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Three-colour bipartite Ramsey number $R_b(G_1, G_2, P_3)$

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Abstract

For simple bipartite graphs G_1, G_2, G_3, the three-colour bipartite graph Ramsey number $R_b(G_1, G_2, G_3)$ is defined as the least positive integer n such that any 3- edge - colouring of $K_{n,n}$ assures a monochromatic copy of G_i in the ith colour for some i, $i \in \{1, 2, 3\}$. In this paper, we consider the three-colour bipartite Ramsey number $R_b(G_1, G_2, P_3)$. Exact values are determined when $G_1 = G_2 = C_4$ and when $(G_1, G_2) =$ (a bistar, a bistar). For integers $m, n \geq 2$, a recursive upper bound, $R_b(K_{m,m}, K_{n,n}, P_3) \leq R_b(K_{m-1,m-1}, K_{n,n}, P_3) + R_b(K_{m,m}, K_{n-1,n-1}, P_3) + 3$, is given. When G_1 and G_2 are even cycles, a lower bound is provided. In addition to these results, we have obtained the relations: $R(G, K_{1,n}) \leq R_b(G, K_{1,n+1})$ and $R(G, H) \leq R_b(G, H, P_3)$.

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1. Introduction

Graph Ramsey theory is one of the widely explored areas in Extremal graph theory. Many interesting books are contributed to its various aspects. Rich development of the theory is well discussed in the book *Ramsey Theory* by Graham, Rothschild and Spencer [8]. For positive integers p,

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q the *classical Ramsey number* is defined as the least positive integer n such that, in every 2-edge-colouring of the complete graph K_n, there is a copy of K_p in colour 1 or a copy of K_q in colour 2. Generalizing this, *graph Ramsey number* was introduced. Given simple graphs $G_1, G_2, \ldots, G_k$, the *graph Ramsey number* $R(G_1, G_2, \ldots, G_k)$ is the smallest positive integer n such that every k-edge-colouring of the complete graph K_n contains a monochromatic copy of G_i in colour i for some i, $i \in \{1, 2, \ldots, k\}$.

Ramsey type problems involve also in the k-edge-colouring of different types of host graphs besides the conventional one K_n. Replacing K_n by $K_{n,n}$ in $R(G_1, G_2, \ldots, G_k)$, we have *bipartite graph Ramsey number* $R_b(G_1, G_2, \ldots, G_k)$. For simple bipartite graphs $G_1, G_2, \ldots, G_k$ the bipartite (graph) Ramsey number $R_b(G_1, G_2, \ldots, G_k)$ is defined as the least positive integer n such that any k-edge-colouring of $K_{n,n}$ assures a copy of G_i in the ith colour for some i. Particular version of this, $R_b(G_1, G_2)$ was initially introduced by Beineke and Schwenk (see [2]) in 1975. Some variants of graph Ramsey number appear as a result of generalisation of the host graphs K_n and $K_{n,n}$, where complete balanced multipartite graph takes the place. For example, *set multipartite Ramsey number* [3] and *size multipartite Ramsey number* [4, 13, 14, 16].

In [2], Beineke and Schwenk showed that $R_b(K_{2,2}, K_{2,2}) = 5$, $R_b(K_{2,4}, K_{2,4}) = 13$ and $R_b(K_{3,3}, K_{3,3}) = 17$. Also, they proved that $R_b(K_{2,n}, K_{2,n}) = 4n - 3$ for n odd and less than 100 except possibly $n = 59$ or $n = 95$.

In [9], Hattingh and Henning have proved a recursive inequality $R_b(K_{m,m}, K_{n,n}) \leq R_b(K_{m-1,m-1}, K_{n,n}) + R_b(K_{m,m}, K_{n-1,n-1}) + 1$ and computed that $R_b(K_{2,2}, K_{3,3}) = 9$ and $R_b(K_{2,2}, K_{4,4}) = 14$. Also, they calculated $R_b\ (K_{1,m}, P_n)$ (see [10]). Hattingh and Joubert calculated the number R_b for pair of bistars (see [11]).

Bipartite graph Ramsey number for the following graph pairs are calculated by Christou, Iliopoulos and Miller [6]: (mP_2, nP_2), (T_m, T_n) (where T_n is a tree on n vertices) and (T_m, nP_2) for certain values of m and n.

When G_1 and G_2 are even cycles, Zhang and Sun [17] gave a lower bound for the number R_b and also calculated the exact value of $R_b(C_{2m}, C_4)$. Zhang, Sun and Wu computed the value of $R_b(C_{2m}, C_6)$ in [18].

Carnielli and Carmelo [5] showed that $R_b(K_{2,n}, K_{2,n}) = 4n-3$ if $4n-3$ is a prime power and $R_b(K_{2,2}, K_{1,n}) = n + q$ for $q^2 - q + 1 \leq n \leq q^2$, where q is a prime power. In [12], Irving showed that $R_b(K_{4,4}, K_{4,4}) \leq 48$.

