Electronic Journal of Graph Theory and Applications **8** (1) (2020), 123–132

Electronic Journal of
Graph Theory and Applications

Congruences and subdirect representations of graphs

Stefan Veldsman

School of Engineering and Mathematical Science,
La Trobe University, Melbourne, Australia, and

Department of Mathematics, Nelson Mandela University,
Port Elizabeth, South Africa.

veldsman@outlook.com

Abstract

A basic tool in universal algebra is that of a congruence. It has been shown that congruences can be defined for graphs with properties similar to their universal algebraic counterparts. In particular, a subdirect product of graphs and hence also a subdirectly irreducible graph, can be expressed in terms of graph congruences. Here the subdirectly irreducible graphs are determined explicitly. Using congruences, a graph theoretic version of the well-known Birkhoff Theorem from universal algebra is given. This shows that any non-trivial graph is a subdirect product of subdirectly irreducible graphs.

Keywords: congruence on a graph, quotient graph, subdirect product of graphs, subdirectly irreducible graph, Birkhoff's theorem
Mathematics Subject Classification : 05C25, 05C76, 08A30
DOI: 10.5614/ejgta.2020.8.1.9

1. Introduction

Recently, a congruence on a graph was defined by Broere, Heidema and Pretorius [2]. The graphs they consider, do not allow loops. Since there is a direct relationship between a congruence on a graph and a homomorphic image of the graph, the absence of loops makes the theory of

Received: 18 April 2019, Revised: 11 June 2019, Accepted: 23 December 2019.

congruences on such graphs rather restrictive. In [3] this restriction was removed and a theory for congruences on graphs which do admit loops was developed. It was shown that the three classical isomorphism theorems from universal algebra each has a graph theoretical version. Moreover, as for algebras, a subdirect product of graphs can be expressed in terms of congruences. As a first application of these new tools available for graph theory, it was also shown in [3] that a Hoehnke radical can be defined for graphs as was done in universal algebras, and that it determines the connectednesses and disconnectednesses of graphs as defined and developed earlier by Fried and Wiegandt [5]. The connectednesses and disconnectednesses theory of graphs is the graph theoretic version of the classical radical theory of algebraic structures. For the latter, one may consult Gardner and Wiegandt [6] for a comprehensive overview of the radical theory of associative rings.

Here we look at another application of graph congruences culminating in a graph theoretic version of the well-known Birkhoff Theorem in universal algebra [1]: every non-trivial graph is a subdirect product of subdirectly irreducible graphs. The notion of a congruence on a graph is still relatively new and we will start by recalling the definition and basic properties of graph congruences from [3] in the next section. Subdirectly irreducible objects in any category can be thought of as the "primes" (they cannot be decomposed as certain products) and it is desirable to express any object in terms of such irreducible objects. In the third section, we explicitly determine the subdirectly irreducible graphs; such a graph must be one of four namely a two-vertex graph (three possibilities) or a three-vertex graph (only one possibility). It is then proven that any graph with at least two vertices is a subdirect product of such subdirectly irreducible graphs. In [3] it was shown that a subdirect product of graphs can be expressed in terms of congruences and congruences are then also the main tool used to establish the results.

The results presented here are for the category of graphs which admit loops (i.e., a vertex of a graph may have a loop). A version of Birkhoff's Theorem for the category of graphs which do not admit loops was claimed by Fawcett [4] and later proved by Sabidussi [8]. Sabidussi also defined and used the notion of a congruence of a graph in his work (as the kernel of a homomorphism), but he did not develop a theory of congruences for graphs. The category of graphs which admit loops has a much richer homomorphism structure and consequently the subdirectly irreducible graphs considered here are different to those considered by Sabidussi in the category of graphs which do not admit loops.

Let us first fix the terminology and notation required. A graph G with vertex set V and edge set E will typically be denoted by $G = (V, E)$; when we are dealing with different graphs, we may use the notation V_G for V and, similarly, E_G for E. When we write $a \in G$, it actually means a is a vertex of G, i.e., $a \in V_G$. By a graph, we mean a non-empty vertex set, edges are not directed, no multiple edges are allowed but loops are. For $a, b \in G$, an edge between a and b will be denoted by ab and aa denotes a loop at a. For a non-empty set D, we use K_D to denote the complete graph with vertex set D and $\mathcal{C}_D$ to denote the set of all possible edges and loops on the set D, i.e., $\mathcal{C}_D = \{ab \mid a, b \in D\}$. If D has cardinality n, we sometimes write K_n and $\mathcal{C}_n$ respectively. A *(graph) homomorphism* is an edge preserving mapping from the vertex set of a graph into the vertex set of a graph. A *strong homomorphism* is a homomorphism that sends "no edges" to "no edges" and if it is also a bijection, it is called an *isomorphism*. Isomorphic graphs G and H will be denoted by $G \cong H$. For a graph $G = (V_G, E_G)$, a *subgraph* $H = (V_H, E_H)$ of G is a graph with $V_H \subseteq V_G$

