overset{3}{\sim} g_lz_2 \overset{1}{\sim} g_jz_2 \overset{4}{\sim} g_lz_1$	$g_jz_1 \overset{3}{\sim} g_lz_2$ $g_jz_1 \overset{2}{\sim} g_lz_1 \overset{1}{\sim} g_jz_3 \overset{4}{\sim} g_lz_2$ $g_jz_1 \overset{4}{\sim} g_lz_3 \overset{3}{\sim} g_jz_2 \overset{1}{\sim} g_lz_2$	$g_jz_1 \overset{4}{\sim} g_lz_3$ $g_jz_1 \overset{2}{\sim} g_lz_1 \overset{4}{\sim} g_jz_2 \overset{3}{\sim} g_lz_3$ $g_jz_1 \overset{3}{\sim} g_lz_2 \overset{4}{\sim} g_jz_3 \overset{2}{\sim} g_lz_3$
$g_jz_2 \overset{4}{\sim} g_lz_1$ $g_jz_2 \overset{1}{\sim} g_lz_2 \overset{3}{\sim} g_jz_1 \overset{2}{\sim} g_lz_1$ $g_jz_2 \overset{3}{\sim} g_lz_3 \overset{2}{\sim} g_jz_3 \overset{1}{\sim} g_lz_1$	$g_jz_2 \overset{1}{\sim} g_lz_2$ $g_jz_2 \overset{4}{\sim} g_lz_1 \overset{2}{\sim} g_jz_1 \overset{3}{\sim} g_lz_2$ $g_jz_2 \overset{3}{\sim} g_lz_3 \overset{2}{\sim} g_jz_3 \overset{4}{\sim} g_lz_2$	$g_jz_2 \overset{3}{\sim} g_lz_3$ $g_jz_2 \overset{4}{\sim} g_lz_1 \overset{1}{\sim} g_jz_3 \overset{2}{\sim} g_lz_3$ $g_jz_2 \overset{1}{\sim} g_lz_2 \overset{3}{\sim} g_jz_1 \overset{4}{\sim} g_lz_3$
$g_jz_3 \overset{1}{\sim} g_lz_1$ $g_jz_3 \overset{4}{\sim} g_lz_2 \overset{3}{\sim} g_jz_1 \overset{2}{\sim} g_lz_1$ $g_jz_3 \overset{2}{\sim} g_lz_3 \overset{3}{\sim} g_jz_2 \overset{4}{\sim} g_lz_1$	$g_jz_3 \overset{4}{\sim} g_lz_2$ $g_jz_3 \overset{1}{\sim} g_lz_1 \overset{2}{\sim} g_jz_1 \overset{3}{\sim} g_lz_2$ $g_jz_3 \overset{2}{\sim} g_lz_3 \overset{3}{\sim} g_jz_2 \overset{1}{\sim} g_lz_2$	$g_jz_3 \overset{2}{\sim} g_lz_3$ $g_jz_3 \overset{4}{\sim} g_lz_2 \overset{1}{\sim} g_jz_2 \overset{3}{\sim} g_lz_3$ $g_jz_3 \overset{1}{\sim} g_lz_1 \overset{2}{\sim} g_jz_1 \overset{4}{\sim} g_lz_3$

All the edge-disjoint paths when $\phi(\{g_i, g_j\}) = 2, \phi(\{g_j, g_l\}) = 2$ and $g_i \sim g_j \sim g_l$ but $g_i \nsim g_l$

$g_iz_1 \sim g_lz_1$	$g_iz_1 \sim g_lz_2$	$g_iz_1 \sim g_lz_3$
$g_iz_1 \overset{1}{\sim} g_jz_1 \overset{2}{\sim} g_lz_1$ $g_iz_1 \overset{2}{\sim} g_jz_2 \overset{4}{\sim} g_lz_1$ $g_iz_1 \overset{4}{\sim} g_jz_3 \overset{1}{\sim} g_lz_2$	$g_iz_1 \overset{4}{\sim} g_jz_3 \overset{2}{\sim} g_lz_3 \overset{3}{\sim} g_iz_2 \overset{1}{\sim} g_lz_2$ $g_iz_1 \overset{2}{\sim} g_jz_2 \overset{1}{\sim} g_iz_3 \overset{3}{\sim} g_jz_3 \overset{4}{\sim} g_lz_2$ $g_iz_1 \overset{1}{\sim} g_jz_1 \overset{3}{\sim} g_lz_2$	$g_iz_1 \overset{1}{\sim} g_jz_1 \overset{4}{\sim} g_lz_3$ $g_iz_1 \overset{2}{\sim} g_jz_2 \overset{3}{\sim} g_lz_3$ $g_iz_1 \overset{4}{\sim} g_jz_3 \overset{2}{\sim} g_lz_3$

