# The consecutively super edge-magic deficiency of graphs and related concepts

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## Abstract

A bipartite graph $G$ with partite sets $X$ and $Y$ is called consecutively super edge-magic if there exists a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ with the property that $f(X) = \{1, 2, \ldots, |X|\}$, $f(Y) = \{|X| + 1, |X| + 2, \ldots, |V(G)|\}$ and $f(u) + f(v) + f(uv)$ is constant for each $uv \in E(G)$. The question studied in this paper is for which bipartite graphs it is possible to add a finite number of isolated vertices so that the resulting graph is consecutively super edge-magic. If it is possible for a bipartite graph $G$, then we say that the minimum such number of isolated vertices is the consecutively super edge-magic deficiency of $G$; otherwise, we define it to be $+\infty$. This paper also includes a detailed discussion of other concepts that are closely related to the consecutively super edge-magic deficiency.

## 1. Introduction

We generally follow the notation and terminology pertaining to graphs of [5]. All graphs considered here are finite, simple and undirected. We will denote the set of vertices and edges of a graph $G$ by $V(G)$ and $E(G)$, respectively. For two graphs $G_1$ and $G_2$ with disjoint vertex sets,

the *union* $G \cong G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If a graph $G$ consists of $m$ disjoint copies of a graph $H$, then we write $G \cong mH$. For integers $a$ and $b$ with $a \leq b$, we will denote the set $\{x \in \mathbb{Z} : a \leq x \leq b\}$ by writing $[a, b]$, where $\mathbb{Z}$ denotes the set of all integers.

In a seminal paper published in 1970, Kotzig and Rosa [21] introduced the notion of edge-magic labelings. These labelings were originally called magic valuations by them. These were rediscovered in 1996 by Ringel and Lladó [28] who coined one of the now popular terms for them: edge-magic labelings. More recently, they have also been referred to as edge-magic total labelings by Wallis [30]. For a graph $G$, a bijective function $f : V(G) \cup E(G) \rightarrow [1, |V(G)| + |E(G)|]$ is called an *edge-magic labeling* if $f(u) + f(v) + f(uv)$ is a constant (called the *valence*) for each $uv \in E(G)$. If such a labeling exists, then $G$ is called an *edge-magic graph*. In 1998, Enomoto et al. [6] defined a slightly restricted version of an edge-magic labeling $f$ of a graph $G$ by requiring that $f(V(G)) = [1, |V(G)|]$. Such a labeling was called by them a *super edge-magic labeling*. Thus, a *super edge-magic graph* is a graph that admits a super edge-magic labeling.

It is worth mentioning that Acharya and Hegde [1] had introduced the concept of strongly indexable graph. It turns out that the concepts of strongly indexable graph and super edge-magic graph are equivalent. Lately, super edge-magic labelings and super edge-magic graphs have been called by Wallis [30] strong edge-magic total labelings and strongly edge-magic graphs, respectively. Moreover, according to the latest version of the survey on graph labelings by Gallian [13] available to the authors, Hegde and Shetty [15] showed that the properties of being super edge-magic and strongly $k$-indexable (see [13] for the definition) are equivalent.

In 2001, Muntaner-Batle [24] introduced the concept of special super edge-magic labeling of a bipartite graph. Let $G$ be a bipartite graph with partite sets $X$ and $Y$. If $G$ has a super edge-magic labeling $f$ with the property that $f(X) = [1, |X|]$ and $f(Y) = [|X| + 1, |V(G)|]$, then $f$ is called a *special super edge-magic labeling*. Oshima [26] subsequently called such labelings *consecutively super edge-magic*. In this paper, we prefer to use the latter terminology to emphasize the property that a consecutively super edge-magic labeling uses consecutive integers in each partite set. We also refer a bipartite graph with a consecutively super edge-magic labeling as a *consecutively super edge-magic graph*.

The following lemma found in [24] is particularly useful in showing that dense bipartite graphs are not consecutively super edge-magic.

**Lemma 1.1.** *If $G$ is a consecutively super edge-magic graph, then*

$$|E(G)| \leq |V(G)| - 1.$$

The following lemma found in [24] provides us with a necessary and sufficient condition for a bipartite graph to be consecutively super edge-magic.

**Lemma 1.2.** *Let $G$ be a bipartite graph with partite sets $X$ and $Y$. Then $G$ is consecutively super edge-magic if and only if there exists a bijective function $f : V(G) \rightarrow [1, |V(G)|]$ such that $f(X) = [1, |X|]$, $f(Y) = [|X| + 1, |V(G)|]$ and the set*

$$\{f(u) + f(v) : uv \in E(G)\}$$

*consists of $|E(G)|$ consecutive integers.*

For every graph $G$, Kotzig and Rosa [21] proved that there exists an edge-magic graph $H$ such that $H \cong G \cup nK_1$ for some nonnegative integer $n$. This motivated them to define the edge-magic deficiency. The *edge-magic deficiency* $\mu(G)$ of a graph $G$ is the smallest nonnegative integer $n$ for which $G \cup nK_1$ is edge-magic. Inspired by Kotzig-Rosa notion, Figueroa-Centeno et al. [11] defined the concept of *super edge-magic deficiency* $\mu_s(G)$ of a graph $G$ to be either the smallest nonnegative integer $n$ with the property that $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such integer $n$. If $G$ is a graph with $\mu_s(G) = 0$, then $G$ is super edge-magic. Thus, the super edge-magic deficiency of a graph $G$ is a measure of how close $G$ is to being super edge-magic.

Kotzig and Rosa [21] found an upper bound for the edge-magic deficiency of a graph of order $p$, namely, $\mu(G) \le F_{p+2} - 2 - \binom{p+1}{2}$, where $F_p$ is the $p$-th term of the Fibonacci sequence. This implies that every graph has finite edge-magic deficiency. However, not all graphs have finite super edge-magic deficiency. In order to see this, it suffices to consider the following lemma taken from [11].

**Lemma 1.3.** *If $G$ is a graph such that the degrees of all vertices are even and $|E(G)| \equiv 2$ (mod 4), then $\mu_s(G) = +\infty$.*

We now provide the definition for a parameter introduced in [27], the consecutively super edge-magic deficiency, used to measure the closeness of a graph to be consecutively super edge-magic. The *consecutively super edge-magic deficiency* $\mu_c(G)$ of a graph $G$ is defined to be either the smallest nonnegative integer $n$ with the property that $G \cup nK_1$ is consecutively super edge-magic or $+\infty$ if there exists no such integer $n$.

As an immediate consequence of the above three definitions, we have the following relations taking place among three parameters.

**Lemma 1.4.** *For every graph $G$,*

$$\mu(G) \le \mu_s(G) \le \mu_c(G).$$

For a thorough study of graph labeling problems, see the survey by Gallian [13]. For more information on super edge-magic graphs and related topics, see the books by Bača and Miller [2], López and Muntaner-Batle [22] and Marr and Wallis [23].

We end this introduction by summarizing the work conducted in this paper. In Section 2, we introduced some additional concepts (called the alpha-number and strong alpha-number), and supply relations among these parameters and the consecutively super edge-magic deficiency. In Section 3, we show that if a graph $G$ admits a consecutively super edge-magic labeling and $m$ is odd, then $mG$ admits a consecutively super edge-magic labeling, and we provide an explicit formula for a consecutively super edge-magic labeling of $mG$. This result allows us to construct infinite families of consecutively super edge-magic graphs using a single graph as a seed. In addition, we present results concerning the consecutively super edge-magic deficiency of the union of graphs. In particular, we show that if $G_1$ and $G_2$ are two graphs with $\mu_c(G_1) \le \mu_c(G_2) < +\infty$, then $\mu_c(G_1 \cup G_2) < +\infty$. Moreover, we give some results on the alpha-number of the union of graphs. In Section 4, we compute the consecutively super edge-magic deficiency and (strong) alpha-number for some classes of forests. In Section 5, we provide a brief discussion of forests regarding to the parameters studied in this paper or in the past, and propose new conjectures on the consecutively super edge-magic deficiency of forests.

