Electronic Journal of Graph Theory and Applications **9** (1) (2021), 113–123

Electronic Journal of
Graph Theory and Applications

Connected domination value in graphs

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Abstract

In a connected graph $G = (V, E)$, a set $D \subset V$ is a *connected dominating set* if for every vertex $v \in V \setminus D$, there exists $u \in D$ such that u and v are adjacent, and the subgraph $\langle D \rangle$ induced by D in G is connected. A connected dominating set of minimum cardinality is called a γ_c-set of G. For each vertex $v \in V$, we define the *connected domination value* of v to be the number of γ_c-sets of G to which v belongs. In this paper, we study the properties of connected domination value of a connected graph G and its relation to other parameters of a connected graph. Finally, we compute the connected domination value and number of γ_c-sets for a few well-known family of graphs.

Keywords: domination value, connected dominating set, maximum degree
Mathematics Subject Classification: 05C69
DOI: 10.5614/ejgta.2021.9.1.11

1. Introduction

[1] The study of dominating sets, domination number and other variants of domination parameters of a graph like [1, 3, 4, 5, 6, 11, 13] forms an integral part of both theoretical as well as practical aspects of graph theory. However, a systematic local study of domination has not been studied extensively. The first step towards this was by Mynhardt [12], who studied the vertices which belong to every minimum dominating set of a tree. Subsequently, Cockayne *et.al.* [2] and Meddah *et.al.* [10] studied the vertices which belong to either every or none of the (k-)total minimum dominating sets of a tree. Yi [15] and Kang [9] introduced a new concept of (total) domination value $(T)DV$

Received: 2 December 2018, Revised: 30 May 2019, Accepted: 16 December 2020.

[1]Dedicated to my dear friend Late Wyatt Jules Desormeaux

of a vertex in a graph. (Total) domination value of a vertex v is the number of minimum (total) dominating sets containing v.

In this paper, we introduce connected domination value of a graph. Let $G = (V, E)$ be a simple, undirected, connected graph of order $|V|$ and size $|E|$. The degree of a vertex v in G, denoted by $deg(v)$, is the number of vertices adjacent to v in G; an *end-vertex* is a vertex of degree one and a *support vertex* is a vertex which is adjacent to an end-vertex. For $v \in V$, $N(v)$ is the set of all vertices in G adjacent to v and $N[v] = N(v) \cup \{v\}$. A set $D \subset V$ is a *connected dominating set* (CDS) of G if for every vertex $v \in V \setminus D$, there exists $u \in D$ such that $uv \in E$, and the subgraph $\langle D \rangle$ induced by D in G is connected. The minimum cardinality of a connected dominating set is called the *connected domination number* of G and is denoted by γ_c. A connected dominating set of minimum cardinality is called a γ_c-set of G. Analogous to the definitions and notations defined in [15, 9], for each vertex $v \in V$, we define the *connected domination value* of v, $CDV(v)$, to be the number of γ_c-sets of G to which v belongs. We also define τ_c to be the number of γ_c-sets of G. Thus for any graph G and any $v \in V$, $0 \leq CDV(v) \leq \tau_c$. For other notations and graph terminology, refer to [14, 7].

There are similarities as well as differences between DV (or TDV) and CDV of a graph. In this paper, we recall results on DV from [15] and TDV from [9] that can be carried out to CDV and prove results of CDV that are different from DV (or TDV).

2. Basic Properties of Connected Domination Value

In this section, we study some basic properties and bounds of connected domination value of a vertex of a graph.

Lemma 2.1. *Let G be a connected graph with $n(> 2)$ vertices. Then every support vertex is contained in each γ_c-set of G.*

Proof. Let v be a support vertex adjacent to an end-vertex u and D be a γ_c-set of G. Since $deg(u) = 1$, D must contain u or v. If D does not contain v, then $\langle D \rangle$ fails to be connected as every path joining u to any other vertex of D must contain v as an intermediate vertex. Hence, the lemma follows. □

We recall a few observations and results from [15] and [9].

Proposition 2.1. *[15] For any graph $G = (V, E)$,*

$$\sum_{v \in V} DV(v) = \tau \cdot \gamma.$$

Proposition 2.2. *[15] If $\varphi : G \to G'$ be a graph isomorphism and $\varphi(v) = v'$. Then $DV_G(v) = DV_{G'}(v')$.*

Proposition 2.3. *[15] For any $v_0 \in V$,*

$$\tau \leq \sum_{v \in N[v_0]} DV(v) \leq \tau \cdot \gamma$$

and the bounds are tight.

Proposition 2.4. *[15] For any $v_0 \in V$,*

$$\sum_{v \in N[v_0]} DV(v) \leq \tau(1 + deg(v_0)),$$

and this bound is tight.