$g_iz_2 \sim g_lz_1$	$g_iz_2 \sim g_lz_2$	$g_iz_2 \sim g_lz_3$
$g_iz_2 \overset{3}{\sim} g_jz_1 \overset{2}{\sim} g_jz_1$ $g_iz_2 \overset{2}{\sim} g_jz_3 \overset{3}{\sim} g_iz_3 \overset{1}{\sim} g_jz_2 \overset{4}{\sim} g_lz_1$ $g_iz_2 \overset{4}{\sim} g_jz_2 \overset{3}{\sim} g_lz_3 \overset{2}{\sim} g_jz_3 \overset{1}{\sim} g_lz_1$	$g_iz_2 \overset{2}{\sim} g_jz_3 \overset{4}{\sim} g_lz_2$ $g_iz_2 \overset{4}{\sim} g_jz_2 \overset{1}{\sim} g_iz_3 \overset{2}{\sim} g_jz_1 \overset{3}{\sim} g_lz_2$ $g_iz_2 \overset{3}{\sim} g_jz_1 \overset{2}{\sim} g_lz_1 \overset{4}{\sim} g_jz_2 \overset{1}{\sim} g_lz_2$	$g_iz_2 \overset{3}{\sim} g_jz_2 \overset{4}{\sim} g_lz_3$ $g_iz_2 \overset{4}{\sim} g_jz_2 \overset{1}{\sim} g_iz_3 \overset{3}{\sim} g_jz_3 \overset{2}{\sim} g_lz_3$ $g_iz_2 \overset{2}{\sim} g_jz_3 \overset{4}{\sim} g_lz_2 \overset{1}{\sim} g_jz_2 \overset{3}{\sim} g_lz_3$

$g_iz_3 \sim g_lz_1$	$g_iz_3 \sim g_lz_2$	$g_iz_3 \sim g_lz_3$
$g_iz_3 \overset{3}{\sim} g_jz_3 \overset{1}{\sim} g_lz_1$ $g_iz_3 \overset{2}{\sim} g_jz_1 \overset{3}{\sim} g_lz_2 \overset{1}{\sim} g_jz_2 \overset{4}{\sim} g_lz_1$ $g_iz_3 \overset{1}{\sim} g_jz_2 \overset{4}{\sim} g_iz_2 \overset{3}{\sim} g_jz_1 \overset{2}{\sim} g_lz_1$	$g_iz_3 \overset{2}{\sim} g_jz_1 \overset{3}{\sim} g_lz_2$ $g_iz_3 \overset{1}{\sim} g_jz_2 \overset{3}{\sim} g_lz_3 \overset{2}{\sim} g_jz_3 \overset{4}{\sim} g_lz_2$ $g_iz_3 \overset{3}{\sim} g_jz_3 \overset{2}{\sim} g_iz_2 \overset{4}{\sim} g_jz_2 \overset{1}{\sim} g_lz_2$	$g_iz_3 \overset{2}{\sim} g_jz_1 \overset{4}{\sim} g_lz_3$ $g_iz_3 \overset{1}{\sim} g_jz_2 \overset{3}{\sim} g_lz_3$ $g_iz_3 \overset{3}{\sim} g_jz_3 \overset{2}{\sim} g_lz_3$

All the edge-disjoint paths when $\psi(\{g_i, g_j\}) = 1$

$g_iz_1 \overset{1}{\sim} g_jz_1$ $g_iz_1 \overset{2}{\sim} g_jz_2 \overset{4}{\sim} g_iz_2 \overset{3}{\sim} g_jz_1$ $g_iz_1 \overset{4}{\sim} g_jz_3 \overset{3}{\sim} g_iz_3 \overset{2}{\sim} g_jz_1$	$g_iz_1 \overset{2}{\sim} g_jz_2$ $g_iz_1 \overset{4}{\sim} g_jz_3 \overset{3}{\sim} g_iz_3 \overset{1}{\sim} g_jz_2$ $g_iz_1 \overset{1}{\sim} g_jz_1 \overset{3}{\sim} g_iz_2 \overset{4}{\sim} g_jz_2$	$g_iz_1 \overset{4}{\sim} g_jz_3$ $g_iz_1 \overset{2}{\sim} g_jz_2 \overset{1}{\sim} g_iz_3 \overset{3}{\sim} g_jz_3$ $g_iz_1 \overset{1}{\sim} g_jz_1 \overset{3}{\sim} g_iz_2 \overset{2}{\sim} g_jz_3$
$g_iz_2 \overset{3}{\sim} g_jz_2$ $g_iz_2 \overset{4}{\sim} g_jz_2 \overset{1}{\sim} g_iz_3 \overset{2}{\sim} g_jz_1$ $g_iz_2 \overset{2}{\sim} g_jz_3 \overset{4}{\sim} g_iz_1 \overset{1}{\sim} g_jz_1$	$g_iz_2 \overset{4}{\sim} g_jz_2$ $g_iz_2 \overset{3}{\sim} g_jz_1 \overset{1}{\sim} g_iz_1 \overset{2}{\sim} g_jz_2$ $g_iz_2 \overset{2}{\sim} g_jz_1 \overset{3}{\sim} g_iz_3 \overset{1}{\sim} g_jz_2$	$g_iz_2 \overset{2}{\sim} g_jz_3$ $g_iz_2 \overset{3}{\sim} g_jz_1 \overset{1}{\sim} g_iz_1 \overset{4}{\sim} g_jz_3$ $g_iz_2 \overset{4}{\sim} g_jz_2 \overset{1}{\sim} g_iz_3 \overset{3}{\sim} g_jz_3$
$g_iz_3 \overset{2}{\sim} g_jz_1$ $g_iz_3 \overset{1}{\sim} g_jz_2 \overset{4}{\sim} g_iz_2 \overset{3}{\sim} g_jz_1$ $g_iz_3 \overset{3}{\sim} g_jz_3 \overset{4}{\sim} g_iz_1 \overset{1}{\sim} g_jz_1$	$g_iz_3 \overset{1}{\sim} g_jz_2$ $g_iz_3 \overset{3}{\sim} g_jz_3 \overset{4}{\sim} g_iz_1 \overset{2}{\sim} g_jz_2$ $g_iz_3 \overset{2}{\sim} g_jz_1 \overset{3}{\sim} g_iz_2 \overset{4}{\sim} g_jz_2$	$g_iz_3 \overset{3}{\sim} g_jz_3$ $g_iz_3 \overset{1}{\sim} g_jz_2 \overset{4}{\sim} g_iz_2 \overset{2}{\sim} g_jz_3$ $g_iz_3 \overset{2}{\sim} g_jz_1 \overset{1}{\sim} g_iz_1 \overset{4}{\sim} g_jz_3$