Regarding $R_b(G_1, G_2, \ldots, G_k)$, Hattingh and Henning [9] gave the following results:

1. For all integers $k \geq 2$, $R_b(\underbrace{K_{2,2}, K_{2,2}, \ldots, K_{2,2}}_{k}) \leq k^2 + k - 1$.

For disjoint copies of $K_{2,2}$, they have established a lower and an upper bound.

2. For all integers $n \geq 2$, $4n - 1 \leq R_b(nK_{2,2}, nK_{2,2}) \leq 4n + 1$.

In this paper, we determine the exact value of the bipartite Ramsey number $R_b(G_1, G_2, P_3)$ for certain families of graph pairs (G_1, G_2). More precisely, in this paper, given simple bipartite graphs G_1 and G_2, we give (i) a recursive upper bound for $R_b(K_{m,m}, K_{n,n}, P_3)$, (ii) a lower bound for $R_b(C_{2m}, C_{2n}, P_3)$, and compute the value of $R_b(G_1, G_2, P_3)$ for the pairs: $(G_1, G_2) = (C_4, C_4)$, and $(B(m,n), B(p,q))$, where $B(r,s)$ denotes a bistar. In addition to these results, we have obtained the inequalities: $R(G, K_{1,n}) \leq R_b(G, K_{1,n+1})$ and $R(G, H) \leq R_b(G, H, P_3)$.

Some of the known results related to the study of $R_b(G_1, G_2, P_3)$ are below:

Let $t \geq 2, s, n_1, n_2, \ldots, n_t$ be positive integers and $r = \sum_{i=1}^{t-1}(n_i - 1)$.

Theorem 1.1. *(see [9]) Let T_m be any tree of order $m \geq 2$. Then*

$$R_b(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_{t-1}}, T_m) \leq r + m - 1.$$

Theorem 1.2. *(see [9])*

$$R_b(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_{t-1}}, sK_2) = \begin{cases} s & \text{for } r \leq \left\lfloor \frac{s-1}{2} \right\rfloor, \\ r + \left\lfloor \frac{s-1}{2} \right\rfloor + 1 & \text{for } r \geq \left\lfloor \frac{s-1}{2} \right\rfloor. \end{cases}$$

Theorem 1.3. *(see [9])* $R_b(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t}) = \sum_{i=1}^{t}(n_i - 1) + 1 = r + n_t$.

Theorem 1.4. *(see [15]) Let m be a positive integer. Then*

$$R_b(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_{t-1}}, P_m) = \begin{cases} \left\lfloor \frac{m+1}{2} \right\rfloor & \text{if } r < \frac{1}{2}\left\lfloor \frac{m}{2} \right\rfloor, \\ 2r + 1 & \text{if } \frac{1}{2}\left\lfloor \frac{m}{2} \right\rfloor \leq r < \frac{1}{2}\left\lfloor \frac{m}{2} \right\rfloor, \\ r + \frac{m}{2} & \text{if } r \geq \frac{m}{2},\ m \text{ even}, \\ r + \frac{m+1}{2} & \text{if } r \geq \frac{m-1}{2},\ m \text{ odd},\ r \equiv 0\ mod(\frac{m-1}{2}), \\ r + \frac{m-1}{2} & \text{if } r \geq \frac{m-1}{2},\ m \text{ odd},\ r \not\equiv 0\ mod(\frac{m-1}{2}). \end{cases}$$

2. Notation

Notation not defined here can be found in [1]. If $T \subseteq V(G)$, the *induced subgraph* $G[T]$ is the subgraph of G whose vertex set is T and whose edge set consists of all edges of G which have both ends in T. For any k-edge-colouring of a simple graph G, let E_i denote the set of edges of colour i and we use $G(E_i)$ to denote the spanning subgraph of G with edge set E_i, where $i = 1, 2, \ldots, k$. A spanning 1-regular subgraph of G is called a 1-*factor* of G.

For any positive integer n, the *stripe graph* nP_2 consists of $2n$ vertices and n independent edges, $K_{1,n}$ denotes the star on $n + 1$ vertices and P_n denotes the path on n vertices. A *bistar* is a tree with diameter three. A *leaf* of a tree is a vertex with degree one. A *support* vertex of a tree is a vertex that is adjacent to a leaf and has degree at least two. For integers r and s with $r, s \geq 2$, $B(r, s)$ denotes the bistar with two support vertices having degrees r and s.

If (X, Y) is a bipartition of $K_{n,n} - I$ with $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{n-1}\}$, then the 1-factor I can be taken as $\{x_0y_0, x_1y_1, \ldots, x_{n-1}y_{n-1}\}$.

3. $R_b(G_1, G_2, P_3)$

We study $R_b(G_1, G_2, P_3)$ by introducing *bipartite minus a* 1-*factor graph Ramsey number* $R_{b-1}(G_1, G_2)$. For simple graphs G_1 and G_2, we define $R_{b-1}(G_1, G_2)$ as the smallest positive integer n such that every 2-edge-colouring of $K_{n,n} - I$ contains a copy of G_1 in colour 1 or a copy of G_2 in colour 2, where I is a 1-factor of $K_{n,n}$.

