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The distance magic property and two families of Cartesian product graphs

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Abstract

Let $G = G(V, E)$ be a simple graph. The graph G is said to be *distance magic* if there exists a bijection $f : V \to \{1, 2, \ldots, |V|\}$ and a constant s such that $\sum_{y \in N(x)} f(y) = s$ for all $x \in V$. In this paper we show that the only distance magic graph of the form $K_n \Box C_m$ is $K_1 \Box C_4$, and that $m = 4$ if $C_m \Box K_{n,t}$ is distance magic. Necessary conditions are given for $C_4 \Box K_{n,t}$ to be distance magic when $n > t$. These conditions are shown to be sufficient when n and t are both even. We conclude with some examples of distance magic graphs of the form $C_4 \Box K_{n,t}$ with $n > t$, in particular constructing an infinite sequence of non-isomorphic distance magic graphs of this type.

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1. Introduction

Let $G = G(V, E)$ be a simple graph. Given $x \in V$, let $N(x)$ be the set of vertices adjacent to x. The graph G is said to be *distance magic* if there exists a bijection $f : V \to \{1, 2, \ldots, |V|\}$ and

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a constant s such that, for all $x \in V$,

$$\sum_{y \in N(x)} f(y) = s.$$

Such an f is a *distance magic labeling* of G. This kind of labeling was first studied by Vilfred [11], who called it a Σ-labeling. The term 1-vertex magic labeling is also used. Surveys include [9] and Section 5.6 of [5].

Let G and H be two graphs with vertex sets $V(G)$ and $V(H)$, respectively. The Cartesian product $G \square H$ is a graph with vertex set $V(G) \times V(H)$. In this graph, (g_1, h_1) is adjacent to (g_2, h_2) if and only if (i) $g_1 = g_2$ and h_1 is adjacent to h_2 in H, or (ii) g_1 is adjacent to g_2 in G and $h_1 = h_2$.

Let K_n be the complete graph on n vertices, let C_m be the cycle graph on m vertices ($m \geq 3$), and let $K_{n,t}$ be the complete bipartite graph on n and t vertices. In [10], Seoud, Abdel Maqsoud, and Aldiban asked which graphs of the types $K_n \square C_m$ and $C_m \square K_{n,t}$ are distance magic. For some small values of the parameters, these graphs have been studied in isomorphic forms. The graph $K_1 \square C_m$ is isomorphic to C_m, known to be distance magic only for $m = 4$ [7]. The graph $K_3 \square C_m$ is isomorphic to $C_3 \square C_m$, which is never distance magic [8]. The graph $C_m \square K_{2,2}$ is isomorphic to $C_m \square C_4$, which also is never distance magic [8]. Seoud, Abdel Maqsoud, and Aldiban showed that $K_n \square C_3$ is not distance magic when n is odd, and that $K_n \square C_m$ is not distance magic when n is even. They also showed that $C_m \square K_{n,t}$ is not distance magic in the following cases:

- $C_m \square K_{1,n}$ for $n \geq 1$
- $C_m \square K_{n,n}$ for m odd and $n \neq 2$
- $C_m \square K_{n,n+1}$ for $m \equiv 1 \pmod 4$ and n even.

A positive result is given in [3], where Cichacz, Froncek, Krop, and Raridan showed that $C_4 \square K_{n,n}$ is distance magic if and only if n is even and $n > 2$. They also showed that $K_n \square C_4$ is not distance magic for $n \geq 2$.

In this paper we show that $K_n \square C_m$ is only distance magic when $n = 1$ and $m = 4$, and that it is necessary to have $m = 4$ for $C_m \square K_{n,t}$ to be distance magic. We establish conditions on n and t for distance magic labelings of $C_4 \square K_{n,t}$ in the case that $n > t$. When n and t are even these conditions are shown to be sufficient. The smallest examples of such distance magic graphs are identified, and we show that there are in fact infinitely many such graphs up to graph isomorphism.

2. $K_n \square C_m$

Let V be the set of vertices of $G = K_n \square C_m$. The elements of V can be given as $v_{i,j}$ for $1 \leq i \leq n$ and $j \in \mathbf{Z}/m\mathbf{Z}$. The vertex $v_{i,j}$ is adjacent to all $v_{l,j}$ with $l \neq i$ and also to $v_{i,j-1}$ and $v_{i,j+1}$.