Proposition 2.5. *[15] Let H be a subgraph of a graph G with $V(G) = V(H)$. If $\gamma(G) = \gamma(H)$, then $\tau(G) \geq \tau(H)$.*

Proposition 2.6. *[9] Let G be a connected graph with $\gamma_t = 2$. Then $TDV(v) \leq deg(v)$ for any vertex v in G.*

All the above propositions proved in [15] and [9] remains true if DV (or TDV) is replaced by CDV, τ, γ, γ_t are replaced by $\tau_c, \gamma_c, \gamma_c$ respectively and if graphs and subgraphs are connected.

Corollary 2.1. *Let G be a connected vertex-transitive graph of order n, where n is a prime. Then τ_c is a multiple of n.*

Proof. Since G is a connected vertex transitive graph, by Proposition 2.2, $CDV(v)$ is a constant, say k, for all $v \in V$. Thus, by Proposition 2.1, $\tau_c \cdot \gamma_c = nk$. Now as G is a connected graph of order n, $\gamma_c < n$ and hence n does not divide γ_c. Thus n, being a prime, divides τ_c. □

3. Connected Domination Value and Maximum Degree

In this section, we study the bounds on connected domination value of the highest degree of the vertices in a connected graph. First we recall some results from [15] and [9].

Proposition 3.1. *[15] Let G be a graph with n vertices and $\Delta = n - 1$. Then $\gamma = 1$ and $DV(v) \leq 1, \forall v \in V$, and equality holds if and only if $deg(v) = n - 1$.*

The above proposition remains true when DV is replaced by CDV (due to the fact that $\gamma = 1$ implies $\gamma_c = 1$.)

Proposition 3.2. *[9] Let G be a graph with $n(\geq 3)$ vertices and $\Delta = n - 2$. Then $\gamma_t = 2$ and $TDV(v) \leq n - 2$. Further, if $deg(v) = n - 2$, then $TDV(v) = |N(w)|$ where w is the unique vertex in G such that $vw \notin E$.*

Proposition 3.3. *[9] Let G be a graph of order n with $\gamma_t = 2$ and $\Delta \leq n - 2$, then $\tau \leq \binom{n}{2} - \lceil \frac{n}{2} \rceil$ and the bound is tight.*

Theorem 3.1. *[9] Let G be a connected graph with $n(\geq 4)$ vertices and $\Delta = n - 3$. Let v be a vertex of G with $deg(v) = n - 3$. Then either $\gamma_t = 2$ and $TDV(v) \leq n - 3$ or $\gamma_t = 3$ and $TDV(v) \leq (\frac{n-3}{2})^2 + 2(n - 4)$.*

The above two Propositions and Theorem remains true for connected graphs when τ, γ_t, and TDV, respectively, is replaced by τ_c, γ_c, and CDV (due to the fact that, for any connected graph with $\gamma_c \neq 1, \gamma_t = 2$ and $\gamma_t = 3$, respectively, implies $\gamma_c = 2$ and $\gamma_c = 3$).

4. Connected Domination Value for Some Graph Families

4.1. Complete n-partite graphs

Let $G = K_{a_1,a_2,\ldots,a_n}$ be a complete n-partite graph with the vertex set V partitioned into partite sets $V_1, V_2, \ldots, V_n$ and let $a_i = |V_i| \geq 1, \forall i \in \{1, 2, \ldots, n\}$ and $n \geq 2$. Again, we recall a few results from [15].

Theorem 4.1. *[15] Let $G = K_{a_1,a_2,\ldots,a_n}$ be a complete n-partite graph with $a_i \geq 2, \forall i \in \{1, 2, \ldots, n\}$. Then*

$$\tau = \frac{1}{2}\left[\left(\sum_{i=1}^{n} a_i\right)^2 - \sum_{i=1}^{n} a_i^2\right] \text{ and } DV(v) = \left(\sum_{i=1}^{n} a_i\right) - a_j, \text{ if } v \in V_j.$$

Theorem 4.2. *[15] Let $G = K_{a_1,a_2,\ldots,a_n}$ be a complete n-partite graph with $a_i = 1$ for some i, i.e., $a_j = 1 \forall j \in \{1, 2, \ldots, k\}$ and $a_j > 1, \forall j \in \{k+1, k+2, \ldots, n\}$. Then $\tau = k$ and*

$$DV(v) = \begin{cases} 1, & \text{if } v \in V_j (1 \leq j \leq k), \\ 0. & \text{if } v \in V_j (k+1 \leq j \leq n). \end{cases}$$

Corollary 4.1. *[15] If G is a complete graph K_n, then $\tau = n$ and $DV(v) = n, \forall v \in G$.*