The following theorem provides a proof on the existence of $R_{b-1}(G_1, G_2)$ by establishing its relation with $R_b(G_1, G_2, P_3)$.

Theorem 3.1. *For simple bipartite graphs G_1 and G_2,*
$$R_b(G_1, G_2, P_3) = R_{b-1}(G_1, G_2).$$

Proof. Let $t = R_{b-1}(G_1, G_2) - 1$. By the definition of $R_{b-1}(G_1, G_2)$, there exists a 2-edge-colouring (E_1, E_2) of $T' = K_{t,t} - I$ such that neither $T'(E_1)$ contains a copy of G_1 nor $T'(E_2)$ contains a copy of G_2. Then, (E_1, E_2, I) is a 3-edge-colouring of $T'' = K_{t,t}$ such that $T''(E_1)$ contains no copy of G_1, $T''(E_2)$ contains no copy of G_2, and $T''(I)$ contains no copy of P_3. Now, by the definition of $R_b(G_1, G_2, P_3)$, we have $R_b(G_1, G_2, P_3) \geq t + 1 = R_{b-1}(G_1, G_2)$.

Let $s = R_b(G_1, G_2, P_3) - 1$. By the definition of $R_b(G_1, G_2, P_3)$, there exists a 3-edge-colouring (E_1, E_2, E_3) of $U' = K_{s,s}$ such that neither $U'(E_1)$ contains a copy of G_1 nor $U'(E_2)$ contains a copy of G_2 nor $U'(E_3)$ contains a copy of P_3. This implies that $U'(E_3)$ is a matching of $K_{s,s}$. So $U'(E_3) \subseteq I$, for some 1-factor I. Now, $(E_1 \setminus I, E_2 \setminus I)$ is a 2-edge-colouring of $U'' = K_{s,s} - I$ such that neither $U''(E_1 \setminus I)$ contains a copy of G_1 nor $U''(E_2 \setminus I)$ contains a copy of G_2. Thus, by the definition of $R_{b-1}(G_1, G_2)$, we have $R_{b-1}(G_1, G_2) \geq s+1 = R_b(G_1, G_2, P_3)$. □

Throughout the paper, all our proofs on $R_b(G_1, G_2, P_3)$ are provided in terms of $R_{b-1}(G_1, G_2)$.

4. Bounds

4.1. A recursive upper bound for $K_{m,m}$ versus $K_{n,n}$

Here, we provide an upper bound for $R_b(K_{m,m}, K_{n,n}, P_3)$.

Theorem 4.1. *For integers $m, n \geq 2$,*
$$R_{b-1}(K_{m,m}, K_{n,n}) \leq R_{b-1}(K_{m-1,m-1}, K_{n,n}) + R_{b-1}(K_{m,m}, K_{n-1,n-1}) + 3.$$

Proof. For the sake of simplicity, we denote $R_{b-1}(K_{p,p}, K_{q,q})$ as $R_{b-1}(p, q)$. Let $t = R_{b-1}(m-1, n) + R_{b-1}(m, n-1) + 3$. Consider a 2-edge-colouring (E_1, E_2) of $T(X, Y) = K_{t,t} - I$, where $X = \{x_0, x_1, \ldots, x_{t-1}\}$, $Y = \{y_0, y_1, \ldots, y_{t-1}\}$ and $I = \{x_0y_0, x_1y_1, \ldots, x_{t-1}y_{t-1}\}$. We need to prove that there is a copy of $K_{m,m}$ in colour 1 or a copy of $K_{n,n}$ in colour 2. We have three cases.

Case 1. $\delta(T(E_1)) \geq R_{b-1}(m-1, n) + 1$.

Then, by symmetry, let x_i be a vertex of X such that $d_{T(E_1)}(x_i) = \delta(T(E_1))$. Let $y_j \in N_{T(E_1)}(x_i)$ $(i \neq j)$. Then $d_{T(E_1)}(y_j) \geq \delta(T(E_1)) \geq R_{b-1}(m-1, n) + 1$. Consider the subgraph $T_1(X_1, Y_1) = T[(N_{T(E_1)}(x_i) \cup N_{T(E_1)}(y_j)) \setminus \{x_i, y_j\}]$, where $X_1 = N_{T(E_1)}(y_j) \setminus \{x_i\}$ and $Y_1 = N_{T(E_1)}(x_i) \setminus \{y_j\}$. Then $|X_1| \geq R_{b-1}(m-1, n)$ and $|Y_1| \geq R_{b-1}(m-1, n)$. By the definition of $R_{b-1}(m-1, n)$, T_1 contains a colour 1 copy of $K_{m-1,m-1}$ or a colour 2 copy of $K_{n,n}$. If we have a colour 2 copy of $K_{n,n}$, then we are done. Otherwise, $T[X_1 \cup Y_1 \cup \{x_i, y_j\}]$ gives us a colour 1 copy of $K_{m,m}$.