## 2. Relations Among Parameters

We start with some necessary definitions for presenting our results included in this section.

The graph labeling method that has received the most attention over the years was originated with a paper by Rosa [29] in 1967 who called them $\beta$-valuations. A few years later, Golomb [14] called these labelings graceful and this is the term that has been used since then. For a graph $G$, an injective function $f : V(G) \to [0, |E(G)|]$ is called a *graceful labeling* if each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting edge labels are distinct. Rosa [29] also introduced the concept of $\alpha$-valuations (a particular type of graceful labelings) as a tool for decomposing the complete graph into isomorphic subgraphs. A graceful labeling $f$ is called an $\alpha$-*valuation* if there exists an integer $\lambda$ so that $\min\{f(u), f(v)\} \leq \lambda < \max\{f(u), f(v)\}$ for each $uv \in E(G)$. Rosa [29] pointed out that a graph that admits an $\alpha$-valuation is necessarily bipartite and therefore cannot contain a cycle of odd length.

It has demonstrated in [18] that $\alpha$-valuations are sometimes useful for computing the super edge-magic deficiency of certain graphs. In particular, they proved that if $G$ is a graph without isolated vertices that has an $\alpha$-valuation, then $\mu_s(G) \leq |E(G)| - |V(G)| + 1$. This bound is sharp in the sense that there are infinitely many graphs $G$ for which $\mu_s(G) = |E(G)| - |V(G)| + 1$. Indeed, all complete bipartite graphs (see [7]) and some 2-regular bipartite graphs (see [10, 11, 18, 19]) attain the bound.

The *gracefulness* grac($G$) of a graph $G$ is the smallest positive integer $n$ for which there exists an injective function $f : V(G) \to [0, n]$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting set of edge labels consists of distinct integers. It is easy to see that grac($G$) $\leq 2^{p-1} - 1$ for every graph $G$ of order $p$. This implies that every graph has finite gracefulness. If $G$ is a graph of size $q$ with grac($G$) = $q$, then $G$ is graceful. Thus, the gracefulness of a graph $G$ is a measure of how close $G$ is to being graceful. This definition first appeared in a paper by Golomb [14]. For further knowledge on the gracefulness of graphs, the authors suggest that the reader consults the results in [3, 20].

The concept of gracefulness motivated the authors to define the beta-number and strong beta-number in [17]. The *beta-number* $\beta(G)$ of a graph $G$ is the smallest positive integer $n$ for which there exists an injective function $f : V(G) \to [0, n]$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting set of edge labels is $[c, c + |E(G)| - 1]$ for some positive integer $c$. The beta-number of $G$ is $+\infty$, otherwise. If $c = 1$, then the resulting beta-number is called the *strong beta-number* of $G$ and it is denoted by $\beta_s(G)$. It is clear that if $G$ is a graph with $\beta(G) = |E(G)|$, then $G$ is graceful. It is also true that if $G$ is a graph with $\beta_s(G) = |E(G)|$, then $G$ is graceful. Thus, the beta-number and strong beta-number of a graph $G$ are measures of how close $G$ is to being graceful.

We now provide the definitions for a new type of parameter (called the alpha-number) introduced in [27] and its restriction. They play an important role in the study of consecutively super edge-magic deficiency. The *alpha-number* $\alpha(G)$ of a graph $G$ is defined to be the smallest positive integer $n$ such that $\beta(G) = n$ with the additional property that there exists an integer $\lambda$ so that $\min\{f(u), f(v)\} \leq \lambda < \max\{f(u), f(v)\}$ for each $uv \in E(G)$. The alpha-number is defined to be $+\infty$, otherwise. If $c = 1$, then the resulting alpha-number is called the *strong alpha-number* of $G$ and it is denoted by $\alpha_s(G)$. These parameters can be regarded as measures of how close a graph

is to having an $\alpha$-valuation.

The four parameters described in this section make up a string of inequalities as the following lemma indicates.

**Lemma 2.1.** *For every graph $G$ of order $p$ and size $q$,*

$$\max\{p-1, q\} \leq \operatorname{grac}(G) \leq \beta(G) \leq \alpha(G) \leq \alpha_s(G).$$

With the above definitions in hand, we can now state the following result established in [27], which shows the connection between the alpha-number of a graph and its consecutively super edge-magic deficiency.

**Theorem 2.1.** *If $G$ is a graph of order $p$, then*

$$\alpha(G) = \mu_c(G) + p - 1.$$

The preceding theorem indicates that the problems of determining the alpha-number and the consecutively super edge-magic deficiency are in fact equivalent. It is also immediate from the same result that for any graph $G$, $\alpha(G) = +\infty$ if and only if $\mu_c(G) = +\infty$.

The following corollary gives us an inequality relating the strong alpha-number of a graph and its consecutively super edge-magic deficiency. This result is easily obtained by substituting $c = 1$ into the equation $c = x + n + p - q - s + 2$ given in the proof of Theorem 2.1 (see [27]).

**Corollary 2.1.** *Assume that a graph $G \cup nK_1$ has a consecutively super edge-magic labeling $f$ for which $\mu_c(G) = n$. If $G \cup nK_1$ has the partite sets $X$ and $Y$ such that $n = |E(G)| - |V(G)| - |X| + s - 1$, where $s = \min\{f(u) + f(v) : uv \in E(G)\}$, then*

$$\alpha_s(G) \leq \mu_c(G) + |V(G)| - 1.$$

It is worth mentioning that the bound presented in Corollary 2.1 is sharp (for instance, see Corollary 4.4 in Section 4).

If $G$ is a graph with an $\alpha$-valuation, then it is clear that $\alpha(G) = |E(G)|$. In this case, this fact together with Theorem 2.1 gives us an exact formula for the consecutively super edge-magic deficiency of such graphs in terms of its order and size.

**Corollary 2.2.** *If $G$ is a graph with an $\alpha$-valuation, then*

$$\mu_c(G) = |E(G)| - |V(G)| + 1.$$

The above corollary is relatively important, since various classes of graphs have been proved to admit $\alpha$-valuations (see [13] for a detailed list of results).

### 3. Results on Unions of Graphs

In their investigation of (super) edge-magic properties of disconnected graphs, Figueroa-Centeno et al. [11] proved that if $G$ is a (super) edge-magic bipartite or tripartite graph, then $mG$ is (super) edge-magic when $m$ is odd. In the following theorem, we present a consecutively super edge-magic analogue of the mentioned result.

**Theorem 3.1.** *If $G$ is a consecutively super edge-magic graph and $m$ is odd, then $mG$ is consecutively super edge-magic.*

*Proof.* Without loss of generality, assume that $m$ is odd and $m \geq 3$. Since $G$ is a consecutively super edge-magic graph, it follows that $G$ is bipartite. Thus, if we let $U$ and $V$ denote the two partite sets of $G$, then we have $E(G) = UV$, where the juxtaposition of two partite sets denotes the edges between those two sets. Take $f : V(G) \to [1, |V(G)|]$ to be an arbitrary consecutively super edge-magic labeling of $G$, and define $H \cong mG$ to be the graph with

$$V(H) = \bigcup_{i=1}^{m} (U_i \cup V_i) \text{ and } E(G) = \bigcup_{i=1}^{m} U_i V_i,$$

where $x_i \in X_i$ for each $i \in [1, m]$ if and only if $x \in X$ ($X$ is one of the sets $U$ or $V$).