Corollary 4.2. *[15] If G is a complete bipartite graph K_{a_1,a_2}, then*

$$\tau = \begin{cases} a_1 \cdot a_2, & \text{if } a_1, a_2 \geq 2, \\ 2, & \text{if } a_1 = a_2 = 1, \\ 1, & \text{if } \{a_1, a_2\} = \{1, x\} \text{ where } x > 1. \end{cases}$$

If $a_1, a_2 \geq 2$, then

$$DV(v) = \begin{cases} a_2, & \text{if } v \in V_1, \\ a_1, & \text{if } v \in V_2. \end{cases}$$

If $a_1 = a_2 = 1$, then $DV(v) = 1$ for any v in $K_{1,1}$. If $\{a_1, a_2\} = \{1, x\}$ with $x > 1$, say $a_1 = 1, a_2 = x$, then

$$DV(v) = \begin{cases} 1, & \text{if } v \in V_1, \\ 0, & \text{if } v \in V_2. \end{cases}$$

The above two theorems and two corollaries remain true when DV and τ, respectively, is replaced by CDV and τ_c.

4.2. Cycles and Paths

Let C_n be a cycle on n vertices, which are labelled 1 to n in anti-clockwise order. As C_n is vertex-transitive, $CDV(v)$ is constant for all vertices $v \in C_n$. Note that, for $n \geq 3$, $\gamma_c(C_n) = n-2$ and the induced subgraph by each minimum connected dominating set is isomorphic to P_{n-2}, a path on $n-2$ vertices.

Theorem 4.3. *For $n \geq 3$, $\tau_c(C_n) = n$ and $CDV(v) = n-2, \forall v \in V(C_n)$.*

Proof. Observe that any $n-2$ consecutively labelled vertices form a minimum connected dominating set of C_n. Thus, $\tau_c(C_n)$ is the number of distinct isomorphic copies of P_{n-2} in C_n, i.e., $\mathcal{C} = \{\{1,2,\ldots,n-3,n-2\},\{2,3,\ldots,n-2,n-1\},\ldots,\{n,1,\ldots,n-3\}\}$ is the collection of all minimum connected dominating sets of C_n. Hence, $\tau_c(C_n) = n$.

As C_n is vertex-transitive, $CDV(v) = CDV(1)$ for all vertices $v \in V(C_n)$. Now, by observing the number of occurrences of 1 in $\mathcal{C}$, we get $CDV(1) = n-2$ and hence the theorem. □

Theorem 4.4. *For $n \geq 2$,*

$$\tau_c(P_n) = \begin{cases} 2, & \text{if } n = 2, \\ 1, & \text{if } n \geq 3. \end{cases}$$

and $CDV(v) = 1$ for each vertex $v \in V(P_2)$. For $n \geq 3$,

$$CDV(v) = \begin{cases} 1, & \text{if } v \text{ is an interior vertex,}. \\ 0, & \text{if } v \text{ is an end vertex.} \end{cases}$$

Proof. Let P_n be a path on n vertices, which are labelled 1 to n consecutively.
Case 1: $n = 2$ In this case, each of the vertices is a minimum connected dominating set and hence $\tau_c = 2$ and $CDV(v) = 1$ for each vertex $v \in P_2$.
Case 2: $n \geq 3$ Since $\{2,3,\ldots,n-1\}$ is the unique minimum connected dominating set of P_n with $n-2$ vertices, we have $\gamma_c(P_n) = n-2$,$\tau_c = 1$ and $CDV(v) \in \{0,1\}$. □

4.3. The Petersen Graph

Let $\mathcal{P}$ be the Petersen graph. It is to be noted that $\gamma_c(\mathcal{P}) = 4$ and for any v in $\mathcal{P}$, $N[v]$ is a minimum connected dominating set. In fact, these are the only minimum connected dominating sets of $\mathcal{P}$. Since for any two vertices u and v, $N[u] \neq N[v]$, the number of minimum connected dominating sets is equal to the order of $\mathcal{P}$, i.e., $\tau_c(\mathcal{P}) = 10$. Also as $\mathcal{P}$ is vertex transitive, $CDV(v)$ is constant for all vertices $v \in \mathcal{P}$. Thus $CDV(v) = CDV(1)$ for any v in $\mathcal{P}$. Now, $CDV(1)$ is equal to the number of $N[v]$'s in which 1 belongs to, i.e., $CDV(1) = |N[1]| = 4$.