Case 2. $\delta(T(E_2)) \geq R_{b-1}(m, n-1) + 1$.

Proof of Case 2 is similar to the proof of Case 1.

Case 3. $\delta(T(E_1)) \leq R_{b-1}(m-1, n)$ and $\delta(T(E_2)) \leq R_{b-1}(m, n-1)$.

By symmetry, let $x_i \in X$ be a vertex such that $d_{T(E_1)}(x_i) = \delta(T(E_1))$. Then $d_{T(E_2)}(x_i) =$

$t - 1 - d_{T(E_1)}(x_i) \geq R_{b-1}(m, n-1) + 2$.

Subcase 3.1. $d_{T(E_2)}(y_j) \geq R_{b-1}(m, n-1) + 1$ for some $y_j \in N_{T(E_2)}(x_i)$.

Let $T_2(X_2, Y_2) = T[(N_{T(E_2)}(x_i) \cup N_{T(E_2)}(y_j))\backslash\{x_i, y_j\}]$, where $X_2 = N_{T(E_2)}(y_j)\backslash\{x_i\}$ and $Y_2 = N_{T(E_2)}(x_i)\backslash\{y_j\}$. Then $|X_2| \geq R_{b-1}(m, n-1)$ and $|Y_2| \geq R_{b-1}(m, n-1) + 1$. By the definition of $R_{b-1}(m, n-1)$, T_2 contains a colour 1 copy of $K_{m,m}$ or a colour 2 copy of $K_{n-1,n-1}$. If we have a colour 1 copy of $K_{m,m}$, then we are done. Otherwise, $T[X_2 \cup Y_2 \cup \{x_i, y_j\}]$ gives us a colour 2 copy of $K_{n,n}$.

Subcase 3.2. $d_{T(E_2)}(y_j) \leq R_{b-1}(m, n-1)$ for all $y_j \in N_{T(E_2)}(x_i)$.

3.2.1. $d_{T(E_2)}(x_r) \geq R_{b-1}(m, n-1) + 1$ for all $x_r \in X$.

Then $|E_2| \geq t(R_{b-1}(m, n-1) + 1)$, and therefore there exists a vertex $y_k \in Y$ such that $d_{T(E_2)}(y_k) \geq R_{b-1}(m, n-1) + 1$. Let $x_\ell \in N_{T(E_2)}(y_k)$. By assumption, $d_{T(E_2)}(x_\ell) \geq R_{b-1}(m, n-1) + 1$. Consider $T[(N_{T(E_2)}(x_\ell) \cup N_{T(E_2)}(y_k))\backslash\{x_\ell, y_k\}]$. As similar to previous cases, we have a colour 1 copy of $K_{m,m}$ or a colour 2 copy of $K_{n,n}$ in T.

3.2.2. There exists $x_s \in X$ such that $d_{T(E_2)}(x_s) \leq R_{b-1}(m, n-1)$.

Then $d_{T(E_1)}(x_s) = t - 1 - d_{T(E_2)}(x_s) \geq R_{b-1}(m-1, n) + 2$. Since $t = R_{b-1}(m-1, n) + R_{b-1}(m, n-1) + 3$, $N_{T(E_2)}(x_i) \cap N_{T(E_1)}(x_s) \neq \emptyset$. Let $y_p \in N_{T(E_2)}(x_i) \cap N_{T(E_1)}(x_s)$. By assumption, $d_{T(E_2)}(y_p) \leq R_{b-1}(m, n-1)$. Then $d_{T(E_1)}(y_p) = t - 1 - d_{T(E_2)}(y_p) \geq R_{b-1}(m-1, n) + 2$. Therefore, $|N_{T(E_1)}(y_p)\backslash\{x_s\}| \geq R_{b-1}(m-1, n) + 1$ and $|N_{T(E_1)}(x_s)\backslash\{y_p\}| \geq R_{b-1}(m-1, n) + 1$. Consider $T[(N_{T(E_1)}(x_s) \cup N_{T(E_1)}(y_p))\backslash\{x_s, y_p\}]$. Again, similar to previous cases, we have a colour 1 copy of $K_{m,m}$ or a colour 2 copy of $K_{n,n}$ in T.