□

Theorem 6.2. *Let G be a finite non-abelian group. Then* $\mathrm{rc_k}(\Gamma(G)) \leq \lceil \frac{k}{2} \rceil + 2$*, for* $3 \leq k \leq s = |Z(G)|$ *with* $|Z(G)| \geq 4$*.*

Proof. We will proof that 4 is a coloring works for our graph.

1. **Case $\mathbf{g_i \sim g_j}$** Let $g_iz_{i_a}, g_jz_{j_b}$ be the end vertices. We want to find the edge-disjoint paths between them. Let 4.1 the rainbow table assigned for this case. From 4.1 it is evident that the first path is given by $g_iz_{i_a} \overset{(i_a,j_b)}{\sim} g_jz_b$ with color (i_a, j_b).

   Let j_1 be the column assigned to the row i_a such that $(i_a, j_1) = f_1$ then, we remove the entries with color f_1 to the column $g_jz_{j_1}$ and, the same happen to column $g_jz_{j_b}$.

   Remark 6.3. When we say *remove the entry* we say that entry is not consider to form the rainbow path.

   Thus, the path for this case is

$$g_iz_{i_a} \overset{f}{\sim} g_jz_{j_1} \overset{(i_{a_1},j_1)}{\sim} g_iz_{i_{a_1}} \overset{(i_{a_1},j_b)}{\sim} g_jz_{j_b} \tag{1}$$

with $(i_{a_1}, j_1) \neq f_1 \neq (i_{a_1}, j_b)$ the colors assigned to remaining entries and $g_j z_{j_1}, g_i z_{i_{a_1}}$ the respective vertices from remaining entries.

Let (i_a, j_2) be the entry with $j_2 \neq j_1$, such that $(i_a, j_2) = f_2$ then, we remove the entries with same color as f_2 in column $g_j z_{j_2}$. We can not use the entry $(g_i z_{a_1}, g_j z_{j_b})$ because is an edge for 1, moreover we remove all the entries with same color as f_2 in column $g_j z_{j_b}$. Thus, the path is the following:

$$g_i z_{i_a} \overset{(i_a, j_2)}{\sim} g_j z_{j_2} \overset{(i_{a_2}, j_2)}{\sim} g_i z_{i_{a_2}} \overset{(i_{a_2}, j_b)}{\sim} g_j z_{j_b} \tag{2}$$

with $(i_{a_2}, j_2), (i_{a_2}, j_b)$ the colors assigned to remaining entries and $g_j z_{j_2}, g_i z_{i_{a_2}}$ the respective vertices from remaining entries.

$$\begin{array}{c} \\ g_i z_{i_{a_1}} \\ \\ g_i z_{i_a} \\ \\ \end{array} \overset{\begin{array}{cccc} & g_j z_{j_b} & g_j z_{j_1} & \end{array}}{\begin{bmatrix} & \vdots & \vdots & \\ \cdots & f & & \cdots \\ & \vdots & \vdots & \\ \cdots & & f & \cdots \\ & \vdots & \vdots & \end{bmatrix}}$$

Under the conditions stated above we apply the same to all the colors assigned to i_a-raw. We take edges from remaining entries to form the rest paths with the same method. Let j'_1 such that $f' = (i_a, j'_1)$ from j_b-column we remove the row with entry same color like f'. The new path is the following:

$$g_i z_{i_a} \overset{(i_a, j'_1)}{\sim} g_j z_{j'_1} \overset{(i_{a'_1}, j'_1)}{\sim} g_i z_{i_{a'_1}} \overset{(i_{a'_1}, j_b)}{\sim} g_j z_{j_b} \tag{3}$$

Take $(i_a, j'_1), (i_{a'_1}, j'_1)$ as remaining entries from all the entries do not removed before with a dofferent color as f'.