4.4. The $2 \times n$ rectangular grid: $P_2 \square P_n$

We consider $P_2 \square P_n (n \geq 2)$ as two copies of P_n with vertices labelled $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ with the additional edges $x_i y_i$ for each $i \in \{1,2,\ldots,n\}$. (See Figure 1.) For later use, we partition the vertices into n sets (or columns as shown in Figure 1) $D_i = \{x_i, y_i\}$ for $i \in \{1,2,\ldots,n\}$

Lemma 4.1. *For $n \geq 2$, $\gamma_c(P_2 \square P_n) = n$ for $n \neq 3$ and $\gamma_c(P_2 \square P_3) = 2$.*

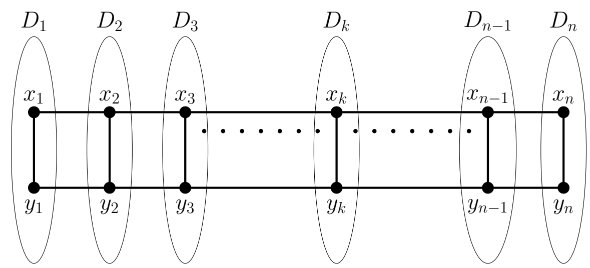

Figure 1. Labelling of vertices in $P_2 \Box P_n$

Proof. It is trivial to observe that $\gamma_c(P_2 \Box P_2) = \gamma_c(P_2 \Box P_3) = 2$. For $n \geq 4$, clearly $\{x_1, x_2, \ldots, x_n\}$ is a connected dominating set of $(P_2 \Box P_n)$, i.e., $\gamma_c(P_2 \Box P_n) \leq n$. If possible, let S be a connected dominating set of $P_2 \Box P_n$ of cardinality $n - 1$.
Case 1: S contains only $n - 1$ x_i's or S contains only $n - 1$ y_i's. Suppose the former holds. Let x_j be the unique vertex not in S. Then y_j is not dominated by any vertex in S. Hence, S cannot contain only x_i's and similarly S can not contain only y_i's.
Case 2: S contains at least one x_i and at least one y_j. Since $\langle S \rangle$ is connected, there exists an index k such that $x_k, y_k \in S$, i.e., both the vertices in D_k are in S. Thus, S contains other $n - 3$ vertices apart from x_k, y_k. Thus there exist at least two columns D_i and D_j which has no vertices in S. Now only options left for $\{D_i, D_j\}$ is $\{D_1, D_2\}$ or $\{D_{n-1}, D_n\}$ or $\{D_1, D_n\}$, as in other cases $\langle S \rangle$ fails to be connected.
Case 2(a): If $\{D_i, D_j\}$ is $\{D_1, D_2\}$ or $\{D_{n-1}, D_n\}$, then the vertices in D_1 or D_n are not dominated by S.
Case 2(b): If $\{D_i, D_j\}$ is $\{D_1, D_n\}$, then both D_2 and D_{n-1} are contained in S, otherwise S will fail to dominate $P_2 \Box P_n$. Thus, in this case, there are at least two columns, namely D_2 and D_{n-1}, with both vertices in S. As S contains $n - 1$ vertices, the number of remaining vertices is $n - 5$, which is distributed among the $n - 4$ columns $D_3, D_4, \ldots, D_{n-2}$. So at least one column among $D_3, D_4, \ldots, D_{n-2}$ has no vertices in S, thereby making $\langle S \rangle$ disconnected.

Thus, $\gamma_c(P_2 \Box P_n) = n$ for $n \geq 4$. □

Lemma 4.2. *For $n \geq 5$, any γ_c-set S in $P_2 \Box P_n$ either contains $\{x_3, x_4, \ldots, x_{n-3}, x_{n-2}\} \subset S$ or $\{y_3, y_4, \ldots, y_{n-3}, y_{n-4}\} \subset S$ (and not both).*

Proof. Let S be a γ_c-set of $P_2 \Box P_n$ of cardinality n, where $n \geq 5$. Note that $S \cap \{x_1, y_1, x_2, y_2\} \neq \emptyset$ and $S \cap \{x_{n-1}, y_{n-1}, x_n, y_n\} \neq \emptyset$. If $D_k \cap S = \emptyset$ for some $k \in \{3, \ldots, n-2\}$, then $\langle S \rangle$ is disconnected, since there is no path connecting a vertex on the left of D_k and a vertex on the right of D_k. Let $x_k \in S$. If possible, $y_k \in S$, then arguing as in Case 2 of Lemma 4.1, other $n - 2$ vertices of S appears in the $n - 1$ columns $D_1, D_2, \ldots, D_{k-1}, D_{k+1}, \ldots, D_n$. Thus there exists $j \in \{1, 2, \ldots, k-1, k+1, \ldots, n\}$ such that $D_j \cap S = \emptyset$. If $D_j \neq D_1$ and $D_j \neq D_n$, then $\langle S \rangle$ is not connected. Thus, $D_j = D_1$ or $D_j = D_n$.