The main idea in this section is to look at the sum of the labels of the neighbors of all the vertices within each copy of K_n. These sums satisfy a recurrence. Methods for solving such recurrences are described in many standard discrete mathematics texts (see Section 7.3 of [6], for instance). The following lemma will be useful in this section and also in Section 3.

Lemma 2.1. *Let $\{r_n\}$ be a sequence of integers satisfying the recurrence $r_n + ar_{n-1} + r_{n-2} = C$, where C is a constant and a is an integer with $|a| > 2$. If $\{r_n\}$ is periodic, then $\{r_n\}$ is constant.*

Proof. A particular solution for the recurrence is $r_n = C/(a+2)$. The characteristic equation $x^2 + ax + 1 = 0$ has two real roots ρ_1 and ρ_2 with $|\rho_1| > 1$ and $0 < |\rho_2| < 1$. Thus, the general solution for the recurrence is $r_n = k_1\rho_1^n + k_2\rho_2^n + \frac{C}{a+2}$. If $k_1 \neq 0$, then $|r_n|$ grows without bound and is not periodic. If $k_1 = 0$, then r_n has limit $\frac{C}{a+2}$. A periodic sequence can only have a limit if each term equals the limit, so $\{r_n\}$ is constant. □

Let f be a real-valued function on V, with $a_{i,j} = f(v_{i,j})$. Let $s_j = \sum_{i=1}^{n} a_{i,j}$ for all j.

Lemma 2.2. *If $\sum_{y\in N(x)} f(y) = s$ for all vertices x of $K_n \square C_m$, then $s_{j-1} + (n-1)s_j + s_{j+1} = ns$ for all $j \in \mathbf{Z}/m\mathbf{Z}$.*

Proof. Consider the sum $\sum_{i=1}^{n}\sum_{y\in N(v_{i,j})} f(y)$. On the one hand, by assumption, the inner sum is always s, so the double sum is ns. On the other hand, each vertex $v_{i,j}$ is adjacent to the other $(n-1)$ vertices $v_{l,j}$ with $l \neq i$, and each vertex $v_{i,j-1}$ or $v_{i,j+1}$ is adjacent to $v_{l,j}$ if and only if $l = i$. Thus, every term of s_j is included in the double sum $(n-1)$ times, every term of s_{j-1} and s_{j+1} is included once, and there are no remaining terms. □

Corollary 2.1. *If $\sum_{y\in N(x)} f(y) = s$ for all vertices x of $K_n \square C_m$, and $n \geq 4$, then all of the s_j are equal.*

Proof. Let $r_j = s_{j \pmod m}$ for all natural numbers j. By Lemmas 2.1 and 2.2, the sequence $\{r_j\}$ is constant. Thus, all of the s_j are equal. □

Lemma 2.3. *Assume $f : V \to \{1, \ldots, nm\}$ is a distance magic labeling of $K_n \square C_m$. If $f(v_{i,j}) = a_{i,j}$, then $a_{i,j} - a_{i,j+6} = a_{i',j} - a_{i',j+6}$ for all $1 \leq i, i' \leq n$ and all $j \in \mathbf{Z}/m\mathbf{Z}$.*

Proof. Assume $i \neq i'$, since otherwise the statement is trivial. The vertices $v_{i,j}$ and $v_{i',j}$ have all of the vertices $v_{l,j}$ with $l \neq i, i'$ as common neighbors. Thus, in a distance magic labeling, the labels of the three remaining neighbors for each vertex must have the same sum:

$$a_{i',j} + a_{i,j-1} + a_{i,j+1} = a_{i,j} + a_{i',j-1} + a_{i',j+1}.$$

Thus,

$$a_{i,j} - a_{i',j} = (a_{i,j-1} - a_{i',j-1}) + (a_{i,j+1} - a_{i',j+1}).$$

Since this is true for all j, we also have

$$a_{i,j+1} - a_{i',j+1} = (a_{i,j} - a_{i',j}) + (a_{i,j+2} - a_{i',j+2}),$$

and

$$a_{i,j+2} - a_{i',j+2} = (a_{i,j+1} - a_{i',j+1}) + (a_{i,j+3} - a_{i',j+3}).$$

Combining these equations leads to

$$a_{i,j+3} - a_{i',j+3} = (a_{i,j+2} - a_{i',j+2}) - (a_{i,j+1} - a_{i',j+1})$$
$$= (a_{i,j+1} - a_{i',j+1}) - (a_{i,j} - a_{i',j}) - (a_{i,j+1} - a_{i',j+1}) = -(a_{i,j} - a_{i',j}).$$