Remark 6.4. Suppose that we can coloring with only $\left\lfloor \frac{k}{2} \right\rfloor + 1$ colors. Let $g_i z_{i_m}$ any start vertex, then there exists a pair of vertices $g_j z_{j_n}, g_j z_{j_{n'}}$ such that $\{(a_{i_r}, b_{j_n}) | (a_{i_r}, b_{j_n}) - \text{color} \neq (\left\lfloor \frac{k}{2} \right\rfloor + 1) - color\}$ identify with $\{(a_{i_r}, b_{j_{n'}}) | (a_{i_r}, b_{j_n}) - \text{color} = \text{the last color}\}$, therefore is impossible to built k paths between any end vertices $g_i z_{i_m}, g_j z_{j_n}$ passes through $g_j z_{j_{n'}}$, just like 5.1.

2. **Case:** $\mathbf{g_i} \sim \mathbf{g_j} \sim \mathbf{g_l}$ with $\mathbf{g_i} \nsim \mathbf{g_l}$ in $S^M_{\Gamma(G)}$.
   (a) **Repetition of different color to the last color**
   Case: repetition of one color between columns. Suppose that f is the repeated color between the columns assigned to the end vertices $g_i z_{i_a}$ and $g_l z_{l_b}$ i.e. $f = (j_c, i_a) = (j_c, l_b)$ in the rainbow table, for some $c = \{1, ..., |Z(G)|\}$, with $l_b \in g_l Z$ and $i_a \in g_i Z$. Suppose that f is in the path passes through $g_j z_{j_c}$, thus for do the rainbow path we need

to find another row $j_{c'}$ such that $(j_{c'}, l_b) = f' \neq f$ then for do the rainbow path, to the row $j_{c'}$ we remove the columns with color f (i.e. 2 columns) and one of color f'. To row j_c remove 2 columns for color f' and 2 columns assigned for i_a and l_b. Then we remove a total of 7 columns. There are in total $2|Z(G)|$ columns in our rainbow table, then it remains $2|Z(G)| - 7$ columns with $|Z(G)| \geq 4$, leaving at least one column for do the path without similar colors. The path is $g_i z_{i_a} \overset{f}{\sim} g_j z_{j_c} \overset{f_1}{\sim} g(j_c) \overset{f_2}{\sim} g(j_{c'}) \overset{f'}{\sim} g_l z_{l_b}$ with f_1, f_2 colors assigned to left column and $g(j_c), g(j_{c'})$ vertices in column assigned to above column.

$$\begin{array}{c} \\ j_c \\ \\ j_{c'} \\ \\ \end{array}\begin{bmatrix} & i_a & & & & l_b & \\ & \vdots & & & & \vdots & \\ \cdots\cdots & f & \cdots\cdots f' \cdots & f_1 & \cdots f' \cdots & f & \cdots\cdots \\ & \vdots & & & & \vdots & \\ \cdots f' \cdots & g & \cdots f \cdots\cdots & f_2 & & f' & \cdots f \cdots \\ & \vdots & & & & \vdots & \end{bmatrix}$$

Now we make the path who starts in $g_i z_{i_a} \overset{g}{\sim} g_j z_{j_{c'}}$

When $\mathbf{g \neq f}$ and $\mathbf{g \neq f'}$. As written above we remove the columns in row $j_{c'}$ with color f and one of color g, i.e. 3 columns, and in the row j_c remove the columns assigned with color g and two of columns i_a and l_b, in total we remove 7 columns and leaving $2|Z(G)| - 7$ columns where we can find the desired path.

Case: repetition of two colors between columns with $\mathbf{g = f'}$. We remove 2 columns with color f' to j_c-row and 2 columns assigned to i_a and l_b. In row $j_{c'}$ remove 2 columns assigned with color f. There are in total $2|Z(G)| - 6$ free columns to find rainbow paths.

Case: repetition of 3 colors Suppose that there are 3 repeated colours between the columns for do the paths with end vertices $g_i z_{i_a}$ and $g_l z_{l_b}$ passes through $g_j z_{j_c}$, $g_j z_{j_{c'}}$ and $g_j z_{j_{c''}}$. For do the paths passes through $g_j z_{j_c}$, just like the first case, we remove columns with color f' to j_c-row and, to row j_c' remove the 2 columns with color f minus the rows assigned i_a and l_b, then for $|Z(G)| \geq 4$ there are $2|Z(G)| - 6$ free columns for do the rainbow path with end vertices $g_i z_{i_a}$ and $g_l z_{l_b}$ cross above $g_j z_{j_c}$ and $g_j z_{j_{c'}}$. The same happens for rainbow path passes through $g_j z_{j_{c'}}, g_j z_{c''}$ and $g_j z_{j_{c''}}, g_j z_{j_c}$. The paths have the following form:

$$\begin{aligned} g_i z_{i_a} &\overset{f}{\sim} g_j z_{j_c} \overset{g_1}{\sim} g_1(j_c) \overset{g_2'}{\sim} g_2'(j_{c'}) \overset{f'}{\sim} g_l z_{l_b}, \\ g_i z_{i_a} &\overset{f'}{\sim} g_j z_{j_{c'}} \overset{g_1'}{\sim} g_1'(j_{c'}) \overset{g_2''}{\sim} g_2''(j_{c''}) \overset{f''}{\sim} g_l z_{l_b} \\ g_i z_{i_a} &\overset{f''}{\sim} g_j z_{j_{c''}} \overset{g_1''}{\sim} g_1''(j_{c''}) \overset{g_2}{\sim} g_2(j_c) \overset{f}{\sim} g_l z_{l_b} \end{aligned}$$