Now, consider the vertex labeling $g : V(H) \to [1, m |V(G)|]$ such that

$$g(x_i) = \begin{cases} mf(x) - m + i & \text{if } x \in U \text{ and } i \in [1, m], \\ mf(x) - (i-1)/2 & \text{if } x \in V \text{ and } i \text{ is odd}, \\ mf(x) - (m - 1 + i)/2 & \text{if } x \in V \text{ and } i \text{ is even}. \end{cases}$$

Then $g$ extends to a consecutively super edge-magic labeling of $H$ with valence $ms - 3(m-1)/2 + m |V(G) \cup E(G)|$, where $s = \min \{f(u) + f(v) : uv \in E(G)\}$. To show this, notice first that

$$g(U) = [1, m|U|] \text{ and } g(V) = [m|U| + 1, m|V(G)|],$$

since

$$f(U) = [1, |U|] \text{ and } f(V) = [|U| + 1, |V(G)|].$$

Next, to see that $g(u) + g(v) + g(uv) = ms - 3(m-1)/2 + m |V(G) \cup E(G)|$ for every $uv \in E(H)$, where $u \in U$ and $v \in V$, notice that if we let $c = ms - 3(m-1)/2$, then we have

$$\{g(u) + g(v) : uv \in E(H)\} = [c, c + |E(H)| - 1].$$

Therefore, we conclude by means of Lemma 1.2 that $mG$ is consecutively super edge-magic when $m$ is odd and $m \geq 3$. $\square$

It is interesting to notice that Theorem 3.1 can be deduced from the results found in [12] with relative ease. However, the proof given in this paper provides us with an explicit formula for the consecutively super edge-magic labeling announced in the statement of Theorem 3.1.

The preceding theorem also yields the following corollary.

**Corollary 3.1.** *If $G$ is a bipartite graph and $m$ is odd, then $\mu_c(mG) \leq m\mu_c(G)$.*

*Proof.* Let $G$ be a bipartite graph and let $m$ be odd. Also, assume that $\mu_c(G) < +\infty$, otherwise the result is trivial. Then $G \cup nK_1$ is consecutively super edge-magic for some nonnegative integer $n$. Since $G \cup nK_1$ is certainly bipartite, it follows from Theorem 3.1 that $m(G \cup nK_1) \cong mG \cup mnK_1$ is consecutively super edge-magic, which implies that $\mu_c(mG) \leq m\mu_c(G)$. □

The following result is an immediate consequence of Theorem 2.1 and Corollary 3.1.

**Corollary 3.2.** *If $G$ is a bipartite graph and $m$ is odd, then*

$$\alpha(mG) \leq m\alpha(G) + m - 1.$$

For a nontrivial tree that has an $\alpha$-valuation, we have the following result.

**Corollary 3.3.** *If $T$ is a nontrivial tree of order $p$ that has an $\alpha$-valuation, then*

$$\alpha(mT) = mp - 1,$$

*where $m$ is odd.*

*Proof.* Let $T$ be a nontrivial tree of order $p$ that has an $\alpha$-valuation, and assume that $m$ is odd. Then $\alpha(T) = |E(T)| = p - 1$. This together with Corollary 3.2 implies that $\alpha(mT) \leq mp - 1$. On the other hand, the reverse inequality quickly follows from Lemma 2.1. □

The following result is easily obtained from Corollary 3.2 and Lemma 2.1.

**Corollary 3.4.** *If $F$ is a forest of order $p$ such that $\alpha(F) = p - 1$, then*

$$\alpha(mF) = mp - 1,$$

*where $m$ is odd.*

The union of two graphs with finite super edge-magic deficiency does not always have finite super edge-magic deficiency. In fact, the cycle $C_3$ of length $3$ is clearly super edge-magic, but we can see from Lemma 1.3 that $\mu_s(2C_3) = +\infty$. However, if two graphs are both consecutively super edge-magic, the situation is different as the next theorem indicates.

**Theorem 3.2.** *If $G_1$ and $G_2$ are consecutively super edge-magic graphs, then*

$$\mu_c(G_1 \cup G_2) < +\infty.$$

*Proof.* Assume that $G_1$ and $G_2$ are consecutively super edge-magic graphs. For each $i \in [1, 2]$, let $X_i$ and $Y_i$ be the partite sets of $G_i$, where $|X_i| = x_i$ and $|Y_i| = y_i$, and let $f_i$ be a consecutively super edge-magic labeling of $G_i$. Now, without loss of generality, assume that $f_i$ has the property that $x_i \geq y_i$ for each $i \in [1, 2]$. Otherwise, take a consecutively super edge-magic labeling $g_i$ given by

$$g_i(v) = x_i + y_i + 1 - f_i(v),$$

where $v \in X_i \cup Y_i$ and $i \in [1, 2]$. Moreover, assume that $f_1$ and $f_2$ have the property that $2x_1 + 2 \geq s_1$ and $s_2 + |E(G_2)| - 1 \geq x_2 + y_2 + 1$, respectively, where $s_i = \min\{f_i(u) + f_i(v) : uv \in E(G_i)\}$ and $i \in [1, 2]$. Otherwise, take a consecutively super edge-magic labeling $g_i$ given by

$$g_i(v) = x_i + 1 - f_i(v) \text{ if } v \in X_i$$

and

$$g_i(v) = 2x_i + y_i + 1 - f_i(v) \text{ if } v \in Y_i$$

for each $i \in [1, 2]$.

With the above properties of $f_1$ and $f_2$ in hand, consider the function $f$ such that

$$f(v) = \begin{cases} f_1(v) & \text{if } v \in X_1, \\ f_1(v) + s_2 - s_1 + 2x_1 + |E(G_2)| & \text{if } v \in Y_1, \\ f_2(v) + x_1 & \text{if } v \in X_2 \cup Y_2. \end{cases}$$

Then $f$ extends to a consecutively super edge-magic labeling of $G_1 \cup G_2 \cup nK_1$ for some nonnegative integer $n$. To show this, notice first that

$$f_i(X_i) = \{j : j \in [1, x_i]\} \text{ and } f_i(Y_i) = \{j : j \in [x_i + 1, x_i + y_i]\}$$

for each $i \in [1, 2]$. This implies that

$$
\begin{aligned}
\{f(v) : v \in X_1\} &= [1, x_1], \\
\{f(v) : v \in X_2\} &= [x_1 + 1, x_1 + x_2], \\
\{f(v) : v \in Y_2\} &= [x_1 + x_2 + 1, x_1 + x_2 + y_2], \\
\{f(v) : v \in Y_1\} &= [s_2 - s_1 + 3x_1 + |E(G_2)| + 1, s_2 - s_1 + 3x_1 + |E(G_2)| + y_1].
\end{aligned}
$$

Next, to see that $f$ is an injective function, notice that

$$s_2 - s_1 + 3x_1 + |E(G_2)| + 1 \geq x_1 + x_2 + y_2 + 1.$$

This is true as we indicate next, since $f_1$ and $f_2$ have the aforementioned properties.