Similarly, $a_{i,j+6} - a_{i',j+6} = -(a_{i,j+3} - a_{i',j+3})$, so $a_{i,j+6} - a_{i',j+6} = a_{i,j} - a_{i',j}$, and thus $a_{i,j} - a_{i,j+6} = a_{i',j} - a_{i',j+6}$. □

Theorem 2.1. *The graph $G = K_n \square C_m$ is distance magic if and only if $n = 1$ and $m = 4$.*

Proof. If $G = K_1 \square C_4$ then G has the distance magic labeling $a_{1,1} = 1$, $a_{1,2} = 2$, $a_{1,3} = 4$, $a_{1,4} = 3$. Otherwise, assume as in the previous theorem that the $a_{i,j}$ give a distance magic labeling of G. Since the cases $n = 1$, $n = 2$, and $n = 3$ are known to satisfy the theorem (see the Introduction), we can assume $n \geq 4$. Applying Lemma 2.3, for a fixed j let d be the common value of $a_{i,j} - a_{i,j+6}$ for all i. Thus,

$$s_j = \sum_{i=1}^{n} a_{i,j} = \sum_{i=1}^{n} (a_{i,j+6} + d) = s_{j+6} + nd.$$

By Corollary 2.1, $s_j = s_{j+6}$, so $d = 0$. Thus, $a_{i,j} = a_{i,j+6}$ for all i, which can only happen if $v_{i,j}$ and $v_{i,j+6}$ are the same vertex. This implies that m divides 6, so either $m = 3$ or $m = 6$. But $m \neq 3$ by [10], so assume $m = 6$, and recall that n cannot be even, also by [10]. Each s_j must be 1/6 of the sum of all of the labels, so

$$s_j = \frac{1}{6} \cdot \frac{6n(6n+1)}{2} = \frac{n(6n+1)}{2},$$

which is not an integer when n is odd. Thus, G cannot be distance magic when $n \geq 4$. □

3. $C_m \square K_{n,t}$

We will describe $C_m \square K_{n,t}$ as having vertex set V consisting of the vertices $v_{i,j}$ for $i \in \mathbf{Z}/m\mathbf{Z}$ and $1 \leq j \leq n$, and the vertices $w_{i,k}$ for $i \in \mathbf{Z}/m\mathbf{Z}$ and $1 \leq k \leq t$. For all i, j, k the vertex $v_{i,j}$ is adjacent to $v_{i-1,j}$ and $v_{i+1,j}$, the vertex $w_{i,k}$ is adjacent to $w_{i-1,k}$ and $w_{i+1,k}$ and the vertex $v_{i,j}$ is adjacent to $w_{i,k}$.

Let f be a real-valued function on V. Let $a_{i,j} = f(v_{i,j})$ and $b_{i,k} = f(w_{i,k})$. For all i, let $c_i = \sum_{j=1}^{n} a_{i,j}$ and $d_i = \sum_{k=1}^{t} b_{i,k}$.

Lemma 3.1. *Suppose $\sum_{y \in N(x)} f(y) = s$ for all vertices x in $C_m \square K_{n,t}$. Then $c_{i-1} + (2 - nt)c_{i+1} + c_{i+3} = n(2-t)s$ and $d_{i-1} + (2 - nt)d_{i+1} + d_{i+3} = t(2-n)s$ for all i.*

Proof. For a fixed i, the n vertices $v_{i,j}$ contain one neighbor of each $v_{i-1,j}$ and each $v_{i+1,j}$ for $1 \leq j \leq n$. Each vertex $v_{i,j}$ is a neighbor of all of the $w_{i,k}$ as well. This accounts for all neighbors of the $v_{i,j}$, so, for all i,

$$\sum_{j=1}^{n} \sum_{y \in N(v_{i,j})} f(y) = c_{i-1} + nd_i + c_{i+1} = ns.$$

Similarly, for all i,

$$\sum_{k=1}^{t} \sum_{y \in N(w_{i,k})} f(y) = d_{i-1} + tc_i + d_{i+1} = ts.$$

To find a recurrence for the c_i, note that eliminating d_i from the system

$$c_{i-1} + nd_i + c_{i+1} = ns$$

$$d_i + tc_{i+1} + d_{i+2} = ts$$

leads to

$$nd_{i+2} = n(t-1)s + c_{i-1} + (1-nt)c_{i+1}.$$

Plugging this into $c_{i+1} + nd_{i+2} + c_{i+3} = ns$ produces the desired equation. The proof for the d_i recurrence is similar. □