and $E_H \subseteq E_G$. When $E_H = \{ab \mid a, b \in V_H \text{ and } ab \in E_G\}$, then H is called a *strong subgraph* (or *induced subgraph*) of G. For a homomorphism $f : G \to H$, the *image graph* $f(G)$ will always be the induced subgraph of H on the vertex set $f(V_G)$. In general, unless mentioned otherwise, if a subset V_H of V_G is regarded as a graph, it will be the subgraph induced by G on V_H. There are two (non-isomorphic) one-vertex graphs, called the *trivial graphs*; the one with a loop T_0 and the one without a loop T. If a graph is not trivial (i.e. with at least two vertices), then it is called *non-trivial*. The six non-isomorphic two-vertex graphs will be denoted by $B_i, i = 1, 2, 3, ..., 6$ where $V_{B_i} = \{0, 1\}$ and $E_{B_1} = \emptyset$ (empty set), $E_{B_2} = \{01\}, E_{B_3} = \{00\}, E_{B_4} = \{00, 11\}, E_{B_5} = \{01, 11\}$ and $E_{B_6} = \{00, 01, 11\}$. We will also need one three-vertex graph A_3 with $V_{A_3} = \{0, 1, 2\}$ and $E_{A_3} = \{00, 11, 22, 01, 21\}$. For an equivalence relation $\sim$ on a vertex set V, we use $[a]$ to denote the equivalence class of $a \in V$. If necessary, it may be written with a subscript as $[a]_\sim$. We use $[a][b]$ to denote the set $[a][b] = \{st \mid s \in [a], t \in [b]\} \subseteq \mathcal{C}_V$.

2. Congruences on graphs

For completeness, in this section we recall the definition and basic properties of graph congruences from [3]:

Definition 1. *Let $G = (V_G, E_G)$ be a graph. A congruence on G is a pair $\theta = (\sim, \mathcal{E})$ where:*
(i) $\sim$ is an equivalence relation on V_G;
(ii) $\mathcal{E}$ is the congruence edge-set with $E_G \subseteq \mathcal{E} \subseteq \mathcal{C}_G$; and
(iii) (Substitution Property of $\mathcal{E}$ with respect to $\sim$) for $x, y \in V_G$, $xy \in \mathcal{E}$ implies $[x][y] \subseteq \mathcal{E}$.
A strong congruence on G is a pair $\theta = (\sim, \mathcal{E})$ where $\sim$ is an equivalence relation on V_G and $\mathcal{E} = \mathcal{E}(\sim) := \{xy \mid x, y \in V_G \text{ with } [x][y] \cap E_G \neq \emptyset\}$.

A strong congruence is also a congruence. Congruences can be partially ordered by the relation "contained in": For two congruences $\alpha = (\sim_\alpha, \mathcal{E}_\alpha)$ and $\beta = (\sim_\beta, \mathcal{E}_\beta)$ on G, α is *contained in* β, written as $\alpha \subseteq \beta$, if $\sim_\alpha \subseteq \sim_\beta$ and $\mathcal{E}_\alpha \subseteq \mathcal{E}_\beta$. Let $\bumpeq$ denote the identity relation (diagonal) on V_G (i.e., $x \bumpeq y$ if and only if $x = y$). The congruence $\iota_G := (\bumpeq, E_G)$ on G is called the *identity congruence* on G. It is a strong congruence and is the smallest congruence on G. The *universal congruence* on G is the pair $\upsilon_G = (\leftrightsquigarrow, \mathcal{C}_G)$ where $\leftrightsquigarrow$ is the universal relation (i.e., $a \leftrightsquigarrow b$ for all $a, b \in V_G$). Any congruence on G is contained in υ_G. These two congruences, the identity and the universal congruence, are sometimes referred to as the trivial congruences on a graph. For any congruence $\theta = (\sim, \mathcal{E})$ on a graph G, it is always the case that $\mathcal{E}(\sim) \subseteq \mathcal{E}$ which means that $\mathcal{E}(\sim)$ is the smallest congruence edge-set on V_G for which $(\sim, \mathcal{E}(\sim))$ is a congruence on G. Also note that for any graph $G = (V_G, E_G)$, $\mathcal{E}(\bumpeq) = E_G$. The next example is the prototype of all graph congruences.