$$
\begin{array}{c}
\\ j_c \\ \\ j_{c'} \\ \\ j_{c''} \\ \\
\end{array}
\overset{\begin{array}{cccccccc} & i_a & & & & l_b & \end{array}}{
\begin{bmatrix}
 & \vdots & & & & \vdots & \\
\cdots & f & g_1 & & g_2 & f & \cdots \\
 & \vdots & | & & | & \vdots & \\
\cdots & f' & g_2' & g_1' & | & f' & \cdots \\
 & \vdots & & | & | & \vdots & \\
\cdots & f'' & & g_2'' & g_1'' & f'' & \cdots \\
 & \vdots & & & & \vdots &
\end{bmatrix}}
$$

Note that g_1, g_2'; g_1', g_2'' and g_1'', g_2 are the colors between free columns with colors assigned f, f'; f', f'' and f'',f respectively, and $g_1(j_c)$, $g_2(j_c)$; $g_1'(j_{c'})$,$g_2'(j_{c'})$; $g_1''(j_{c''})$, $g_2''(j_{c''})$ are vertices associated to the colors in the free columns with its rows $j_c, j_{c'}, j_{c''}$ respectively.

(b) **Repetition of last color between columns**
Case: repeat the last color $\lceil \frac{\mathbf{k}}{\mathbf{2}} \rceil + \mathbf{2}$ one time. Let $g_i z_{i_a}$ and $g_l z_{l_b}$ be the end vertices and suppose that only is repeated the last color $\lceil \frac{k}{2} \rceil + 2$ only one time. Let $f = \lceil \frac{k}{2} \rceil + 2$ be the last color and let $B = 2\big[k - \big(\lceil \frac{k}{2} \rceil + 1\big)\big]$ be the number of entries with the last color in each row of the rainbow table. Let $j_{c'}$ be a row associated with different color to f in the entries $(j_{c'}, i_a)$ and $(j_{c'}, l_b)$.

For make the rainbow path passes through j_c, to row $j_{c'}$ remove B columns associated to the last color f and one column designated to color f', i.e., we remove $B+1$ columns. Further in row j_c we remove $B - 2$ columns associated to f, 2 columns associated to color f' and 2 columns for columns associated to i_a and l_b, thus we remove from row j_c $B + 2$ columns. If the columns removed are all different from each other then we keep $C = 2k - (2B + 3)$ free columns, in the extreme case that we eliminate the same columns for each case, evaluate in f and f', thus we would have $2k - (B + 2)$ free columns, then the value of free columns is $2k - (2B + 3) \leq C \leq 2k - (B + 2)$ for $k \leq 4$. The same happens to do a path passes through $g_j z_{j_{c'}}$. Thus, we have enough free columns to do the rainbow path.

$$
\begin{array}{c}
\\ j_c \\ \\ j_{c'} \\ \\
\end{array}
\overset{\begin{array}{cccc} & i_a & & l_b \end{array}}{
\begin{bmatrix}
 & \vdots & & \vdots & \\
\cdots & f & & f & \cdots \\
 & \vdots & & \vdots & \\
\cdots & g & & f' & \cdots \\
 & \vdots & & \vdots &
\end{bmatrix}}
$$

Later, for make the rainbow path from $g_i z_{i_a} \overset{g}{\sim} g_j z_{z_{j_{c'}}}$ we remove 2 columns assigned to color g to j_c-row, $B - 2$ columns assigned to color f and 2 for the columns i_a, l_b, i.e., remove $B+2$ columns. Moreover from $j_{c'}$-row remove B columns for last color f plus 1 column for color g, i.e. $B+1$ columns. In total the amount of free columns is between:

$$2k-(2B+3) \leq C \leq 2k-(B+2) \qquad k \geq 4 \tag{4}$$

Then, there are enough free columns for do the rainbow path.

Case: repeat two colors, one of them the last color, i.e., $\mathrm{g} = \mathrm{f}' \neq \mathrm{f}$. To the row $j_{c'}$ we remove B columns associated to last color f and the row j_c we remove $B-2$ columns associated to last color f, 2 columns associated to color f' and 2 columns associated to columns i_a and l_b, i.e. we remove $B+2$ columns. In total there are $2k-(2B+2) \leq C \leq 2k-(B+2)$

$$2(k-B-1) \leq C \leq 2k-(B+2) \qquad \text{for } k \geq 4 \tag{5}$$

Since $k-B-1>0$ for all k we always have a minimum, two columns to form two paths.

Case: repeat at most $\frac{\mathbf{B}}{\mathbf{2}}$ entries between columns. Suppose that between columns i_a and l_b assigned to end vertices $g_i z_{i_a}, g_l z_{l_b}$ there are, at most $D = k - (\lceil \frac{k}{2} \rceil + 1)$ entries with the last color f in each column, since $D < \lceil \frac{k}{2} \rceil + 1$ we can proceed like the previous cases.