If $D_j = D_1$, then as S dominates x_1 and y_1 we have $D_2 \subset S$. Thus x_2, y_2, x_k, y_k are four distinct vertices of S. Thus other $n-4$ vertices appear in $n-3$ columns $D_3, D_4, \ldots, D_{k-1}, D_{k+1}, \ldots, D_n$. Again arguing in the same way, there exists $i \in \{3, 4, \ldots, k-1, k+1, \ldots, n\}$ such that $D_i \cap S = \emptyset$. If $D_i \neq D_3$ and $D_i \neq D_n$, then $\langle S \rangle$ is not connected. Thus, $D_i = D_3$ or $D_i = D_n$. Also, if $D_i = D_3$, then $\langle S \rangle$ is not connected as there does not exist any path from x_2 to x_k (or from y_2 to y_k) in $\langle S \rangle$. Thus, $D_i = D_n$. This implies that $D_{n-1} \subset S$ (to dominate x_n and y_n). Hence, $x_2, y_2, x_k, y_k, x_{n-1}, y_{n-1}$ are six distinct vertices of S. Thus other $n-6$ vertices appear in $n-5$ columns $D_3, D_4, \ldots, D_{k-1}, D_{k+1}, \ldots, D_{n-2}$. Again arguing in the same way, there exists $l \in \{3, 4, \ldots, k-1, k+1, \ldots, n-2\}$ such that $D_l \cap S = \emptyset$. This implies $\langle S \rangle$ is not connected as there is no path joining x_2 and x_{n-1} in $\langle S \rangle$, which is a contradiction.

Similarly, it can be shown that starting with $D_j = D_n$ will also lead to disconnectedness of $\langle S \rangle$, which is a contradiction. Thus, the assumption $y_k \in S$ is invalid.

Hence, if $x_k \in S$ for any $k \in \{2, 3, \ldots, n-1\}$, then to maintain connectedness of $\langle S \rangle$, $\{x_3, x_4, \ldots, x_{n-2}\} \subset S$. In a similar way, if $y_k \in S$ for any $k \in \{2, 3, \ldots, n-1\}$, then $\{y_3, y_4, \ldots, y_{n-2}\} \subset S$. Finally the lemma follows from the observation that to dominate $P_2 \square P_n$, at least one of x_k or y_k with $k \in \{2, 3, \ldots, n-1\}$ must belong to S. □

Theorem 4.5. $\tau_C(P_2 \square P_n) = \begin{cases} 4, & \textit{if } n = 2. \\ 1, & \textit{if } n = 3, \\ 8, & \textit{if } n \geq 4. \end{cases}$.

Proof. Let S be a γ_c-set of $P_2 \square P_n$ of cardinality n where $n \geq 2$. If $n = 2$, then $P_2 \square P_2 \cong C_4$ and any two adjacent vertices form a γ_c-set, i.e., $\{x_1, y_1\}, \{x_1, x_2\}, \{y_1, y_2\}, \{x_2, y_2\}$ are all possible γ_c-sets of $P_2 \square P_2$. If $n = 3$, there is a unique γ_c-set $\{x_2, y_2\}$. So, let $n \geq 4$. By Lemma 4.2, either $\{x_3, x_4, \ldots, x_{n-3}, x_{n-2}\} \subset S$ or $\{y_3, y_4, \ldots, y_{n-3}, y_{n-2}\} \subset S$ (and not both). Let $\{x_3, x_4, \ldots, x_{n-3}, x_{n-2}\} \subset S$. As $y_3 \notin S$, to maintain connectedness of $\langle S \rangle$ and to dominate x_1, we have $x_2 \in S$. In the same way, $x_{n-1} \in S$. Thus, $\{x_2, x_3, \ldots, x_{n-2}, x_{n-1}\} \subset S$. Since, S contains n elements, let the other 2 vertices in S be a, b. To dominate x_1 and y_1, one of a and b (say a) must be either x_1 or y_2. Similarly b is either x_n or y_{n-1}. Since there are two choices each for a and b such that S forms a γ_c-set, the number of γ_c-sets containing $x_3, x_4, \ldots, x_{n-3}, x_{n-2}$ is 4. Similarly, the number of γ_c-sets containing $y_3, y_4, \ldots, y_{n-3}, y_{n-2}$ is 4. Hence, by Lemma 4.2, we get $\tau_c(P_2 \square P_n) = 8$ for $n \geq 4$. □

Theorem 4.6. *Let $P_2 \square P_n$ be a rectangular grid with $n \geq 2$ and let $u_i = x_i$ or y_i. If $n = 2$, then $CDV(v) = 2$ for all $v \in V(P_2 \square P_2)$. If $n = 3$, then $CDV(u_1) = CDV(u_3) = 0$ and $CDV(u_2) = 1$. If $n \geq 4$, then*

$$CDV(u_i) = \begin{cases} 2, & \textit{if } i = 1 \textit{ or } n, \\ 6, & \textit{if } i = 2 \textit{ or } n-1, \\ 4, & \textit{otherwise.} \end{cases}$$

Proof. The proof is obvious for $n = 2$ and 3, by Theorem 4.5. So, we assume that $n \geq 4$. Let v be a vertex in $P_2 \square P_n$.