The kernel of a homomorphism: Given any graph homomorphism $f : G \longrightarrow H$, the *kernel of* f, written as $\ker f = (\sim_f, \mathcal{E}_f)$, is defined by $\sim_f = \{(x, y) \mid x, y \in V_G, f(x) = f(y)\}$ and $\mathcal{E}_f = \{uv \mid u, v \in V_G, f(u)f(v) \in E_H\}$. This is a congruence on G. With f is also associated the *strong kernel of* f, written as $sker\ f = (\sim_f, \mathcal{E}_{sf})$ with the same equivalence relation but $\mathcal{E}_{sf} = \mathcal{E}(\sim_f)$. This is a strong congruence on G and *sker* $f \subseteq \ker f$; in fact, if $\theta = (\sim_f, \mathcal{E})$ is any congruence on G for some $\mathcal{E}$, then *sker* $f \subseteq \theta$. If f is a strong homomorphism, then $\ker f =$ *sker* f. Injectivity of a homomorphism is equivalent to the equivalence relation $\sim_f$ coinciding with $\bumpeq$.

Moreover, the kernel of f is the identity congruence on G if and only if f is an injective strong homomorphism. If f is a surjective strong homomorphism, then it is an isomorphism if and only if the kernel of f is the identity congruence.

Quotients: Given any congruence $\theta = (\sim, \mathcal{E})$ on a graph $G = (V_G, E_G)$, a new graph, denoted by $G/\theta = (V_{G/\theta}, E_{G/\theta})$ and called *the quotient of G modulo θ*, is defined by taking $V_{G/\theta} := \{[x] \mid x \in V_G\}$ and $E_{G/\theta} := \{[x][y] \mid xy \in \mathcal{E}\}$. In this context, the natural (= canonical) mapping $p_\theta : G \to G/\theta$ given by $p_\theta(x) = [x]$ is a surjective homomorphism with $\ker p_\theta = \theta$. In particular, for $\theta = \iota_G$ we have G/ι_G isomorphic to G. If θ is a strong congruence, then p_θ is a strong homomorphism with *sker* $p_\theta = \theta$. In general, if we take $\theta = (\eqcirc, \mathcal{E})$ where $\mathcal{E}$ is any set with $E_G \subseteq \mathcal{E} \subseteq \mathcal{C}_G$, then θ is a congruence on G and G/θ is the graph with vertex set $V_{G/\theta} = V_G$ and edge set $E_{G/\theta} = \mathcal{E}$ (here we identify $[x] = \{x\}$ with x). If υ_G is the universal congruence on G, then G/υ_G is isomorphic to the trivial graph T_0 (one vertex with a loop).

The lattice of congruences: For a given graph G, the set of all congruences on G is denoted by $\mathcal{C}(G)$. This set $\mathcal{C}(G)$ is a partially ordered set with respect to containment $\subseteq$ as defined above. We can say more. Any collection of congruences $\{\theta_i = (\sim_i, \mathcal{E}_i) \mid i \in I\} \subseteq \mathcal{C}(G)$ has a greatest lower bound in $\mathcal{C}(G)$ given by $\bigcap\limits_{i\in I} \theta_i = (\sim, \mathcal{E})$ where $a \sim b \Leftrightarrow a \sim_i b$ for all $i \in I$ and $ab \in \mathcal{E} \Leftrightarrow ab \in \mathcal{E}_i$ for all $i \in I$. Moreover, $\{\theta_i = (\sim_i, \mathcal{E}_i) \mid i \in I\}$ also has a least upper bound in $\mathcal{C}(G)$ given by the congruence $(\sim, \mathcal{E})$ where $a \sim b \Leftrightarrow$ there are $i_1, i_2, \ldots, i_n \in I$ and $a_{i_1}, a_{i_2}, ..., a_{i_n} \in V, n \geq 2$, such that $a = a_{i_1} \sim_{i_1} a_{i_2} \sim_{i_2} a_{i_3} \sim_{i_3} \cdots \sim_{i_{n-2}} a_{i_{n-1}} \sim_{i_{n-1}} a_{i_n} = b$ and $\mathcal{E} : = \mathcal{E}(\sim) = \{ab \mid a, b \in V$ and there is an $i \in I$ and $a'b' \in \mathcal{E}_i$ with $a' \sim a$ and $b' \sim b\}$. It can easily be verified that $(\sim, \mathcal{E})$ is the least upper bound in $\mathcal{C}(G)$ for $\{\theta_i = (\sim_i, \mathcal{E}_i) \mid i \in I\}$. We write this least upper bound as $\bigcup\limits_{i\in I} \theta_i = (\sim, \mathcal{E})$.

Isomorphism theorems for congruences:

For two graphs G and H, let $f : G \to H$ be a graph homomorphism with $\theta = (\sim, \mathcal{E})$ a congruence on G and $\alpha = (\sim_\alpha, \mathcal{E}_\alpha)$ a congruence on H. By $f(\theta)$ we mean the pair $(f(\sim), f(\mathcal{E}))$ where $f(\sim) = \{(f(a), f(b)) \mid a, b \in V_G, a \sim b\}$ and $f(\mathcal{E}) = \{f(a)f(b) \mid ab \in \mathcal{E}\}$. This need not be a congruence on H, but nevertheless it will be compared to α in the usual sense, meaning $f(\theta) \subseteq \alpha$ if and only if $f(\sim) \subseteq \sim_\alpha$ and $f(\mathcal{E}) \subseteq \mathcal{E}_\alpha$. We start with two auxiliary results and remind the reader that all the results from this section are from [3] which also contains the proofs.