3. **Case: any vertices of same class** We can do the paths directly, if we want to go from $g_i z_{i_a}$ to $g_i z_{i_b}$ the paths are of the following form $g_i z_{i_a} \overset{(i_a,p)}{\sim} g_j z_p \overset{(i_j,p)}{\sim} g_i z_{i_b}$ for $p = \{1, ..., s = |Z(G)|\}$. We note that we can only find up to $(\lceil \frac{k}{2} \rceil + 2)$ edge disjoint paths for any pair of vertices.

$$\begin{array}{cc} & \begin{array}{ccccc} g_jz_1 & g_jz_2 & \cdots & \cdots & g_jz_s \end{array} \\ \begin{array}{c} g_iz_{i_a} \\ \\ g_iz_{i_b} \end{array} & \left[\begin{array}{ccccc} (i_a, j_1) & (i_a, j_2) & \cdots & \cdots & (i_a, j_s) \\ | & | & & & | \\ (i_b, j_1) & (i_b, j_2) & \cdots & \cdots & (i_b, j_s) \end{array} \right] \end{array}$$

□

Corollary 6.5. *Let G be a finite non-abelian group. If $g_i \sim g_j$ then $\lfloor \frac{k}{2} \rfloor + 1 < \mathrm{rc}_k(\Gamma(G))$.*

Proof. From 6.4. □

Corollary 6.6. *Let G be a finite non-abelian group. If $g_i \sim g_j \sim g_l$ with $g_i \nsim g_l$ then $\lceil \frac{k}{2} \rceil + 1 < \mathrm{rc}_k(\Gamma(G))$.*

Proof. Suppose that $B = 2(k - \lceil \frac{k}{2} \rceil)$ then, for any value of k, $B = 2m$ ($k = \{2m, 2m+1\}$). For the case where only repeat one time the last color f, from 4

$$\begin{aligned} -3 \leq C \leq 2m-2 \qquad & \text{for } k = 2m \\ -1 \leq C \leq 2m \qquad & \text{for } k = 2m+1 \end{aligned}$$

Thus, there are cases when we have not free columns for do the rainbow paths. The same happens for case 5:

$$\begin{array}{rl} -2 \leq C \leq 2m-2 & \text{for } k = 2m \\ 0 \leq C \leq 2m-1 & \text{for } k = 2m+1 \end{array}$$

Therefore, we can not form k rainbow paths with $\lceil \frac{k}{2} \rceil + 1$ different colors. □

Theorem 1.3 *Let G be a finite non-abelian group. Then $\mathrm{rc_k}(\Gamma(G)) = \lceil \frac{k}{2} \rceil + 2$, for $3 \leq k \leq s = |Z(G)|$ with $|Z(G)| \geq 4$.*

Proof. From 6.2, 6.5 and 6.6. □

Given the structure of $\Gamma(G)$, it could be considered a generalization of study in [5] to find the Harary index of $\Gamma(G)$.

Example 6.7. Let G be the Heisenberg group for $p = 3$ with presentation

$$\langle x, a, b | x^3 = a^3 = b^3 = 1, ab = ba, xax^{-1} = ab, xbx^{-1} = b \rangle.$$

We know that $|G| = 27$, $|G \setminus Z(G)| = 24$ and $|G/Z(G)| = 9$, i.e. the partition for $V(\Gamma(G)) = \{Z, aZ, a^2Z, xZ, axZ, a^2xZ, x^2Z, ax^2Z, a^2x^2Z\}$ by $[x, a] = b$ we have $xa = bax$, then $xaZ = axZ$. The following is the graph for $S^M_{\Gamma(G)}$

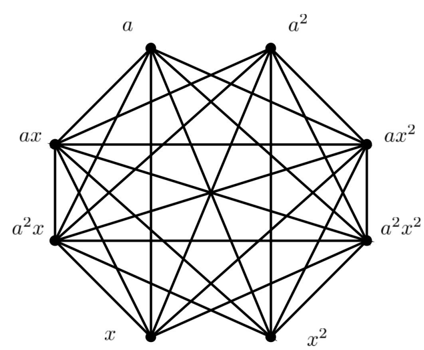
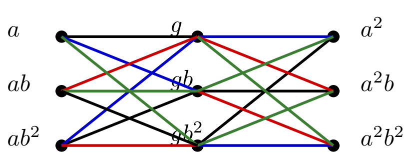


Figure 1. Heisenberg skeleton graph for $p = 3$.

In $S^M_{\Gamma(G)}$ the only vertices with distance 2 are a with a^2 and x with x^2. Suppose without loss of generality that $\psi(\{g, a\}) = 1$. The edge-disjoint paths for end vertices a and a^2 are the following



	a	ab	ab^2	a^2	a^2b	a^2b^2
g	1	3	2	2	3	4
gb	2	4	1	4	1	3
gb^2	4	2	3	1	4	2

And all the paths are given in 6.1.