$$
\begin{aligned}
& s_2 - s_1 + 3x_1 + |E(G_2)| + 1 - (x_1 + x_2 + y_2 + 1) \\
=\ & s_2 - s_1 + 2x_1 + |E(G_2)| - x_2 - y_2 \\
=\ & (s_2 + |E(G_2)| - 1) + 1 - x_2 - y_2 - s_1 + 2x_1 \\
\geq\ & (x_2 + y_2 + 1) + 1 - x_2 - y_2 - s_1 + 2x_1 \\
=\ & 2x_1 + 2 - s_1 \\
\geq\ & s_1 - s_1 = 0.
\end{aligned}
$$

Finally, notice that the set

$$\{f(u) + f(v) : uv \in E(G_1 \cup G_2)\}$$

consists of $|E(G_1 \cup G_2)|$ consecutive integers, since

$$
\begin{aligned}
\{f_1(u) + f_1(v) : uv \in E(G_1)\} &= [s_1, s_1 + |E(G_1)| - 1], \\
\{f_2(u) + f_2(v) : uv \in E(G_2)\} &= [s_2, s_2 + |E(G_2)| - 1]
\end{aligned}
$$

and

$$
f(u) + f(v) = \begin{cases} f_1(u) + f_1(v) + s_2 - s_1 + 2x_1 + |E(G_2)| & \text{if } uv \in E(G_1), \\ f_2(u) + f_2(v) + 2x_1 & \text{if } uv \in E(G_2). \end{cases}
$$

$\square$

As an immediate consequence of Theorems 2.1 and 3.2, we have the following result.

**Corollary 3.5.** *If $G_1$ and $G_2$ are two graphs such that $\alpha(G_1) = |V(G_1)| - 1$ and $\alpha(G_2) = |V(G_2)| - 1$, then $\alpha(G_1 \cup G_2) < +\infty$.*

Theorem 3.2 yields the following corollary as well.

**Corollary 3.6.** *If $G_1$ and $G_2$ are two graphs such that $\mu_c(G_1) \leq \mu_c(G_2) < +\infty$, then*

$$
\mu_c(G_1 \cup G_2) < +\infty.
$$

*Proof.* Assume that $G_1$ and $G_2$ are two graphs such that $\mu_c(G_1) \leq \mu_c(G_2) < +\infty$. Then there exists some nonnegative integers $m$ and $n$ such that $G_1 \cup mK_1$ and $G_2 \cup nK_1$ are consecutively super edge-magic. It follows from Theorem 3.2 that $\mu_c(G_1 \cup G_2 \cup (m+n)K_1) < +\infty$, which implies that $\mu_c(G_1 \cup G_2) < +\infty$. $\square$

Combining Theorem 2.1 with Corollary 3.6, we obtain the following result.

**Corollary 3.7.** *If $G_1$ and $G_2$ are two graphs such that $\alpha(G_1) \leq \alpha(G_2) < +\infty$, then*

$$
\alpha(G_1 \cup G_2) < +\infty.
$$

## 4. Results on Forests

In their 2006 paper, Figueroa-Centeno et al. [11] provided a constructive proof that nontrivial trees and forests have finite super edge-magic deficiencies. In fact, they have verified those facts by showing the stronger statement that nontrivial trees and forests have finite consecutively super edge-magic deficiencies. In light of Theorem 2.1, this implies that nontrivial trees and forests have finite alpha-numbers. In a series of papers [8, 9, 11], Figueroa-Centeno et al. studied the super edge-magic properties of forests. Afterwards, they conjectured in [10] that $\mu_s(F) \leq 1$ for every forest $F$ with two components. In this section, we present some consecutively super edge-magic deficiency and (strong) alpha-number analogues of the super edge-magic deficiency results on forests found in [10, 11].

**Theorem 4.1.** *For every positive integer $m$,*

$$\mu_c(mP_2) = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ 1 & \text{if } m \text{ is even.} \end{cases}$$

*Proof.* It is straightforward that $P_2$ is a consecutively super edge-magic graph. It follows from Theorem 3.1 that the forest $mP_2$ is consecutively super edge-magic when $m$ is odd. This implies that $\mu_c(mP_2) = 0$ when $m$ is odd.

Now, recall that Kotzig and Rosa [21] proved that the forest $mP_2$ is super edge-magic if and only if $m$ is odd. This implies that $\mu_s(mP_2) \geq 1$ when $m$ is even. This together with Lemma 1.4 implies that $\mu_c(mP_2) \geq 1$ when $m$ is even. To establish the reverse inequality, define the forest $F \cong mP_2 \cup K_1$ with

$$V(F) = \{x_i : i \in [1, m]\} \cup \{y_i : i \in [1, m]\} \cup \{z\} \text{ and } E(F) = \{x_i y_i : i \in [1, m]\},$$

and consider the vertex labeling $f : V(F) \to [1, 2m+1]$ such that

$$f(w) = \begin{cases} i & \text{if } w = x_i \text{ and } i \in [1, m], \\ 3m/2 + 1 + i & \text{if } w = y_i \text{ and } i \in [1, m/2], \\ m/2 + i & \text{if } w = y_i \text{ and } i \in [m/2 + 1, m], \\ 3m/2 + 1 & \text{if } w = z. \end{cases}$$

It follows from Lemma 1.2 that $f$ extends to a consecutively super edge-magic labeling of $F$ with valence $9m/2 + 3$. To show this, notice first that the two partite sets of the vertex labels in $F$ are $\{f(x_i) : i \in [1, m]\} = [1, m]$ and

$$\{f(y_i) : i \in [1, m]\} \cup \{z\} = [m+1, 2m+1].$$

Next, to see that $f(u) + f(v) + f(uv) = 9m/2 + 3$ for all $uv \in E(F)$, notice that

$$\{f(u) + f(v) : uv \in E(F)\} = [3m/2 + 2, 5m/2 + 1]$$

or, equivalently,

$$\{f(u) + f(v) : uv \in E(F)\} = [s, s + |E(F)| - 1],$$

where $s = 3m/2 + 2$. Consequently, we conclude that $\mu_c(F) \leq 1$ when $m$ is even, completing the proof. $\square$

For every positive integer $m$, it is known from [16] that $\beta(mP_2) = 2m$ when $m \equiv 2 \pmod 4$ and $\beta(mP_2) = 2m - 1$ in all other cases. By letting $G \cong mP_2$ in Theorem 2.1, we now obtain the following result from Theorem 4.1.

**Corollary 4.1.** *For every positive integer $m$,*

$$\alpha(mP_2) = \begin{cases} 2m - 1 & \text{if } m \text{ is odd,} \\ 2m & \text{if } m \text{ is even.} \end{cases}$$

We next present a formula for the strong alpha-number of the forest $mP_2$.