This proves the theorem. □

4.2. Lower bound for even cycle versus even cycle

Theorem 4.2. *For $m, n \geq 2$, $R_{b-1}(C_{2m}, C_{2n}) \geq m + n - 1$.*

Proof. Consider $T(X, Y) = K_{m+n-2, m+n-2} - I$ with $X = \{x_0, x_1, \ldots, x_{m+n-3}\}$, $Y = \{y_0, y_1, \ldots, y_{m+n-3}\}$ and $I = \{x_i y_i : i = 0, 1, \ldots, m+n-3\}$. Let the colour 1 graph be the subgraph induced by $\{x_0, x_1, \ldots, x_{m-2}\} \cup Y$ and the colour 2 graph be that of $\{x_{m-1}, x_m, \ldots, x_{m+n-3}\} \cup Y$. As one of the partite sets of colour 1 and colour 2 graphs contain, respectively, $m-1$ and $n-1$ vertices, neither $T(E_1)$ contains a C_{2m} nor $T(E_2)$ contains a C_{2n}. □

Corollary 4.1. *For $m \geq 2$, $R_{b-1}(C_{2m}, C_4) \geq m + 1$.*

5. Exact values

5.1. C_4 versus C_4

A *decomposition* of a graph G is a collection $\{H_i\}$ of nonempty subgraphs of G such that each edge of G appears in exactly one subgraph in the collection. If $\{H_i\}$ is a decomposition of G such that, for each i, $H_i \cong H$ for some graph H, then G is said to be *H-decomposable,* and we denote it by $H \mid G$.

The *Fano plane* has point set $P = \{1, 2, 3, 4, 5, 6, 7\}$ and line set $L = \{\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{1, 5, 6\}, \{2, 6, 7\}, \{1, 3, 7\}\}$. The *Heawood graph* is the incidence graph of the Fano plane, in otherwords, it is a bipartite graph with bipartition (P, L) in which $p \in P$ is adjacent to $\ell \in L$ if, and only if, $p \in \ell$. Figure 1 is a diagram of Heawood graph.

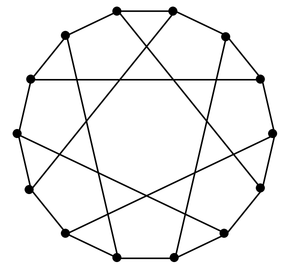

Figure 1. Heawood graph

Lemma 5.1. *If H denotes the Heawood graph, then $H \mid (K_{7,7} - I)$, where I is a 1-factor of $K_{7,7}$. Furthermore, $R_{b-1}(C_4, C_4) \geq 8$.*

Proof. Let $X = \{x_0, x_1, \dots, x_6\}$ and $Y = \{y_0, y_1, \dots, y_6\}$ be the partite sets of $K_{7,7} - I$, where $I = \{x_0y_0, x_1y_1, \dots, x_6y_6\}$. Let $H_1 = (C_{14} : x_0y_1x_6y_0x_5y_6x_4y_5x_3y_4x_2y_3x_1y_2x_0) \oplus \{x_0y_4, x_1y_5, x_2y_6, x_3y_0, x_4y_1, x_5y_2, x_6y_3\}$. Then $H_2 = (C_{14} : x_0y_6x_1y_0x_2y_1x_3y_2x_4y_3x_5y_4x_6y_5x_0) \oplus \{x_0y_3, x_1y_4, x_2y_5, x_3y_6, x_4y_0, x_5y_1, x_6y_2\}$. Observe that $H_1 \cong H \cong H_2$ and therefore $H \mid (K_{7,7} - I)$. Since H is of girth 6, $R_{b-1}(C_4, C_4) \geq 8$. □

Theorem 5.1. $R_{b-1}(C_4, C_4) = 8$.

Proof. By Lemma 5.1, $R_{b-1}(C_4, C_4) \geq 8$. We prove the other inequality $R_{b-1}(C_4, C_4) \leq 8$ by contradiction. Suppose that there exists a 2-edge-colouring (E_1, E_2) of $T(X, Y) = K_{8,8} - I$ such that neither $T(E_1)$ nor $T(E_2)$ contains a C_4. Since $|X| = 8$ and T is 7-regular, we have, by pigeonhole principle, at least four vertices in X, say, x_0, x_1, x_2 and x_3, with at least four incident edges of the same colour, say, colour 1. As $T(E_1)$ does not contain C_4, $|N_{T(E_1)}(x_i) \cap N_{T(E_1)}(x_j)| \leq 1$, for distinct x_i and x_j, where $0 \leq i, j \leq 3$. For distinct x_i and x_j, $0 \leq i, j \leq 3$, if $|N_{T(E_1)}(x_i) \cap N_{T(E_1)}(x_j)| = 0$, then $|Y| \geq 16$, a contradiction. Hence, there exists a pair, say, x_0 and x_1 having one common neighbour in $T(E_1)$, say, y_2. Without loss of generality, let $N_{T(E_1)}(x_0) \supseteq \{y_1, y_2, y_3, y_4\}$. Then $N_{T(E_1)}(x_1) \supseteq \{y_0, y_2, y_5, y_6\}$ or $N_{T(E_1)}(x_1) \supseteq \{y_2, y_5, y_6, y_7\}$. As $d_{T(E_1)}(x_2) \geq 4$, x_2 is adjacent to at least two vertices of $N_{T(E_1)}(x_0)$ or at least two vertices of $N_{T(E_1)}(x_1)$ in $T(E_1)$. But then, we have a C_4 in $T(E_1)$, with either $\{x_0, x_2\} \subseteq V(C_4)$ or $\{x_1, x_2\} \subseteq V(C_4)$, a contradiction. □