Case 1: $[v \in \{x_1, y_1, x_n, y_n\}]$ Let $v = x_1$, then using the line of proof of Theorem 4.5, the γ_c-sets containing x_1 are precisely those where $a = x_1$ and b is either x_n or y_{n-1}, i.e., $CDV(v) = 2$. Same is the case when $v = y_1$ or $v = x_n$ or $v = y_n$.

Case 2: $[v \in \{x_2, y_2, x_{n-1}, y_{n-1}\}]$ Let $v = x_2$. Note that any connected dominating set contains either x_2, y_2. Also total number of minimum connected dominating sets is 8, out of which only

two does not contain x_2, namely $\{y_1, y_2, \ldots, y_n\}$ and $\{y_1, y_2, \ldots, y_{n-1}, x_{n-1}\}$. Thus $CDV(x_2) = 8 - 2 = 6$. Now, as there exist isomorphisms which maps x_2 to y_2, x_{n-1}, y_{n-1} respectively, by Proposition 2.2, we have $CDV(x_2) = CDV(y_2) = CDV(x_{n-1}) = CDV(y_{n-1}) = 6$.
Case 3: $[v \notin \{x_1, y_1, x_2, y_2, x_{n-1}, y_{n-1}, x_n, y_n\}]$ In this case, from the proof of Theorem 4.5, we have $CDV(v) = 4$. □

4.5. The $2 \times n$ cylindrical grid: $P_2 \square C_n$

We consider $P_2 \square C_n (n \geq 3)$ as two copies of C_n with vertices labelled $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ with the additional edges $x_i y_i$ for each $i \in \{1, 2, \ldots, n\}$. (See Figure 2.) For later use, we partition the vertices into n sets (or columns as shown in Figure 2) $D_i = \{x_i, y_i\}$ for $i \in \{1, 2, \ldots, n\}$.

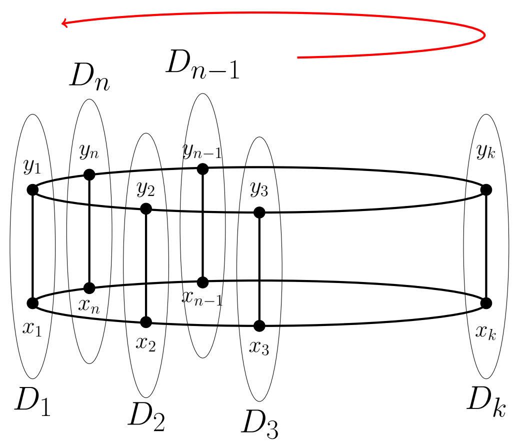

Figure 2. Labelling of vertices in $P_2 \square C_n$

Lemma 4.3. *For $n \geq 3$,*

$$\gamma_c(P_2 \square C_n) = \begin{cases} 2, & \text{if } n = 3, \\ n, & \text{if } n > 3. \end{cases}$$

Proof. The lemma is trivially true for $n = 3$. For $n > 3$, clearly $\{x_1, x_2, \ldots, x_n\}$ is a connected dominating set of $P_2 \square C_n$ and hence $\gamma_c(P_2 \square C_n) \leq n$. Suppose there exists a connected dominating set S with $n - 1$ vertices. Since there are n columns $D_1, D_2, \ldots, D_n$, then $D_i \cap S = \emptyset$ for some $i \in \{1, 2, \ldots, n\}$.
Case 1: [Either D_{i-1} or D_{i+1} contains no vertices from S.] Let $D_{i-1} \cap S = \emptyset$. Then both $D_{i+1} \subset S$ and $D_{i-2} \subset S$. Thus other $n-5$ vertices of S appear in $n-4$ columns $D_1, D_2, \ldots, D_{i-3}, D_{i+2}, \ldots, D_n$. Thus there exists $j \in \{1, 2, \ldots, i-3, i+2, \ldots, n\}$ such that $D_j \cap S = \emptyset$. This implies that there does not exist any path from x_{i+1} to x_{i-2} in $\langle S \rangle$ which is a contradiction to the connectedness of $\langle S \rangle$. The case for $D_{i+1} \cap S = \emptyset$ is similar.
Case 2: [Both D_{i-1} and D_{i+1} contains at least one vertex from S.] As there are two vertices in both D_{i-1} and D_{i+1}, 4 possibilities are there:

Case 2A: $[x_{i-1}, y_{i+1} \in S]$ Since $D_i \cap S = \emptyset$, the shortest path joining x_{i-1} and y_{i+1} should pass through at least one vertex of each D_k for $k \in \{1, 2, \ldots, i-2, i+2, \ldots, n\}$ and since $\langle S \rangle$ in connected, at least one D_k contains two vertices x_k and y_k. This makes the total count of vertices to be n which is more than $n-1$ and hence a contradiction.
Case 2B: $[x_{i+1}, y_{i-1} \in S]$ Same as Case 2A.
Case 2C: $[x_{i-1}, x_{i+1} \in S]$ In this case, to dominate y_i, at least one of y_{i-1} and y_{i+1} belong to S. Without loss of generality, let $y_{i-1} \in S$. Thus $D_{i-1} \subset S$ and $x_{i+1} \in S$. Therefore, other $n-4$ vertices of S appears in the $n-3$ columns $D_1, D_2, \ldots, D_{i-2}, D_{i+1}, D_{i+2}, \ldots, D_n$.[2] Thus $\exists j \in \{1, 2, \ldots, i-2, i+2, \ldots, n\}$ such that $D_j \cap S = \emptyset$. As D_i and D_j are not consecutive columns, there does not exist any path joining x_{i-1} and x_{i+1} in $\langle S \rangle$. This implies $\langle S \rangle$ is disconnected which is a contradiction.
Case 2D: $[y_{i-1}, y_{i+1} \in S]$ Same as Case 2C.

Combining all the cases, we see that $P_2 \square C_n$ can not have a connected dominating set of cardinality $n-1$ and hence $\gamma_c(P_2 \square C_n) = n$ for $n \geq 4$. □

Theorem 4.7. *For $n \geq 3$,*

$$\tau_c(P_2 \square C_n) = \begin{cases} 3, & \text{if } n = 3, \\ 30, & \text{if } n = 4, \text{ and} \\ 2(n^2+1), & \text{if } n > 4. \end{cases}$$

and for $v \in V(P_2 \square C_n)$ and $n \geq 3$,

$$CDV(v) = \begin{cases} 1, & \text{if } n = 3, \\ 15, & \text{if } n = 4, \text{ and} \\ n^2+1, & \text{if } n > 4. \end{cases}$$

Proof. First, we deal with the case when $n = 3$. In this case, the only 3 γ_c-sets are $\{x_1, y_1\}, \{x_2, y_2\}$ and $\{x_3, y_3\}$. Thus $\tau_c = 3$ and $CDV(v) = 1$ for each vertex v in $V(P_2 \square C_3)$.

Now, we deal with the case when $n > 3$. Let S be a γ_c-set of $P_2 \square C_n$ of cardinality n.
Case 1:[Each D_i contains one element of S.] Let $x_1 \in D_1 \cap S$. We claim that $y_i \notin S$, for all i. If possible, let $y_i \in S$ for some $i \in \{1, 2, \ldots, n\}$. As $\langle S \rangle$ is connected, there exists a path joining x_1 and y_i in $\langle S \rangle$. However, that path will contain x_j and y_j as consecutive vertices for some j. Thus D_j contains two vertices in S, a contradiction. Thus $S = \{x_1, x_2, \ldots, x_n\}$. Similarly, $y_1 \in D_1 \cap S$ implies $S = \{y_1, y_2, \ldots, y_n\}$.
Case 2:[There exists at least one D_i with no element of S.]
Case 2A:[There exists more than one D_i's with no element of S.] We first note that if the number of columns not intersecting S is more than 2, then $\langle S \rangle$ is disconnected. Thus, let D_i and D_j be two columns which do not intersect S. As $\langle S \rangle$ is connected, D_i and D_j are consecutive columns, i.e., let the two columns be D_i and D_{i+1}. Then $D_{i-1} \subset S$ and $D_{i+2} \subset S$. Thus other

[2]Note that D_{i+1} has one vertex x_{i+1} in S, but it is also included in the list of $n-3$ columns as y_{i+1} may belong to S.