Proposition 2.1. *Let $f : G \to H$ be a homomorphism. Then $f(\ker f) \subseteq \iota_H$ and if $\rho = (\sim_\rho, \mathcal{E}_\rho)$ is a congruence on G with $f(\rho) \subseteq \iota_H$, then $\rho \subseteq \ker f$.*

Proposition 2.2. *Let $f : G \to H$ and $g : G \to K$ be surjective homomorphisms. Then $\ker f = (\sim_f, \mathcal{E}_f) \subseteq \ker g = (\sim_g, \mathcal{E}_g)$ if and only if there is a homomorphism $h : H \to K$ such that $h \circ f = g$.*

This brings us to:

Theorem 2.1. *(First Isomorphism Theorem) Let $f : G \to H$ be a homomorphism. Then $G/\ker f$ is isomorphic to $f(G)$ where $f(G)$ is the induced subgraph of H on $f(V_G)$. If f is surjective, then $G/\ker f$ is isomorphic to H. Moreover, if f is a surjective strong homomorphism, then $G/skerf$ is isomorphic to H.*

Let G be a graph with induced subgraph H. Then a congruence $\theta = (\sim, \mathcal{E})$ on G induces a congruence $H \cap \theta = (\sim_H, \mathcal{E}_H)$ on H with $\sim_H = (V_H \times V_H) \cap \sim = \{(a,b) \mid a,b \in V_H \text{ and } a \sim b\}$ and $\mathcal{E}_H = \{ab \mid a,b \in V_H\} \cap \mathcal{E} = \{ab \mid a,b \in V_H \text{ with } ab \in \mathcal{E}\}$. The mapping $f : H \to G/\theta$ defined by $f(a) = [a]$ for all $a \in V_H$ is a homomorphism with $\ker f = H \cap \theta$. Now $f(V_H)$ is a set of vertices of G/θ on which we form the induced subgraph of G/θ, denoted by $(H+\theta)/\theta$. Then, by the First Isomorphism Theorem, we have:

Theorem 2.2. *(Second Isomorphism Theorem) Let H be an induced subgraph of a graph G. Let θ be a congruence on G. Then $H \cap \theta$ is a congruence on H and $H/H \cap \theta$ is isomorphic to $(H+\theta)/\theta$ where the latter graph is the induced subgraph of G/θ on the vertex set $\{[a] \mid a \in V_H\}$.*

Theorem 2.3. *(Third Isomorphism Theorem) Let G be a graph and let $\theta_1 = (\sim_1, \mathcal{E}_1)$ and $\theta_2 = (\sim_2, \mathcal{E}_2)$ be two congruences on G with $\theta_1 \subseteq \theta_2$. Then $\theta_2/\theta_1 := (\sim, \mathcal{E})$ is a congruence on G/θ_1 where the equivalence is given by $[a]_1 \sim [b]_1 \Leftrightarrow a \sim_2 b$ and the congruence edge-set is given by $[a]_1[b]_1 \in \mathcal{E} \Leftrightarrow ab \in \mathcal{E}_2$. Moreover, the quotient graph $(G/\theta_1)/(\theta_2/\theta_1)$ is isomorphic to the quotient graph G/θ_2.*

This theorem gives the expected connection between the congruences on a graph and the congruences on an associated quotient graph.

Theorem 2.4. *Let G be a graph with θ a fixed congruence on G. Any congruence ξ of the graph G/θ is of the form α/θ for some congruence α on G with $\theta \subseteq \alpha$. Moreover, there is a one-to-one correspondence between $\{\alpha \mid \alpha$ is a congruence on G with $\theta \subseteq \alpha\}$ and $\mathcal{C}(G/\theta)$ which preserves inclusions, intersections and unions of congruences.*

Products and subdirect products of graphs:

For an index set I, let $G_i = (V_i, E_i)$ be a graph for all $i \in I$. The product $\prod_{i\in I} G_i$ of the graphs G_i is the graph $\prod_{i\in I} G_i := (\prod_{i\in I} V_i, E)$ where $\prod_{i\in I} V_i$ is just the usual Cartesian product of the sets V_i and $E = \{fg \mid f,g \in \prod_{i\in I} V_i \text{ with } f(i)g(i) \in E_i \text{ for all } i \in I\}$. For every $j \in I$, the j-th *projection* $\pi_j : \prod_{i\in I} G_i \to G_j$ defined by $\pi_j(f) = f(j)$ for all $f \in \prod_{i\in I} V_i$ is a surjective homomorphism. An induced subgraph H of $\prod_{i\in I} G_i$ is called a *subdirect product of the graphs $G_i, i \in I$*, provided the restriction of each projection π_j to H is a surjective mapping onto G_j. As in universal algebra, subdirect products can be expressed in terms of congruences and quotients:

Theorem 2.5. *For each $i \in I$, let θ_i be a congruence on a graph G with $\theta := \bigcap_{i\in I} \theta_i$. Then G/θ is isomorphic to a subdirect product of the quotient graphs $G/\theta_i, i \in I$.*

In particular, it then follows that:

Corollary 2.1. *A graph G is a subdirect product of graphs $G_i, i \in I$, if and only if for every $i \in I$ there are congruences θ_i on G with G_i isomorphic to G/θ_i and $\bigcap_{i\in I} \theta_i = \iota_G$.*

3. Subdirectly irreducible graphs.

A graph G is called *subdirectly irreducible* if the following condition is fulfilled: whenever G is a subdirect product of graphs $G_i, i \in I$ for some index set I, then $G \cong G_j$ for at least one $j \in I$. In terms of congruences, this means that G is subdirectly irreducible if and only if whenever θ_i is a congruence on G for all i from some some index set I with $\cap\theta_i = \iota_G$, then $\theta_j = \iota_G$ for some $j \in I$. If G is not subdirectly irreducible, it is called *subdirectly reducible*. In layman's terms, this means that a subdirectly reducible graph sits tightly in a product of graphs, none of which coincides with G.
The two-vertex graph B_6 has only the two trivial congruences ι_{B_6} and υ_{B_6} and is subdirectly irreducible; B_5 and B_4 each has three congruences, the two trivial congruences as well as $(\Bumpeq, \{00, 01, 11\})$ and both are subdirectly irreducible. The graphs B_3 and B_2 each has five congruences and B_1 has nine. These three graphs are subdirectly reducible: B_1 is a subdirect product of B_4 and two copies of B_5 using the congruences $\gamma_1 = (\Bumpeq, E_1 = \{00, 11\}), \gamma_2 = (\Bumpeq, E_2 = \{00, 01\})$ and $\gamma_3 = (\Bumpeq, E_3 = \{01, 11\})$; B_2 is a subdirect product of two copies of B_5 (using the congruences γ_2 and γ_3) and B_3 is a subdirect product of B_4 and B_5 (using γ_1 and γ_3). For ease of reference, we record this here as our first result.

Proposition 3.1. *The two-vertex graphs B_4, B_5 and B_6 are subdirectly irreducible. The two-vertex graphs B_1, B_2 and B_3 are subdirectly reducible being subdirect products of copies of B_4 and B_5.*

We also need:

Proposition 3.2. *Suppose the graph G is a subdirect product of the graphs $G_i, i \in I$, and for some $i_0 \in I$, G_{i_0} is a subdirect product of graphs H_j for $j \in J$ (take $I \cap J = \emptyset$). Then G is a subdirect product of the graphs K_r for $r \in R := J \cup (I - \{i_0\})$ where*

$$K_r = \begin{cases} G_r \text{ if } r \in I - \{i_0\} \\ H_r \text{ if } r \in J \end{cases}.$$

Proof. By Corollary 2.9 there are congruences θ_i on G with G_i isomorphic to G/θ_i and $\bigcap_{i\in I} \theta_i = \iota_G$ and congruences η_j on G_{i_0} with H_j isomorphic to G_{i_0}/η_j and $\bigcap_{j\in J} \eta_j = \iota_{G_{i_0}}$. By Theorem 2.7, there are congruence α_j on G with $\theta_{i_0} \subseteq \alpha_j$ such that $\bigcap_{j\in J} \alpha_j = \theta_{i_0}$ and $\eta_j = \alpha_j/\theta_{i_0}$. Then $\left(\bigcap_{i\in I-\{i_0\}} \theta_i\right) \cap \left(\bigcap_{j\in J} \alpha_j\right) = \left(\bigcap_{i\in I-\{i_0\}} \theta_i\right) \cap \theta_{i_0} = \bigcap_{i\in I} \theta_i = \iota_G$. The result follows from Theorem 2.6 since $G/\alpha_j \cong (G/\theta_{i_0})/(\alpha_j/\theta_{i_0}) \cong G_{i_0}/\eta_j \cong H_j$ for all $j \in J$. $\square$

We now determine the subdirectly irreducible graphs in a series of intermediate results. For this we often use the following congruences: For a graph $G = (V_G, E_G)$, let $a \in G$ and let $\rho_a := (\sim_a, \mathcal{E}_a)$ be the congruence on G where $\sim_a$ is the equivalence on V_G with two equivalence classes $[a] = \{a\}$ and $[b] = V_G - \{a\}$ for $b \neq a$, and $\mathcal{E}_a = \mathcal{E}(\sim_a)$. If G has more than three vertices, then $\rho_a \neq \iota_G$. Clearly $aa \in \bigcap_{b\in G} \mathcal{E}_b \Leftrightarrow aa \in E_G \Leftrightarrow aa \in \mathcal{E}_a$. Note also that $\bigcap_{a\in G} \rho_a = (\Bumpeq, \mathcal{E})$ where $E_G \subseteq \mathcal{E} = \bigcap_{a\in G} \mathcal{E}_a$.