**Theorem 4.2.** *For every positive integer $m$,*

$$\alpha_s (mP_2) = \begin{cases} 2m - 1 & \text{if } m = 1, \\ +\infty & \text{if } m \neq 1. \end{cases}$$

*Proof.* The result is trivial for $m = 1$. Thus, assume that $m \geq 2$ and suppose, to the contrary, that $\alpha_s (mP_2) = n$ for some positive integer $n$. Then there exists an injective function $f : V (mP_2) \to [0, n]$ such that $\{|f (x) - f (y)| : xy \in E (mP_2)\} = [1, m]$and there exists an integer $\lambda$ so that $\min \{f (x), f (y)\} \leq \lambda < \max \{f (x), f (y)\}$ for each $xy \in E (mP_2)$. It follows that such a $\lambda$ must be the smaller of the two vertex labels that yield the edge labeled 1. Let $k_1$ be the second largest vertex label in the smaller of the two partite sets of $mP_2$. On the other hand, let $k_2$ be the second smallest vertex label in the larger of the two partite sets of $mP_2$. Then we have $k_1 \leq \lambda - 1$ and $k_2 \geq \lambda + 2$. This implies that $k_2 - k_1 \geq 3$, that is, the second smallest edge label is at least 3. Consequently, $2 \notin \{|f (x) - f (y)| : xy \in E (mP_2)\}$, which contradicts the fact that $2 \in \{|f (x) - f (y)| : xy \in E (mP_2)\}$. Therefore, we conclude that $\alpha_s (mP_2) = +\infty$ for all $m \geq 2$. □

The preceding theorem indicates that extending the range of vertex labels of a forest does not need to produce a finite strong alpha-number. This is unexpected in light of the aforementioned fact that every forest has a finite alpha-number (see beginning of this section).

We next present a result on forests that consist of two disjoint copies of a star. For this purpose, let $S_m$ denote the star with $m + 1$ vertices.

**Theorem 4.3.** *For every two positive integers $m$ and $n$, $\mu_c (S_m \cup S_n) = 1$.*

*Proof.* First, we show that $\mu_c (S_m \cup S_n) \leq 1$ for every two positive integers $m$ and $n$. To do this, define the forest $F_1 \cong S_m \cup S_n \cup K_1$ with

$$V (F_1) = \{x, y, z\} \cup \{x_i : i \in [1, m]\} \cup \{y_i : i \in [1, n]\}$$

and

$$E (F_1) = \{xx_i : i \in [1, m]\} \cup \{yy_i : i \in [1, n]\},$$

and consider the vertex labeling $f : V (F_1) \to [1, m + n + 3]$ such that

$$f (w) = \begin{cases} 2 & \text{if } w = x, \\ 1 & \text{if } w = y, \\ 2 + i & \text{if } w = x_i \text{ and } i \in [1, m], \\ m + 3 & \text{if } w = z, \\ m + 3 + i & \text{if } w = y_i \text{ and } i \in [1, n]. \end{cases}$$

It follows from Lemma 1.2 that $f$ extends to a consecutively super edge-magic labeling of $F_1$ with valence $m + n + 4$. To show this, notice first that the two partite sets of the vertex labels in $F_1$ are $\{f (x), f (y)\} = [1, 2]$ and

$$\{f (x_i) : i \in [1, m]\} \cup \{f (y_i) : i \in [1, m]\} \cup \{f (z)\} = [3, m + n + 3].$$

Next, to see that $f(u) + f(v) + f(uv) = m + n + 4$ for all $uv \in E(F_1)$, notice that

$$\{f(u) + f(v) : uv \in E(F_1)\} = [5, m + n + 4]$$

or, equivalently,

$$\{f(u) + f(v) : uv \in E(F_1)\} = [5, 5 + |E(F_1)| - 1].$$

This indicates that $\mu_c(F_1) \leq 1$ for every two positive integers $m$ and $n$.

To verify the inequality in the other direction, let $F_2 \cong S_m \cup S_n$ be the forest with

$$V(F_2) = \{x, y\} \cup \{x_i : i \in [1, m]\} \cup \{y_i : i \in [1, n]\}$$

and

$$E(F_2) = \{xx_i : i \in [1, m]\} \cup \{yy_i : i \in [1, n]\},$$

and suppose, to the contrary, that $F_2$ has a consecutively super edge-magic labeling $f$. Then there are two cases to concern with the possibilities of partite sets of $F_2$.

**Case 1.** The sets $A = \{x_i : i \in [1, m]\} \cup \{y\}$ and $B = \{x\} \cup \{y_i : i \in [1, n]\}$ form the partite sets of $F_2$.

**Case 2.** The sets $A = \{x, y\}$ and $B = \{x_i : i \in [1, m]\} \cup \{y_i : i \in [1, n]\}$ form the partite sets of $F_2$.

According to the vertex labels of each partite set of $F_2$, each of Cases 1 and 2 divides into two subcases as indicate next.

**Subcase 1-1.** The set of vertex labels of each partite set of $F_2$ is

$$f(A) = [1, m + 1] \text{ and } f(B) = [m + 2, m + n + 2].$$

**Subcase 1-2.** The set of vertex labels of each partite set of $F_2$ is

$$f(A) = [n + 2, m + n + 2] \text{ and } f(B) = [1, n + 1].$$

**Subcase 2-1.** The set of vertex labels of each partite set of $F_2$ is

$$f(A) = [1, 2] \text{ and } f(B) = [3, m + n + 2].$$

**Subcase 2-2.** The set of vertex labels of each partite set of $F_2$ is

$$f(A) = [m + n + 1, m + n + 2] \text{ and } f(B) = [1, m + n].$$

Furthermore, according to the induced edge labels of $F_2$, each of Subcases 1-1 and 1-2 in turn divides into two subcases as indicate next.

**Subcase 1-1-1.** The set of induced edge labels of $F_2$ is

$$\{f(u) + f(v) : uv \in E(F_2)\} = [m + 3, 2m + n + 2].$$

**Subcase 1-1-2.** The set of induced edge labels of $F$ is

$$\{f(u) + f(v) : uv \in E(F_2)\} = [m + 4, 2m + n + 3].$$

**Subcase 1-2-1.** The set of induced edge labels of $F_2$ is

$$\{f(u) + f(v) : uv \in E(F_2)\} = [n + 4, m + 2n + 3].$$

**Subcase 1-2-2.** The set of induced edge labels of $F_2$ is

$$\{f(u) + f(v) : uv \in E(F_2)\} = [n + 3, m + 2n + 2].$$

Similarly, according to the induced edge labels of $F_2$, each of Subcases 2-1 and 2-2 in turn divides into two subcases as indicate next.

**Subcase 2-1-1.** The set of induced edge labels of $F_2$ is

$$\{f(u) + f(v) : uv \in E(F_2)\} = [4, m + n + 3].$$

**Subcase 2-1-2.** The set of induced edge labels of $F_2$ is

$$\{f(u) + f(v) : uv \in E(F_2)\} = [5, m + n + 4].$$

**Subcase 2-2-1.** The set of induced edge labels of $F_2$ is

$$\{f(u) + f(v) : uv \in E(F_2)\} = [m + n + 3, 2m + 2n + 2].$$

**Subcase 2-2-2.** The set of induced edge labels of $F_2$ is

$$\{f(u) + f(v) : uv \in E(F_2)\} = [m + n + 2, 2m + 2n + 1].$$

Subcases 1-1-1, 1-1-2, 1-2-1 and 1-2-2 hold the following claim.