5.2. Bistar versus bistar

Hattingh and Henning gave a lower bound for $R_b(G_1, G_2, \dots, G_k)$ as follows:

Lemma 5.2. *(see [9]) Let $n_1, n_2, \dots, n_k$ ($k \geq 2$) be positive integers, and let G_i be a bipartite graph of maximum degree at least n_i for $i = 1, 2, \dots, k$. Then*

$$R_b(G_1, G_2, \dots, G_k) \geq \sum_{i=1}^{k}(n_i - 1) + 1.$$

In our consideration of $R_b(G_1, G_2, P_3)$, for some G_1 and G_2, this bound is attained. In the places, we make use of the above lemma.

Hattingh and Jourbert gave the following result on bipartite Ramsey numbers of k copies of bistars.

Theorem 5.2. *(see [11]) If there are k copies of bistars, where $k \geq 2$ and $s \geq 2$, then* $R_b(B(s,s), \ldots, B(s,s)) \leq \left\lceil k(s-1) + \sqrt{(s-1)^2(k^2-k) - k(2s-4)} \right\rceil$.

Here, we wish to determine $R_b(B(m,n), B(p,q), P_3)$. For the sake, first we prove the following theorem.

Theorem 5.3. *For positive integers m and n with $m \geq n \geq 2$,*
$$R_{b-1}(B(m,m), B(n,n)) = m + n.$$

Proof. By Lemma 5.2, $R_{b-1}(B(m,m), B(n,n)) = R_b(B(m,m), B(n,n), P_3) \geq m + n$. Let $t = m+n$. To prove $R_{b-1}(B(m,m), B(n,n)) \leq t$, we contrarily assume that there exists a 2- edge - colouring (E_1, E_2) of $T(X,Y) = K_{t,t} - I$, where $X = \{x_0, x_1, \ldots, x_{t-1}\}$, $Y = \{y_0, y_1, \ldots, y_{t-1}\}$ and $I = \{x_0y_0, x_1y_1, \ldots, x_{t-1}y_{t-1}\}$ such that neither $T(E_1)$ has a $B(m,m)$ nor $T(E_2)$ has a $B(n,n)$. Let $A = \{x_i \in X \mid d_{T(E_1)}(x_i) \geq m\}$ and $B = \{y_i \in Y \mid d_{T(E_1)}(y_i) \geq m\}$. This implies, for $x_i \in X \backslash A$, $d_{T(E_1)}(x_i) \leq m-1$ and $d_{T(E_2)}(x_i) \geq n$; and for $y_i \in Y \backslash B$, $d_{T(E_1)}(y_i) \leq m-1$ and $d_{T(E_2)}(y_i) \geq n$.

Claim 1. $T[A \cup B] \subseteq T(E_2)$.

Otherwise, there exists an edge $x_iy_j \in E_1$ with $x_i \in A$ and $y_j \in B$. Then we have a $B(m,m)$ in $T(E_1)$ with support vertices x_i and y_j.

Claim 2. $T[(X \backslash A) \cup (Y \backslash B)] \subseteq T(E_2)$.

Otherwise, there exists an edge $x_iy_j \in E_1$ with $x_i \in X \backslash A$ and $y_j \in Y \backslash B$. Note that $d_{T(E_2)}(x_i) \geq n$ and $d_{T(E_2)}(y_j) \geq n$. Let $V \subseteq N_{T(E_2)}(x_i)$ and $U \subseteq N_{T(E_2)}(y_j)$ be such that $|U| = |V| = n$.

For some $x_p \in U$, if $d_{T(E_2)}(x_p) \geq n$, then $x_p \in X \backslash A$ and so we have a $B(n,n)$ in colour 2 with support vertices x_p and y_j, which is a contradiction. Therefore, for every $x_p \in U$, $d_{T(E_2)}(x_p) \leq n-1$ and so $d_{T(E_1)}(x_p) \geq m$. Similarly, for every $y_q \in V$, $d_{T(E_1)}(y_q) \geq m$. Consequently, $U \subseteq A$ and $V \subseteq B$. The facts $V \subseteq N_{T(E_2)}(x_i)$, $U \subseteq N_{T(E_2)}(y_j)$, $|U| = |V| = n$ and Claim 1 together imply that every edge of $T[U \cup V]$ is a non-pendant edge of a $B(n,n)$ in $T(E_2) \cap T[(A \cup \{x_i\}) \cup (B \cup \{y_j\})]$ and every such edge provides two support vertices, say, $u \in A$ and $v \in B$ with $d_{T(E_2)}(u) \geq n$ and $d_{T(E_2)}(v) \geq n$. This gives a contradiction to the way in which A and B are chosen. This proves Claim 2.

Now we divide the proof into four cases.