Proposition 3.3. *Let G be a non-trivial graph with $E_G = \emptyset$. Then G is a subdirect product of $|G|$-copies of the two-vertex graph B_1 and hence also a subdirect product of the subdirectly irreducible graphs B_4 and B_5.*

Proof. For every $a \in G$, the congruence ρ_a defined above is given here by $\rho_a = (\sim_a, E_G)$ and $G/\rho_a \equiv B_1$. Moreover, $\bigcap\limits_{a \in G} \rho_a = \iota_G$ and so we have G is a subdirect product of copies of B_1. The last part follows from the previous two propositions. □

Proposition 3.4. *Let G be a non-trivial graph with $E_G = \mathcal{C}_G$; i.e., G is complete with all loops. Then G is a subdirect product of $|G|$-copies of the subdirectly irreducible two-vertex graph B_6; hence G is subdirectly reducible.*

Proof. Here we have for every $a \in G$, the congruence $\rho_a = (\sim_a, E_G)$ and $G/\rho_a \cong B_6$. Moreover, $\bigcap\limits_{a \in G} \rho_a = \iota_G$ and hence G is a subdirect product of copies of B_6. □

Because any induced subgraph of a product of copies of the graph B_6 will be complete with all loops, we have:

Corollary 3.1. *A non-trivial graph G is complete with all loops if and only if G is a subdirect product of copies of the graph B_6.*

Theorem 3.1. *Let G be a non-trivial graph. Then G is subdirectly irreducible if and only if G is one of the four graphs B_4, B_5, B_6 or A_3.*

Proof. Already we know that the two-vertex graphs B_4, B_5 and B_6 are subdirectly irreducible. A direct verification shows that $A_3 = (\{0, 1, 2\}, \{00, 11, 22, 01, 12\})$ has 6 congruences and any intersection of congruences which is the identity must already contain one that is the identity; hence A_3 is subdirectly irreducible. The converse will be shown in a number of steps:

(1) Let G be a graph with at least three vertices. If G is subdirectly irreducible, then every vertex of G has a loop.

Proof of (1): Suppose G has a vertex p which does not have a loop. Let γ be the congruence $\gamma = (\Bumpeq, \mathcal{E}_\gamma)$ where $\mathcal{E}_\gamma = E_G \cup \{pp\}$). Then $\gamma \neq \iota_G$ and also $\rho_a \neq \iota_G$ for all $a \in G$. Now $\gamma \cap \left(\bigcap\limits_{a \in G} \rho_a \right) = (\Bumpeq, E_G) = \iota_G$. Indeed, for the first equality, we note that $\mathcal{E}_\gamma \cap \left(\bigcap\limits_{a \in G} \mathcal{E}_a \right) = E_G$: if $st \in \mathcal{E}_\gamma \cap \left(\bigcap\limits_{a \in G} \mathcal{E}_a \right)$, then $st \in E_G$ or $st = pp$. But st is also in $\mathcal{E}_a$ for all $a \in G$. In particular, if $st = pp$ we get for $a = p$, that $pp = st \in \mathcal{E}_p$ and so $pp \in E_G$; a contradiction. Thus $st \in E_G$ and hence $\gamma \cap \left(\bigcap\limits_{a \in G} \rho_a \right) = \iota_G$. But this contradicts the fact that G is subdirectly irreducible.

(2) Let G be a non-trivial complete graph with all loops. Then G is subdirectly irreducible if and only if G is a two-vertex graph.

Proof of (2): By assumption, G has at least two-vertices and $E_G = \mathcal{C}_G$. If G has only two vertices, then $G = B_6$ which we already know is subdirectly irreducible. Conversely, suppose G is subdirectly irreducible. For every $a \in G$, the congruence $\rho_a = (\sim_a, \mathcal{E}_a) = (\sim_a, \mathcal{C}_G)$ and

$\left(\bigcap_{a\in G}\rho_a\right) = \iota_G$. By the assumption on G, there is an $a \in G$ with $\rho_a = \iota_G$. This implies that $|V_G| = 2$ and we are done.

(3) Let G be a graph with at least three vertices. If G is subdirectly irreducible, then G has all loops and all but one edge; i.e., $E_G = \mathcal{C}_G - \{ab\}$ for two distinct vertices $a, b \in G$.