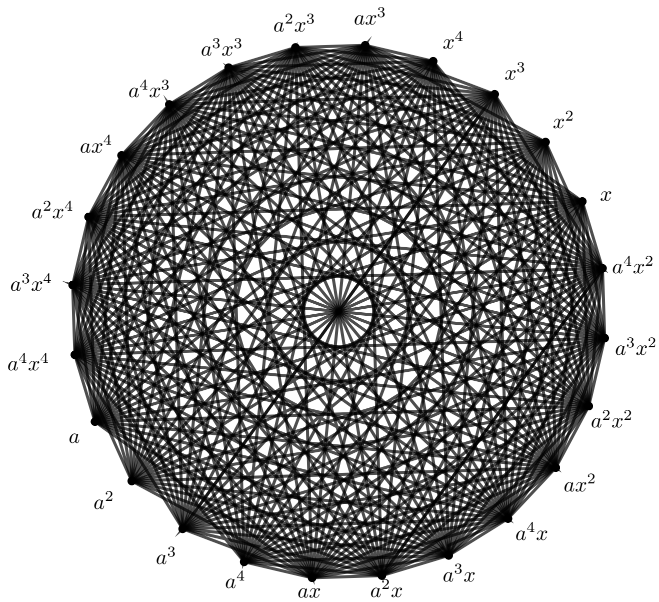


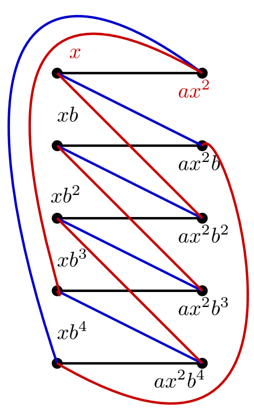
Figure 2. Heisenberg skeleton graph for $p = 5$.

Example 6.8. Let G be the Heisenberg group for $p = 5$ with presentation

$$\langle x, a, b | x^5 = a^5 = b^5 = 1, ab = ba, xax^{-1} = ab, xbx^{-1} = b\rangle.$$

We know that $|G| = 125$, $|G \setminus Z(G)| = 120$ and $|G/Z(G)| = 25$. Since $[x, a] = b$ we have $xa = bax$, then $xaZ = axZ$. The graph 2 is the skeleton $S^M_{\Gamma(G)}$ of G.

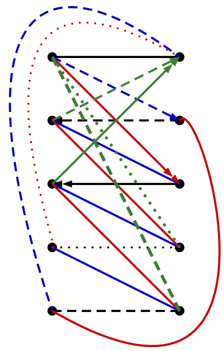
By 3.2 we know that we can found 5 edge-disjoint paths for any pair of vertices then, without loss of generality we give the 5 edge-disjoint paths for end vertices $x, ax^2 \in S^M_{\Gamma(G)}$. By 1.3 we know that we need $\left(\lfloor \frac{5}{2} \rfloor + 2\right)$-color. The rainbow table is given below



	ax^2	ax^2b	a^2b^2	ax^2b^3	ax^2b^4
x	1	2	3		
xb		1	2	3	
xb^2			1	2	3
xb^3	3			1	2
xb^4	2	3			1

Rainbow table for $x \sim ax^2 \in S^M_{\Gamma(G)}$

Then, the 5 edge-disjoin paths are given by:



$x \overset{1}{\sim} ax^2$

$x \overset{2}{\sim} ax^2b \overset{1}{\sim} \quad xb^4 \overset{4}{\sim} \quad ax^2$

$x \overset{3}{\sim} axb^2 \overset{1}{\sim} \quad xb^2 \overset{4}{\sim} \quad ax^2$

$x \overset{4}{\sim} ax^2b^3 \overset{1}{\sim} \quad xb^3 \overset{3}{\sim} \quad ax^2$

$x \overset{4}{\sim} ax^2b^4 \overset{1}{\sim} \quad xb^4 \overset{2}{\sim} \quad ax^2$

We can give 4 paths with 4 colors. The rainbow and the 4 edge-disjoint paths with ends vertices x^4, x^3b^3 are the following

$$\begin{array}{c} \\ a^3 \\ a^3b \\ a^3b^2 \\ a^3b^3 \\ a^3b^4 \end{array} \begin{array}{cccccccccc} x^4 & x^4b & x^4b^2 & x^4b^3 & x^4b^4 & x^3 & x^3b & x^3b^2 & x^3b^3 & x^3b^4 \\ 1 & & & 3 & 2 & 2 & 1 & 3 & & \\ 2 & 1 & & & 3 & & 2 & 1 & 3 & \\ 3 & 2 & 1 & & & & & 2 & 1 & 3 \\ & 3 & 2 & 1 & & 3 & & & 2 & 1 \\ & & 3 & 2 & 1 & 1 & 3 & & & 2 \end{array}$$

$x^4 \overset{1}{\sim} a^3 \overset{4}{\sim} \quad x^3b^3$

$x^4 \overset{2}{\sim} a^3b \overset{3}{\sim} \quad x^3b^3$

$x^4 \overset{3}{\sim} a^3b^2 \overset{1}{\sim} \quad x^3b^3$

$x^4 \overset{4}{\sim} a^3b^3 \overset{2}{\sim} \quad x^3b^3$

If we note, we can not find 5 edge-disjoint paths with only 4 colors, for example, for the end vertices x^4b^4 and x^3b^2 we have the following paths:

Start with color 1	Start with color 2
$x^4b^4 \overset{1}{\sim} a^3b^4 \overset{4}{\sim} x^3b^2$ $x^4b^4 \overset{1}{\sim} a^3b^4 \overset{3}{\sim} x^4b^2 \overset{2}{\sim} a^3b^3 \overset{4}{\sim} x^3b^2$ $x^4b^4 \overset{1}{\sim} a^3b^4 \overset{4}{\sim} x^4 \overset{3}{\sim} a^3b^2 \overset{2}{\sim} x^3b^2$ $x^4b^4 \overset{1}{\sim} a^3b^4 \overset{2}{\sim} x^3b^3 \overset{4}{\sim} a^3 \overset{3}{\sim} x^3b^2$	$x^4b^4 \overset{2}{\sim} a^3 \overset{3}{\sim} x^3b^2$ $x^4b^4 \overset{2}{\sim} a^3 \overset{3}{\sim} x^4b^3 \overset{4}{\sim} a^3b \overset{1}{\sim} x^3b^2$ $x^4b^4 \overset{2}{\sim} a^3 \overset{3}{\sim} x^4b^3 \overset{1}{\sim} a^3b^3 \overset{4}{\sim} x^3b^2$ $x^4b^4 \overset{2}{\sim} a^3 \overset{1}{\sim} x^3b \overset{3}{\sim} a^3b^4 \overset{4}{\sim} x^3b^2$

Start with color 3	Start with color $x^4b^4 \overset{4}{\sim} a^3b^2$	Start with color 4 from $x^4b^4 \overset{4}{\sim} a^3b^3$
$x^4b^4 \overset{3}{\sim} a^3b \overset{1}{\sim} x^3b^2$	$x^4b^4 \overset{4}{\sim} a^3b^2 \overset{2}{\sim} x^3b^2$	$x^4b^4 \overset{4}{\sim} a^3b^3 \overset{1}{\sim} x^3b^4 \overset{3}{\sim} x^3b^2 \overset{2}{\sim} x^3b^2$
$x^4b^4 \overset{3}{\sim} a^3b \overset{4}{\sim} x^4b^2 \overset{1}{\sim} a^3b^2 \overset{2}{\sim} x^3b^2$	$x^4b^4 \overset{4}{\sim} a^3b^2 \overset{3}{\sim} x^4 \overset{2}{\sim} a^3b \overset{1}{\sim} x^3b^2$	$x^4b^4 \overset{4}{\sim} a^3b^3 \overset{2}{\sim} x^3b^3 \overset{3}{\sim} x^3b \overset{1}{\sim} x^3b^2$
Color 3 can not came to color 4	Color 4 can not came to color $a^3 \overset{3}{\sim} x^3b^2$	Color $x^4b^4 \overset{4}{\sim} a^3b^3$ can not came to color $a^3 \overset{3}{\sim} x^3b^2$

Thus, we have not columns for do the rainbow path from $x^4b^4 \overset{3}{\sim} a^3b$ to $a^3b^3 \overset{4}{\sim} x^3b^2$

	x^4	x^4b	x^4b^2	x^4b^3	x^4b^4	x^3	x^3b	x^3b^2	x^3b^3	x^3b^4
a^3	1			3	2	2	1	3		
a^3b	2	1	/	/	3	/	2	1	$\not 3$	/
a^3b^2	3	2	1					2	1	3
a^3b^3	/	$\not 3$	2	1		$\not 3$	/		2	1
a^3b^4			3	2	1	1	3			2

Then, we can not find a path from x^4b^4 to x^3b^2 passes through a^3b, because the last color from x^4b^4 only can came to x^3b^2 passes through a^3b and a^3b^2. Then we need one more color.

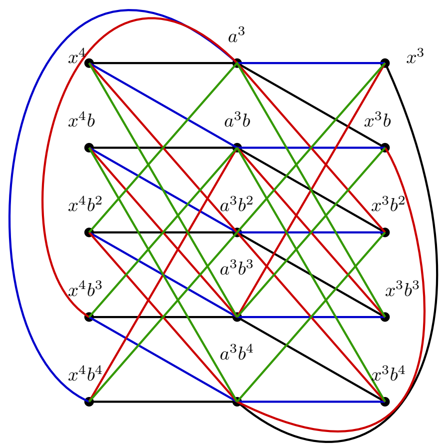


Figure 3. Graph in $\Gamma(G)$

	x^4	x^4b	x^4b^2	x^4b^3	x^4b^4	x^3	x^3b	x^3b^2	x^3b^3	x^3b^4
a^3	1		4	3	2	2	1	3	4	
a^3b	2	1		4	3		2	1	3	4
a^3b^2	3	2	1		4	4		2	1	3
a^3b^3	4	3	2	1		3	4		2	1
a^3b^4		4	3	2	1	1	3	4		2

Rainbow table for found the 5 edge-disjoin paths between x^4 and x^3

With the given structure, we could ask about the meaning of d-coloring redundant as a generalization of [4]. For example, in Figure 3 we could considered a particular case of Turán graph with $T(m|Z|, m)$.

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