Case 1. $A = \emptyset = B$.

Then $\delta(T(E_2)) \geq n$. Hence $T(E_2)$ contains a $B(n,n)$.

Case 2. $A \neq \emptyset$ and $B = \emptyset$.

Subcase 2.1. $A \neq X$.

Then, by Claim 2, we have a $B(n, n)$ in $T(E_2)$.

Subcase 2.2. $A = X$.

We obtain a contradiction by counting $|E_1|$ in two different ways. For every $x_i \in A = X$, $d_{T(E_1)}(x_i) \geq m$ implies $|E_1| \geq m(m+n)$, and for every $y_j \in Y \backslash B = Y$, $d_{T(E_1)}(y_j) \leq m-1$ implies $|E_1| \leq (m-1)(m+n)$.

Case 3. $A = \emptyset$ and $B \neq \emptyset$.

By symmetry, proof follows from Case 2.

Case 4. $A \neq \emptyset$ and $B \neq \emptyset$.

Therefore, $|X \backslash A| \geq m$ and $|Y \backslash B| \geq m$. As $d_{T(E_2)}(x_i) \geq n$ and $d_{T(E_2)}(y_j) \geq n$ for $x_i \in X \backslash A$ and $y_j \in Y \backslash B$, every edge $x_i y_j \in T[(X \backslash A) \cup (Y \backslash B)]$ acts as a non-pendant edge of some $B(n, n)$ in $T(E_2)$.

In all the cases, we obtain a contradiction.

Hence $R_{b-1}(B(m, m), B(n, n)) = m + n$. □

Theorem 5.4. *For $m \geq n \geq 2$ and $p \geq q \geq 2$, $R_{b-1}(B(m, n), B(p, q)) = m + p$.*

Proof. As $m \geq n$ and $p \geq q$, $R_{b-1}(B(m, n), B(p, q)) \leq R_{b-1}(B(m, m), B(p, p))$. From the above theorem, $R_{b-1}(B(m, m), B(p, p)) \leq m+p$ and thus, $R_{b-1}(B(m, n), B(p, q)) \leq m+p$. By Lemma 5.2, $R_{b-1}(B(m, n), B(p, q)) \geq m + p$. Hence $R_{b-1}(B(m, n), B(p, q)) = m + p$. □

Corollary 5.1. *For positive integers m and n with $m \geq n \geq 2$, $R_{b-1}(B(m, n), B(m, n)) = 2m$.*

6. Relations among different Ramsey numbers

Let $X = \{x_0, x_1, \ldots, x_{t-1}\}$, $Y = \{y_0, y_1, \ldots, y_{t-1}\}$ and $I = \{x_0 y_0, x_1 y_1, \ldots, x_{t-1} y_{t-1}\}$. Assume $T(X, Y) = K_{t,t}$ and $T'(X, Y) = K_{t,t} - I$.

$\mathscr{C}$: A coloring transformation. To each blue-red colouring $(\mathscr{B}, \mathscr{R})$ of $S = K_t$ with $V(S) = \{v_0, v_1, \ldots, v_{t-1}\}$, there corresponds a 2-edge-colouring (E_b, E'_r) to $T'(X, Y) = K_{t,t} - I$ and (E_b, E_r) to $T(X, Y) = K_{t,t}$ as follows: Edges $x_i y_j, x_j y_i \in E_b$ if, and only if, $v_i v_j \in \mathscr{B}$; $x_i y_j, x_j y_i \in E'_r$ if, and only if, $v_i v_j \in \mathscr{R}$; and, $E_r = E'_r \cup \{x_i y_i : i = 0, 1, \ldots, t-1\}$.

A result of Gonçalves and Carmelo relating $R(K_{2,m}, K_{1,n})$ and $R_b(K_{2,m}, K_{1,n+1})$ is as below:

Proposition 6.1. *[7] For every $m \geq 2$ and $n \geq 2$*

$$R(K_{2,m}, K_{1,n}) \leq R_b(K_{2,m}, K_{1,n+1}).$$

This is extended as follows.

Theorem 6.1. *Let G be a simple bipartite graph. For any positive integer n, $R_b(G, K_{1,n+1}) \geq R(G, K_{1,n})$.*

Proof. Let $t = R(G, K_{1,n}) - 1$. By the definition of $R(G, K_{1,n})$, there exists a blue-red colouring of $S = K_t$, say $(\mathscr{B}, \mathscr{R})$, such that neither $S(\mathscr{B})$ contains a copy of G nor $S(\mathscr{R})$ contains a copy of $K_{1,n}$. By $\mathscr{C}$, we have a 2-edge-colouring (E_b, E_r) to $T(X, Y) = K_{t,t}$. We show that $T(E_b)$ contains no copy of G and $T(E_r)$ contains no copy of $K_{1,n+1}$. If $T(E_b)$ contains a copy of G, then, as $x_i y_i \in E_r$, we have a copy of G in $S(\mathscr{B})$, a contradiction. Therefore, $T(E_b)$ does not contain G as a subgraph. Also, there is no $K_{1,n+1}$ in $T(E_r)$. Otherwise, $\Delta(T(E_r)) \geq n + 1$. This together with $d_{T(E_r)}(x_i) = d_{T(E_r)}(y_i) = d_{\mathscr{R}}(v_i) + 1$, for every $i \in \{1, 2, \ldots, t\}$, implies that $\Delta(S(\mathscr{R})) \geq n$, a contradiction to the fact that $S(\mathscr{R})$ does not contain $K_{1,n}$ as a subgraph. □