Proof of (3): By (1) we already know that $aa \in E_G$ for all $a \in G$. By (2) G is not complete, say $a, b \in G$ with $ab \in \mathcal{C}_G - E_G$ and $a \neq b$. For any $pq \in \mathcal{C}_G - E_G$, we have $p \neq q$. Let θ_{pq} be the congruence $\theta_{pq} = (\backsimeq, \mathcal{E}_{pq})$ with $\mathcal{E}_{pq} = E_G \cup \{pq\}$. Clearly $\theta_{pq} \neq \iota_G$. If $pq \neq ab$, then $\theta_{pq} \cap \theta_{ab} = \iota_G$ which contradicts G subdirectly irreducible. Thus $E_G = \mathcal{C}_G - \{ab\}$ as required.

This gives:

(4) Let G be a graph with three vertices. Then G is subdirectly irreducible if and only if $G = A_3$.

(5) Let G be a graph with at least four vertices. Then G is subdirectly reducible.

Proof of (5): If not, then by (3) we have $E_G = \mathcal{C}_G - \{pq\}$ for two distinct $p, q \in G$. Let $\alpha = (\sim_\alpha, \mathcal{E}_\alpha)$ be the congruence with $\sim_\alpha$the equivalence on G with equivalence classes $\{p, q\}$ and $\{a\}$ for all $a \in G - \{p, q\}$ and $\mathcal{E}_\alpha = \mathcal{C}_G$. Clearly $\alpha \neq \iota_G$. Let $\sim_\beta$ be the equivalence on G with the three equivalence classes $\{p\}, \{q\}$ and $V_G - \{p, q\}$ and let $\mathcal{E}_\beta = E_G$. Then $\beta = (\sim_\beta, \mathcal{E}_\beta)$ is a congruence on G. Indeed, for this we verify the Substitution Property: $pp \in E_G \Rightarrow [p]_\beta[p]_\beta = \{pp\} \subseteq E_G$; $qq \in E_G \Rightarrow [q]_\beta[q]_\beta = \{qq\} \subseteq E_G$; for $a \in V_G - \{p, q\}$, $pa \in E_G \Rightarrow [p]_\beta[a]_\beta = \{pt \mid t \neq p, t \neq q\} \subseteq E_G$ and likewise $qa \in E_G \Rightarrow [q]_\beta[a]_\beta \subseteq E_G$ and lastly, for $a, b \in V_G - \{p, q\}$, $ab \in E_G \Rightarrow [a]_\beta[b]_\beta = \{st \mid s, t \in V_G - \{p, q\}\} \subseteq E_G$. Thus β is a congruence and since G has at least four vertices, $\beta \neq \iota_G$. But $\alpha \cap \beta = \iota_G$; hence G is subdirectly reducible.

We now show the converse of the statement in the theorem. Let G be a non-trivial graph which is subdirectly irreducible. By (5), G must have two or three vertices. If $|G| = 2$, G can only be one of B_4, B_5 or B_6 and if G has three vertices, then by (4) above it must be the graph A_3. □

4. Birkhoff's Theorem for graphs

We conclude with the graph theoretic version of Birkhoff's Theorem.

Theorem 4.1. *Every non-trivial graph is a subdirect product of subdirectly irreducible graphs; i.e., every non-trivial graph is an induced subgraph of a product of copies of the graphs B_4, B_5, B_6 and A_3.*

Proof. If G has only two vertices, then the statement follows from Proposition 3.1. Suppose thus G has at least three vertices. Then the congruence $\rho_a = (\sim_a, \mathcal{E}_a)$ is not the identity and G/ρ_a is one of the two-vertex graphs B_i for every $a \in G$. Thus, if $\left(\bigcap_{a\in G}\rho_a\right) = \iota_G$, i.e. if $E_G = \bigcap_{a\in G}\mathcal{E}_a$, we are done by Propositions 3.1 and 3.2. Suppose thus $E_G \subset \bigcap_{a\in G}\mathcal{E}_a$, say $p, q \in G$ with $pq \in \mathcal{E}_a$ for all $a \in G$ but $pq \notin E_G$. In particular, $pq \in \mathcal{E}_p$. This means $p \neq q$, for if $p = q$, then $pp \in \mathcal{E}_p$ implies $pq = pp \in E_G$; a contradiction. Since $pq \in \mathcal{E}_p$, we know $[p]_p[q]_p \subseteq \mathcal{E}_p$ and so there is an $r \in V_G - \{p\}$ with $pr \in E_G$. Note that $r \neq q$. Likewise, since $pq \in \mathcal{E}_q$ there is an $s \in V_G - \{p, q\}$ with $qs \in E_G$ (we could have $r = s$). Let $\sim_{pq}$ be the equivalence on V_G with three equivalence classes $\{p\}, \{q\}$ and $V_G - \{p, q\}$. Let $\mathcal{E}_{pq} := \mathcal{C}_G - \{pq\}$. Clearly $E_G \subseteq \mathcal{E}_{pq}$ and to justify the claim that $\gamma_{pq} := (\sim_{pq}, \mathcal{E}_{pq})$ is a congruence on G, we will verify the Substitution Property. Let $ab \in \mathcal{E}_{pq}$. Then $ab \neq pq$ and