**Claim 1.** If one of Subcases 1-1-1, 1-1-2, 1-2-1 or 1-2-2 is satisfied, then all of these subcases are satisfied.

Fist of all, we show that if Subcase 1-1-1 is true, then so is Subcase 1-1-2. To see this, define the vertex labeling $g$ such that

$$g(v) = \begin{cases} m + 2 - f(v) & \text{if } v \in A, \\ 2m + n + 4 - f(v) & \text{if } v \in B. \end{cases}$$

Then $g$ extends to a consecutively super edge-magic labeling of $F_2$ such that

$$g(A) = [1, m + 1] \text{ and } g(B) = [m + 2, m + n + 2],$$

producing that

$$\{g(u) + g(v) : uv \in E(F_2)\} = [m + 4, 2m + n + 3].$$

Next, to see that Subcase 1-1-1 implies Subcase 1-2-1, define the vertex labeling $g$ such that

$$g(v) = m + n + 3 - f(v)$$

for all $v \in A \cup B$. Then $g$ extends to a consecutively super edge-magic labeling of $F_2$ such that

$$g(A) = [n + 2, m + n + 2] \text{ and } g(B) = [1, n + 1],$$

producing that

$$\{g(u) + g(v) : uv \in E(F_2)\} = [n+4, m+2n+3].$$

Finally, to see that Subcase 1-1-1 implies Subcase 1-2-2, define the vertex labeling $g$ such that

$$g(v) = \begin{cases} f(v) + n + 1 & \text{if } v \in A, \\ f(v) - m - 1 & \text{if } v \in B. \end{cases}$$

Then $g$ extends to a consecutively super edge-magic labeling of $F_2$ such that

$$g(A) = [n+2, m+n+2] \text{ and } g(B) = [1, n+1],$$

producing that

$$\{g(u) + g(v) : uv \in E(F_2)\} = [n+3, m+2n+2].$$

Conversely, if one of Subcases 1-1-2, 1-2-1 or 1-2-2 is satisfied, then Subcase 1-1-1 is satisfied by the aforementioned vertex labelings $g$. This completes the proof of Claim 1.

Subcases 2-1-1, 2-1-2, 2-2-1 and 2-2-2 hold the following claim.

**Claim 2.** If one of Subcases 2-1-1, 2-1-2, 2-2-1 or 2-2-2 is satisfied, then all of these subcases are satisfied.

First of all, we show that if Subcase 2-1-1 is true, then so is Subcase 2-1-2. To see this, define the vertex labeling $g$ such that

$$g(v) = \begin{cases} 3 - f(v) & \text{if } v \in A, \\ m+n+5 - f(v) & \text{if } v \in B. \end{cases}$$

Then $g$ extends to a consecutively super edge-magic labeling of $F_2$ such that

$$g(A) = [1, 2] \text{ and } g(B) = [3, m+n+2],$$

producing that

$$\{g(u) + g(v) : uv \in E(F_2)\} = [5, m+n+4].$$

Next, to see that Subcase 2-1-1 implies Subcase 2-2-1, define the vertex labeling $g$ such that

$$g(v) = m + n + 3 - f(v)$$

for all $v \in A \cup B$. Then $g$ extends to a consecutively super edge-magic labeling of $F_2$ such that

$$g(A) = [m+n+1, m+n+2] \text{ and } g(B) = [1, m+n],$$

producing that

$$\{g(u) + g(v) : uv \in E(F_2)\} = [m+n+3, 2m+2n+2].$$

Finally, to see that Subcase 2-1-1 implies Subcase 2-2-2, define the vertex labeling $g$ such that

$$g(v) = \begin{cases} f(v) + m + n & \text{if } v \in A, \\ f(v) - 2 & \text{if } v \in B. \end{cases}$$

Then $g$ extends to a consecutively super edge-magic labeling of $F_2$ such that

$$g(A) = [m + n + 1, m + n + 2] \text{ and } g(B) = [1, m + n],$$

producing that

$$\{g(u) + g(v) : uv \in E(F_2)\} = [m + n + 2, 2m + 2n + 1].$$

Conversely, if one of Subcases 2-1-2, 2-2-1 or 2-2-2 is satisfied, then Subcase 2-1-1 is satisfied by the aforementioned vertex labelings $g$. This completes the proof of Claim 2.

The logically equivalent contrapositive of Claim 2 states that if one of Subcases 1-1-1, 1-1-2, 1-2-1 or 1-2-2 is not satisfied, then all of these subcases are not satisfied. Thus, we particularly show the following claim.

**Claim 3.** Subcase 1-1-1 is not satisfied.

First, assume that $f(y) = 1$. Then $f(x) \neq m + 2$, and the sum of the induced edge labels of $F_2$ is

$$\sum_{uv \in E(F_2)} (f(u) + f(v)) = \sum_{i=m+3}^{2m+n+2} i = \frac{(m+n)(3m+n+5)}{2}.$$

Since $\deg v = 1$ for all $v \in F_2 - \{x, y\}$, the sum of the vertex labels of $F_2$ is

$$\sum_{v \in V(F_2)} f(v) = \sum_{i=1}^{m+n+2} i = \frac{(m+n+2)(m+n+3)}{2}.$$

Our assumption together with the fact that

$$\sum_{uv \in E(F_2)} (f(u) + f(v)) = \sum_{v \in V(F_2)} f(v) + (m-1)f(x) + (n-1)f(y)$$

implies that $m^2 + mn - n - 2 = (m-1)f(x)$. Let $h(m) = m^2 + mn - n - 2$. To satisfy the last equation with the property that $f(x)$ is a positive integer, it is necessary that $h(1) = 0$; however, $h(1) = -1$. Thus, Claim 3 is true when $f(y) = 1$.

Next, assume that $f(x) = m + 2$. Then $f(y) \neq 1$, and our assumption and the fact that

$$\sum_{uv \in E(F_2)} (f(u) + f(v)) = \sum_{v \in V(F_2)} f(v) + (m-1)f(x) + (n-1)f(y)$$

implies that $mn - m - 1 = (n-1)f(y)$. Let $h'(n) = mn - m - 1$. To satisfy the last equation with the property that $f(y)$ is a positive integer, it is necessary that $h'(1) = 0$; however, $h'(1) = -1$. Thus, Claim 3 is true when $f(x) = m + 2$. Hence, Claim 3 is established by the preceding arguments.

The logically equivalent contrapositive of Claim 2 states that if one of Subcases 2-1-1, 2-1-2, 2-2-1 or 2-2-2 is not satisfied, then all of these subcases are not satisfied. Thus, we particularly show the following claim.

**Claim 4.** Subcase 2-1-1 is not satisfied.

If $f(x) = 1$ and $f(y) = 2$, then the sum of the induced edge labels of $F_2$ is

$$\sum_{uv \in E(F_2)} (f(u) + f(v)) = \sum_{i=4}^{m+n+3} i = \frac{(m+n)(m+n+7)}{2}.$$

Since $\deg v = 1$ for all $v \in V(F_2) - \{x, y\}$, the sum of the vertex labels of $F_2$ is

$$\sum_{v \in V(F_2)} f(v) = \sum_{i=1}^{m+n+2} i = \frac{(m+n+2)(m+n+3)}{2}.$$

The equation $m + n - 3 = (m-1)f(x) + (n-1)f(y)$ is also obtained from the fact that

$$\sum_{uv \in E(F_2)} (f(u) + f(v)) = \sum_{v \in V(F_2)} f(v) + (m-1)f(x) + (n-1)f(y).$$

However, it follows from our assumption that

$$(m-1)f(x) + (n-1)f(y) = m + 2n - 3.$$

This leads us to conclude that $m + n - 3 = m + 2n - 3$ so that $n = 0$, which is impossible. Thus, Claim 4 is true when $f(x) = 1$ and $f(y) = 2$. It remains to consider the case that $f(x) = 2$ and $f(y) = 1$. However, the argument for this case is entirely analogous to that of the preceding. This completes the proof of Claim 4.

Therefore, it follows from all the claims that $\mu_c(F_2) \geq 1$ for every two positive integers $m$ and $n$, proving the result. $\qquad\square$

The following corollary is an immediate consequence of Theorems 2.1 and 4.3.