Theorem 6.2. *For simple bipartite graphs G and H, $R_{b-1}(G, H) \geq R(G, H)$.*

Proof. Let $t = R(G, H) - 1$. By the definition of $R(G, H)$, there exists a blue-red colouring of $S = K_t$, say $(\mathscr{B}, \mathscr{R})$, such that neither $S(\mathscr{B})$ contains a copy of G nor $S(\mathscr{R})$ contains a copy of H. Again, by $\mathscr{C}$, we have a 2-edge-colouring (E_b, E_r') to $T'(X, Y) = K_{t,t} - I$. We show that $T'(E_b)$ contains no copy of G and $T'(E_r')$ contains no copy of H. If $T'(E_b)$ contains a copy of G, then, as $x_i y_i \in I$, we have a copy of G in $S(\mathscr{B})$, a contradiction. Therefore, $T'(E_b)$ does not contain G as a subgraph. By a similar argument, we have that $T'(E_r)$ does not contain H as a subgraph. □

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References

[1] R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*, Second Edition, Springer, New York, 2012.

[2] L.W. Beineke and A.J. Schwenk, On a bipartite form of the Ramsey problem, *Proc. 5th British Combin. Conf. 1975, Congressus Numer.* **XV** (1975), 17–22.

[3] A.P. Burger and J.H. van Vuuren, Ramsey numbers in complete balanced multipartite graphs. Part I: Set numbers, *Discrete Math.* **283** (2004), 37–43.

[4] A.P. Burger and J.H. van Vuuren, Ramsey numbers in complete balanced multipartite graphs. Part II: Size numbers, *Discrete Math.* **283** (2004), 45–49.

[5] W.A. Carnielli and E.L. Monte Carmelo, $K_{2,2} - K_{1,n}$ and $K_{2,n} - K_{2,n}$ bipartite Ramsey numbers, *Discrete Math.* **223** (2000), 83–92.

[6] M. Christou, C.S. Iliopoulos and M. Miller, Bipartite Ramsey numbers involving stars, stripes and trees, *Electron. J. Graph Theory Appl.* **1** (2) (2013), 89–99.

[7] A. Gonçalves and E.L. Monte Carmelo, Some geometric structures and bounds for Ramsey numbers, *Discrete Math.* **280** (2004), 29–38.

[8] R.L. Graham, B.L. Rothschild and J.H. Spencer, *Ramsey Theory*, Second Edition, John Wiley and Sons, New York, 1990.

[9] J.H. Hattingh and M.A. Henning, Bipartite Ramsey theory, *Utilitas Math.* **53** (1998), 217–230.

[10] J.H. Hattingh and M.A. Henning, Star-path bipartite Ramsey numbers, *Discrete Math.* **185** (1998), 225–258.

[11] J.H. Hattingh and E.J. Joubert, Some bistar bipartite Ramsey numbers, *Graphs and Combin.* **30** (5) (2014), 1175–1181.

[12] R.W. Irving, A bipartite Ramsey problem and the Zarankiewicz numbers, *Glasgow Math. J.* **19** (1978), 13–26.

[13] C. Jayawardene, E.T. Baskoro, L. Samarasekara and S. Sy, Size multipartite Ramsey numbers for stripes versus small cycles, *Electron. J. Graph Theory Appl.* **4** (2) (2016), 157–170.

[14] A. Lusiani, E.T. Baskoro and S.W. Saputro, On size multipartite Ramsey numbers for stars versus paths and cycles, *Electron. J. Graph Theory Appl.* **5** (1) (2017) 43–50.

[15] G. Raeisi, Star-path and star-stripe bipartite Ramsey numbers in multicoloring, *Trans. Combin.* **4** (3) (2015), 37–42.

[16] D.R. Silaban, E.T. Baskoro and S. Uttunggadewa, Restricted size Ramsey number for path of order three versus graph of order five, *Electron. J. Graph Theory Appl.* **5** (1) (2017), 155–162.

[17] R. Zhang and Y.Q. Sun, The bipartite Ramsey numbers $b(C_{2m}; K_{2,2})$, *The Electron. J. Combin.* **18** (2011), P. 51.

[18] R. Zhang, Y.Q. Sun and Y. Wu, The bipartite Ramsey numbers $b(C_{2m}; C_{2n})$, *International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering*, **7** (1) (2013), 152–155.