$$[a]_{pq}[b]_{pq} = \begin{cases} \{pp\} \text{ if } a = b = p \\ \{qq\} \text{ if } a = b = q \\ \{pt \mid t \in V_G - \{p,q\}\} \text{ if } a = p \text{ and } b \in V_G - \{p,q\} \\ \{qt \mid t \in V_G - \{p,q\}\} \text{ if } a = q \text{ and } b \in V_G - \{p,q\} \\ \{tu \mid t,u \in V_G - \{p,q\}\} \text{ if } a,b \in V_G - \{p,q\} \end{cases}$$

from which we have $[a]_{pq}[b]_{pq} \subseteq \mathcal{E}_{pq}$. Hence γ_{pq} is a congruence on G.

We now distinguish two cases. If $\gamma_{pq} = \iota_G$, then $V_G = \{p,q,r\}$ and $E_G = \mathcal{E}_{pq} = \mathcal{C}_G - \{pq\}$. This means $G = A_3$ and we are done. Suppose thus $\gamma_{pq} \neq \iota_G$ for all $pq \in \bigcap\limits_{a\in G} \mathcal{E}_a - E_G$. Note that for all these pq's we have $G/\gamma_{pq} \cong A_3$. We also have $\left(\bigcap\limits_{a\in G} \mathcal{E}_a\right) \cap \left(\bigcap\{\mathcal{E}_{pq} \mid pq \in \bigcap\limits_{a\in G} \mathcal{E}_a - E_G\}\right) = E_G$ for if $st \in \left(\bigcap\limits_{a\in G} \mathcal{E}_a\right) \cap \left(\bigcap\{\mathcal{E}_{pq} \mid pq \in \bigcap\limits_{a\in G} \mathcal{E}_a - E_G\}\right)$ and $st \notin E_G$, then $st \in \bigcap\limits_{a\in G} \mathcal{E}_a - E_G$ and $st \in \bigcap\{\mathcal{E}_{pq} \mid pq \in \bigcap\limits_{a\in G} \mathcal{E}_a - E_G\}$. This means $st \in \mathcal{E}_{st} = \mathcal{C}_G - \{st\}$; a contradiction. Hence $\left(\bigcap\limits_{a\in G} \mathcal{E}_a\right) \cap \left(\bigcap\{\mathcal{E}_{pq} \mid pq \in \bigcap\limits_{a\in G} \mathcal{E}_a - E_G\}\right) = E_G$ and so $\left(\bigcap\limits_{a\in G} \rho_a\right) \cap \left(\bigcap\{\gamma_{pq} \mid pq \in \bigcap\limits_{a\in G} \mathcal{E}_a - E_G\}\right) = \iota_G$ from which we may conclude that G is a subdirect product of copies of two-vertex graphs and A_3. The result then follows from Proposition 3.1. $\square$

5. Conclusion

Graph congruences is a relatively new concept. A few applications to demonstrate how this new tool can be used in graph theory have been given. The flavor and results for graphs follow closely the motivating influences from universal algebra. Already it has been shown that the classical radical theory of algebras which has also been developed for graphs under the guise of connectednesses and diconnectednesses, can be obtained from congruences as for their algebraic counterparts. Here another classical algebraic application of congruences for graphs, namely Birkhoff's Theorem, is given: Every graph with at least two vertices is a subdirect product of subdirectly irreducible graphs. In addition, all the subdirectly irreducible graphs are determined explicitly. It is shown that there are exactly four such graphs; three with two vertices each and one with three vertices.

References

[1] G. Birkhoff, Subdirect unions in universal algebra, *Bull. Amer. Math. Soc*. **50** (1944) 764 – 768.

[2] I. Broere, J. Heidema and L.M. Pretorius, Graph congruences and what they connote, *Quaest. Math.* **41** (2018), 1045–1058.

[3] I. Broere, J. Heidema and S. Veldsman, Congruences and Hoehnke Radicals on Graphs, *Discuss. Math. Graph Theory*. To appear.

[4] B. Fawcett, *Graphs and ultrapowers*. PhD Thesis, McMaster University (1969).

[5] E. Fried and R. Wiegandt, Connectednesses and disconnectednesses of graphs, *Algebra Universalis* **5** (1975), 411–428.

[6] B.J. Gardner and R. Wiegandt, *The Radical Theory of Rings*, Marcel Dekker Inc., New York (2004).

[7] H.-J Hoehnke, Radikale in algemeinen Algebren, *Mathematische Nachrichten* **32** (1966), 347–383.

[8] G. Sabidussi, Subdirect representations of graphs. Coll. Math. Soc. Janos Bolyai 10 Infinite and finite sets (Keszthelely, Hungary, 1973), North Holland, Amsterdam 1199 - 1226 (1975).