**Corollary 4.2.** *For every two positive integers $m$ and $n$,*

$$\alpha(S_m \cup S_n) = m + n + 2.$$

In light of the preceding result, it is now natural to explore the strong alpha-number of the forest $S_m \cup S_n$, which is contained in the following theorem.

**Theorem 4.4.** *For every two positive integers $m$ and $n$, $\alpha_s(S_m \cup S_n) = +\infty$.*

*Proof.* Define the forest $F \cong S_m \cup S_n$ as in the proof of the preceding theorem. Now, assume, to the contrary, that $\alpha_s(F) = k$ for some positive integer $k$ when $m$ and $n$ are positive integers. Then there exists an injective function $f : V(F) \to [0, k]$ such that

$$\{|f(x) - f(y)| : xy \in E(F)\} = [1, m+n]$$

and there exists an integer $\lambda$ so that $\min\{f(x), f(y)\} \leq \lambda < \max\{f(x), f(y)\}$ for every $xy \in E(F)$. This means that one vertex in some pairs of adjacent vertices is labeled $\lambda$ and the

other vertex in the pair is labeled $\lambda + 1$. Also, the integers $0$, $\lambda$, $\lambda + 1$ are necessary in the elements of the set $\{f(v) : v \in V(F)\}$. From these observations, we distinguish two cases.

**Case 1.** Let $F$ have partite sets $U$ and $V$, where

$$U = \{x, y\} \text{ and } V = \{x_i : i \in [1, m]\} \cup \{y_i : i \in [1, n]\},$$

and, without loss of generality, assume that $f(x) = 0$ and $f(y) = \lambda$. It is true that if $\lambda + 1 \in \{f(x_i) : i \in [1, m]\}$, then $|f(x) - f(x_i)| \geq \lambda + 1 \geq 2$; so $1 \notin \{|f(x) - f(y)| : xy \in E(F)\}$. Thus, $\lambda + 1 \in \{f(y_i) : i \in [1, n]\}$, implying that $1 \in \{|f(y) - f(y_i)| : i \in [1, n]\}$. It is also true that if $\lambda + 2 \in \{f(x_i) : i \in [1, m]\}$, then $|f(x) - f(x_i)| \geq \lambda + 2 \geq 3$. It follows that $2 \notin \{|f(x) - f(y)| : xy \in E(F)\}$. Thus, $\lambda + 2 \in \{f(y_i) : i \in [1, n]\}$, implying that $2 \in \{|f(y) - f(y_i)| : i \in [1, n]\}$. Continuing in this manner, we see that $l \in \{|f(y) - f(y_i)| : i \in [1, n]\}$ for any $l \in [1, n]$. Consequently, we obtain

$$\{f(y_i) : i \in [1, n]\} = [\lambda + 1, \lambda + n]$$

and

$$\{|f(y) - f(y_i)| : i \in [1, n]\} = [1, n],$$

which implies that $f(x_i) \geq \lambda + n + 1$ for any $i \in [1, m]$. This together with our assumption that $f(x) = 0$ implies that $|f(x) - f(x_i)| \geq \lambda + n + 1$ for any $i \in [1, m]$. However, $\lambda \geq 1$; so $|f(x) - f(x_i)| \geq n + 2$ for any $i \in [1, m]$. Hence, $n + 1 \notin \{|f(x) - f(y)| : xy \in E(F)\}$, which is impossible.

**Case 2.** Let $F$ have partite sets $U$ and $V$, where

$$U = \{x\} \cup \{y_i : i \in [1, n]\} \text{ and } V = \{x_i : i \in [1, m]\} \cup \{y\}.$$

Then there are two subcases to pursue.

**Subcase 2-1.** Let $f(x) = \lambda$. Then $f(y_i) \leq \lambda - 1$ ( $i \in [1, n]$ ). It is true that if $f(y) = \lambda + 1$, then $|f(x) - f(x_i)| \geq 2$ ( $i \in [1, m]$ ) and $|f(y) - f(y_i)| \geq 2$ ( $i \in [1, n]$ ). It follows that $1 \notin \{|f(x) - f(y)| : xy \in E(F)\}$. Thus, $\lambda + 1 \in \{f(x_i) : i \in [1, m]\}$, implying that $1 \in \{|f(x) - f(x_i)| : i \in [1, m]\}$. It is also true that if $f(y) = \lambda + 2$, then $|f(x) - f(x_i)| \geq 3$ ( $i \in [1, m]$ ) and $|f(y) - f(y_i)| \geq 3$ ( $i \in [1, n]$ ). It follows that $2 \notin \{|f(x) - f(y)| : xy \in E(F)\}$. Thus, $\lambda + 2 \in \{f(x_i) : i \in [1, m]\}$, implying that $2 \in \{|f(x) - f(x_i)| : i \in [1, m]\}$. Continuing in this manner, we see that $l \in \{|f(x) - f(x_i)| : i \in [1, m]\}$ ( $l \in [1, m]$ ). Consequently, we obtain

$$\{f(x_i) : i \in [1, m]\} = [\lambda + 1, \lambda + m]$$

and

$$\{|f(x) - f(x_i)| : i \in [1, m]\} = [1, m],$$

which implies that $f(y) \geq \lambda + m + 1$. This together with the fact that $f(y_i) \leq \lambda - 1$ ( $i \in [1, n]$ ) implies that $|f(y) - f(y_i)| \geq m + 2$ ( $i \in [1, n]$ ). Hence, $m + 1 \notin \{|f(x) - f(y)| : xy \in E(F)\}$, which is impossible.

**Subcase 2-2.** Let $f(x) = \lambda - l$, where $l \in [1, \lambda]$. Then $\lambda \in \{f(y_i) : i \in [1, n]\}$. It is true that if $\lambda + 1 \in \{f(x_i) : i \in [1, m]\}$, then $|f(x) - f(x_i)| \geq l + 1 \geq 2$ ( $i \in [1, m]$ ) and $|f(y) - f(y_i)| \geq 2$ ( $i \in [1, n]$ ). It follows that $1 \notin \{|f(x) - f(y)| : xy \in E(F)\}$. Thus, $f(y) = \lambda + 1$.

Now, consider the function $g : V(F) \to [1, \alpha_s(F)]$ such that $g(v) = \alpha_s(F) - f(v)$ for all $v \in V(F)$. Then $g(u) \geq \alpha_s(F) - \lambda$ for all $u \in U$ and $g(v) \leq \alpha_s(F) - \lambda - 1$ for all $v \in V$. In particular, we have $g(y) = \alpha_s(F) - \lambda - 1$. It remains only to observe that an analogous argument to the preceding subcase can be applied to obtain a contradiction. $\square$

Before presenting our next result, we mention the consecutively super edge-magic property of paths obtained by Cattell [4]. For this purpose, let $P_n$ denote the path with $n$ vertices.

**Theorem 4.5.** *For an end-vertex $v$ of the path $P_n$, there exists a consecutively super edge-magic labeling $f$ of $P_n$ such that $f(v) = i$ for any $i \in [1, n]$ if and only if $n = 4s + 1$ and $f(v) \notin \{s + 1, 3s + 1\}$.*

As the concept of induced subgraph will appear in the proof of our next result, we define this concept for the sake of clarity. If $S$ is a nonempty subset of the vertex set $V(G)$ of a graph $G$, then the subgraph $\langle S \rangle$ of $G$ *induced* by $S$ is the graph having vertex set $S$ and whose edge set consists of those edges of $G$ incident with two elements of $S$. A subgraph $H$ of $G$ is called *induced* if $H \cong \langle S \rangle$ for some subset $S$ of $V(G)$.

We are now prepared to present the following theorem.