$n-4$ (provided $n > 4$) vertices of S appears in $n-4$ columns $D_1, D_2, \dots, D_{i-2}, D_{i+3}, \dots, D_n$. Since $\langle S \rangle$ is connected, each of these $n-4$ columns contains exactly on element of S. Moreover to maintain connectedness of $\langle S \rangle$, either all the x_i's or all the y_i's of these $n-4$ columns belong to S. Thus, S is of the form $\{y_{i+2}, x_{i+2}, x_{i+3}, \dots, x_n, x_1, x_2, \dots, x_{i-1}, y_{i-1}\}$ or of the form $\{x_{i+2}, y_{i+2}, y_{i+3}, \dots, y_n, y_1, y_2, \dots, y_{i-1}, x_{i-1}\}$.

However, if $n = 4$, the two forms of S given above are identical, i.e., $S = \{x_{i+2}, y_{i+2}, y_{i-1}, x_{i-1}\}$.

Case 2B:[There exists exactly one D_i with no element of S.] Let $D_i \cap S = \emptyset$. Thus, to dominate x_i, y_i, exactly one of the following cases should occur.

Case 2B(i):[$x_{i-1}, y_{i-1} \in S$.] In this case, the other $n-2$ vertices of S appears in the $n-2$ columns $\{D_1, D_2, \dots, D_{i-2}, D_{i+1}, \dots, D_n\}$. Moreover, as D_i is the only column that does not intersect S, each of the $n-2$ columns contains exactly one element from S. Let $x_1 \in S$. Then to preserve connectedness of $\langle S \rangle$, $S = \{y_{i-1}, x_{i-1}, x_{i-2}, \dots, x_1, x_n, \dots, x_{i+1}\}$. Similarly, if $y_1 \in S$, then $S = \{x_{i-1}, y_{i-1}, y_{i-2}, \dots, y_1, y_n, \dots, y_{i+1}\}$.

Case 2B(ii):[$x_{i+1}, y_{i+1} \in S$.] Similar to that of Case-2B(i). In this case, either $S = \{y_{i+1}, x_{i+1}, x_{i+2}, \dots, x_n, x_1, \dots, x_{i-1}\}$ or $S = \{x_{i+1}, y_{i+1}, y_{i+2}, \dots, y_n, y_1, \dots, y_{i-1}\}$.

Case 2B(iii):[$x_{i-1}, y_{i+1} \in S$.] Similarly, in this case, $\exists j \in \{1, 2, \dots, i-2, i+2, \dots, n\}$ such that $S = \{y_{i+1}, y_{i+2}, \dots, y_j, x_j, \dots, x_{i-2}, x_{i-1}\}$.

Case 2B(iv):[$x_{i+1}, y_{i-1} \in S$.] Similarly, in this case, $\exists j \in \{1, 2, \dots, i-2, i+2, \dots, n\}$ such that $S = \{x_{i+1}, x_{i+2}, \dots, x_j, y_j, \dots y_{i-2}, y_{i-1}\}$.

While classifying the γ_c-sets, we see that there are mainly three types of γ_c-sets of $P_2 \square C_n$:

- The types given by Case-1: $S = \{x_1, x_2, \dots, x_n\}$ and $S = \{y_1, y_2, \dots, y_n\}$. Thus total number of γ_c-sets of this type is 2.
- The types given by Case-2A: S's which do not contain vertices from two consecutive columns D_i and D_{i+1}. As the number of ways in which we can drop two consecutive columns is n, the total number of γ_c-sets of this type is equal to $2n$, if $n > 4$ and is equal to 4, if $n = 4$.
- The types given by Case-2B: In Case-2B(i), we have two choices for S for each i. Thus Case-2B(i) contribute $2n$ many γ_c-sets. Similarly, Case-2B(ii) contribute $2n$ many γ_c-sets. In Case-2B(iii), we have n choices for i and $n-3$ choices for j. Thus Case-2B(iii) contribute $n(n-3)$ many γ_c-sets. Similarly, Case-2B(ii) contribute $n(n-3)$ many γ_c-sets.

Thus the total number of distinct γ_c-sets of $P_2 \square C_n$ is $2(n^2+1)$, i.e., $\tau_c = 2(n^2+1)$, if $n > 4$. If $n = 4$, then $\tau_c = 30$. Now, as $P_2 \square C_n$ is vertex transitive, $CDV(u) = CDV(v)$ for all $u, v \in P_2 \square C_n$. Hence, by continuous analogue of Proposition 2.1, we have $2n \cdot CDV(v) = 2n(n^2+1)$, i.e., $CDV(v) = n^2 + 1$ for $n > 4$. For $n = 4$, by Proposition 2.1, we have $8 \cdot CDV(v) = 4 \cdot 30$, i.e., $CDV(v) = 15$.

Hence, the theorem follows. □

Acknowledgement

The author is grateful to the anonymous referee for providing fruitful suggestions and pointing out a few mistakes in the original manuscript. The funding of DST-SERB-SRG Sanction no. $SRG/2019/000475$, Govt. of India is acknowledged.

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