**Theorem 4.6.** *For all positive integers $m$ and $n$,*

$$\mu_c(P_m \cup S_n) = \begin{cases} 1 & \text{if } m = 2 \text{ or } 3, \\ 0 & \text{if } m \neq 2 \text{ or } 3. \end{cases} .$$

*Proof.* Let $F \cong P_m \cup S_n$ be the forest with

$$V(F) = \{x_i : i \in [1, m]\} \cup \{y\} \cup \{y_i : i \in [1, n]\}$$

and

$$E(F) = \{x_i x_{i+1} : i \in [1, m-1]\} \cup \{yy_i : i \in [1, n]\}.$$

For $m = 1$, label the vertices of degree $1$ of $F$ with the first $n$ positive integers, the isolated vertex of $F$ with $n + 1$ and the remaining vertex with $n + 2$. This gives a consecutively super edge-magic labeling of $F$ when $m = 1$. The result also follows from Theorem 4.3 when $m = 2$ or $3$. Thus, assume that $m$ is an integer with $m \geq 4$ and $m \neq 6$, and define the induced subgraphs of $F$ as follows. Let $F_1 \cong P_{m-1}$ and $F_2 \cong S_n$ be the induced subgraphs of $F$ with

$$V(F_1) = \{x_i : i \in [1, m]\} \text{ and } V(F_2) = \{y\} \cup \{y_i : i \in [1, n]\}.$$

For $m \geq 4$ and $m \neq 6$, it follows from Theorem 4.5 that there exists a consecutively super edge-magic labeling $f_1$ of $F_1$ with $f_1(x_2) = 2$. On the other hand, there exists a consecutively super edge-magic labeling $f_2$ of $F_2$ such that $f_2(y) = n + 1$ and $f_2(y_i) = i$ for each $i \in [1, n]$. With these knowledge in hand, consider the vertex labeling $f : V(F) \to [1, m + n + 1]$ such that

$$f(v) = \begin{cases} \lfloor m/2 \rfloor + n + 1 & \text{if } v = x_1, \\ n + 2 + f_1(v) & \text{if } v = x_{2i-1} \text{ and } i \in [2, \lceil m/2 \rceil], \\ n + f_1(v) & \text{if } v = x_{2i} \text{ and } i \in [1, \lfloor m/2 \rfloor], \\ \lfloor m/2 \rfloor + 1 + f_2(v) & \text{if } v = y, \\ f_2(v) & \text{if } v = y_i \text{ and } i \in [1, n]. \end{cases}$$

Then $f$ extends to a consecutively super edge-magic labeling of $F$ with valence $\lfloor m/2 \rfloor + 2m + 3n + 3$. In order to show this, notice first that if we let

$$U = \{x_{2i} : i \in [1, \lfloor m/2 \rfloor]\} \cup \{y_i : i \in [1, n]\}$$

and

$$V = \{x_{2i-1} : i \in [2, \lceil m/2 \rceil]\} \cup \{y\},$$

then $U$ and $V$ form the partite sets of $F$, and

$$f(U) = [1, \lfloor m/2 \rfloor + n] \text{ and } f(V) = [\lfloor m/2 \rfloor + n + 1, m + n + 1].$$

Next, to see that $f(u) + f(v) + f(uv) = \lfloor m/2 \rfloor + 2m + 3n + 3$ for all $uv \in E(F)$, where $u \in U$ and $v \in V$, notice that

$$\{f(u) + f(v) : uv \in E(F)\} = [\lfloor m/2 \rfloor + n + 3, \lfloor m/2 \rfloor + m + 2n + 1]$$

or, equivalently,

$$\{f(u) + f(v) : uv \in E(F)\} = [s, s + |E(F)| - 1],$$

where $s = \lfloor m/2 \rfloor + n + 3$. It remains to notice that if $m = 6$, then the vertex labeling $g : V(F) \to [1, n + 7]$ such that $(g(x_i))_{i=1}^{6} = (1, n + 5, 4, n + 7, 3, n + 4)$, $g(y) = 2$, $g(y_1) = n + 6$ and $g(y_i) = i + 3$ $(i \in [2, n])$ induces a consecutively super edge-magic labeling of $F$ with valence $2n + 19$.

Therefore, we conclude by means of Lemma 1.2 that $F$ is consecutively super edge-magic for all positive integers $m$ and $n$ with $m \geq 4$, which completes the proof. $\qquad\square$

Applying Theorems 2.1 and 4.6, we obtain the following result.

**Corollary 4.3.** *For all positive integers $m$ and $n$,*

$$\alpha(P_m \cup S_n) = \begin{cases} m + n + 1 & \text{if } m = 2 \text{ or } 3, \\ m + n & \text{if } m \neq 2 \text{ or } 3. \end{cases}$$

Let $F \cong P_m \cup S_n$, and consider the next cases. For the cases that $m = 2$ or 3, it follows by Theorem 4.4 that $\alpha_s(F) = +\infty$. For the cases that $m \neq 2$ or 3, the consecutively super edge-magic labeling of $F$ found in the proof of Theorem 4.6 satisfies the hypothesis of Corollary 2.1. This gives us the upper bound for $\alpha_s(F)$, whereas the lower bound is obtained from Lemma 2.1 and Corollary 4.3 in this case. Therefore, we have the following result.

**Corollary 4.4.** *For every two positive integers $m$ and $n$,*

$$\alpha_s(P_m \cup S_n) = \begin{cases} +\infty & \text{if } m = 2 \text{ or } 3, \\ m + n & \text{if } m \neq 2 \text{ or } 3. \end{cases}$$

## 5. Conclusions

We conclude this paper with some remarks on the conjectures stated in [10, 16] and three new conjectures.

The authors conjectured in [16] that if $F$ is a forest of order $p$, then $\operatorname{grac}(F)$ is either $p - 1$ or $p$. It follows from Lemma 2.1 that all the results on alpha-numbers of forests obtained in the last section validate this conjecture. Moreover, as we mentioned in the beginning of the last section, $\mu_c(F)$ is finite for any forest $F$. Thus, we suspect the following conjecture to be true.

**Conjecture 1.** *If $F$ is a forest, then $\mu_c(F) \leq 1$.*

From the work found in the literature, the authors suspect that Conjecture 1 may be very hard to solve. Hence, we also propose the following two conjectures that are weaker versions of Conjecture 1. To solve any of these two conjectures would be of great interest in the subject of graph labelings.

**Conjecture 2.** *If $F$ is a forest, then there exists a nonnegative integer $k$ such that $\mu_c(F) \leq k$.*

Analogous to the conjecture proposed by Figueroa-Centeno et al. [10] that $\mu_s(F) \leq 1$ for every forest $F$ with two components, we now propose the following conjecture.

**Conjecture 3.** *If $F$ is a forest with two components, then $\mu_c(F) \leq 1$.*

The authors conjectured in [16] that $\beta(F) \leq p$ and $\beta_s(F) \leq p$ for every forest $F$ of order $p$. Of course, if Conjecture 1 is true, then it follows from Theorem 2.1 that $\alpha(F) \leq p$ for every forest $F$ of order $p$. Indeed, it follows from Lemma 2.1 that if Conjecture 1 is true, so are the aforementioned conjectures on beta-number and strong beta-number. Moreover, by Lemma 1.4, the truth of Conjecture 3 implies the truth of the aformentioned conjecture by Figueroa-Centeno et al. [10].

For the reader interested in different research directions in the same topic, the authors would like to recommend the paper by Ngurah and Simanjuntak [25].

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