Lemma 3.2. *If $\sum_{y \in N(x)} f(y) = s$ for all vertices x in $C_m \square K_{n,t}$, and $nt > 4$, then all of the c_i are equal, and all of the d_i are equal.*

Proof. If $nt > 4$, then $2 - nt < -2$, so we can apply Lemma 2.1 as in the proof of Corollary 2.1 to the recurrence

$$c_{i-1} + (2-nt)c_{i+1} + c_{i+3} = n(2-t)s$$

to see that $c_{i-1} = c_{i+1}$ for all i. Similarly, the recurrence for the d_i shows that $d_{i-1} = d_{i+1}$ for all i. But then the equations of Lemma 3.1 can be rewritten for all i as

$$(4-nt)c_{i-1} = n(2-t)s,$$

$$(4-nt)d_{i-1} = t(2-n)s.$$

Thus, for all i,

$$c_i = \frac{ns(t-2)}{nt-4} \text{ and } d_i = \frac{ts(n-2)}{nt-4}$$

□

If the assumptions in the lemma hold, we can take c to be the shared value of the c_i, and d to be the shared value of the d_i.

Theorem 3.1. *If $m \neq 4$, then $C_m \square K_{n,t}$ is not distance magic.*

Proof. All of the cases with $nt \leq 4$ have already been examined (see the Introduction). Thus, assume $nt > 4$ so that Lemma 3.2 applies. In a distance magic labeling, the sum of the labels of the neighbors of $v_{i,j}$ would be $a_{i-1,j} + a_{i+1,j} + d$. The sum of the labels of the neighbors of $v_{i+2,j}$ would be $a_{i+1,j} + a_{i+3,j} + d$. For these two sums to be equal, $a_{i-1,j}$ must equal $a_{i+3,j}$, so $v_{i-1,j}$ and $v_{i+3,j}$ are the same vertex. This is only possible if m divides 4. Since $m \geq 3$, we conclude that $m = 4$. □

4. The Case $C_4 \square K_{n,t}$

Given that the case $n = t$ has been studied in [3], and the case $t = 1$ in [10], in this section we will assume that $n > t \geq 2$. To consider what a distance magic labeling of $C_4 \square K_{n,t}$ must look like, first note the following:

Lemma 4.1. *In a distance magic labeling of $C_4 \square K_{n,t}$, the sums $a_{i-1,j} + a_{i+1,j}$ are equal for all i, j, and the sums $b_{i-1,k} + b_{i+1,k}$ are equal for all i, k.*

Proof. The sum of the labels of the neighbors of $v_{i,j}$ is $a_{i-1,j} + a_{i+1,j} + d$, and this must be the same sum for all i, j if the labeling is distance magic. Similarly, $b_{i-1,k} + b_{i+1,k} + c$ is the same for all i, k. □

We can give expressions for these sums:

Lemma 4.2. *In a distance magic labeling of $C_4 \square K_{n,t}$,*

$$a_{i-1,j} + a_{i+1,j} = \frac{(t-2)(n+t)(4n+4t+1)}{2nt-2n-2t}$$

for all i, j, and

$$b_{i-1,k} + b_{i+1,k} = \frac{(n-2)(n+t)(4n+4t+1)}{2nt-2n-2t}$$

for all i, k.

Proof. The sum $4c + 4d$ must be the sum of the labels of all the vertices, and thus the sum of all the integers from 1 to $4n + 4t$, so using the formulas found in the proof of Lemma 3.2 we get

$$\frac{4nts-8ns}{nt-4} + \frac{4nts-8ts}{nt-4} = \frac{(4n+4t)(4n+4t+1)}{2}.$$

Solving this for s gives

$$s = \frac{(n+t)(4n+4t+1)(nt-4)}{4nt-4n-4t}.$$

We can plug this back in to get an expression for $4c$ in terms of just n and t:

$$4c = \frac{4nt-8n}{nt-4}s = \frac{(nt-2n)(n+t)(4n+4t+1)}{nt-n-t}.$$

But $4c$ is the sum of all of the $a_{i,j}$, and thus the sum of the $2n$ equal sums $a_{i-1,j} + a_{i+1,j}$, so

$$a_{i-1,j} + a_{i+1,j} = \frac{(t-2)(n+t)(4n+4t+1)}{2nt-2n-2t}$$

for all i, j. The proof of the other equation is similar. □

Corollary 4.1. *In a distance magic labeling of $C_4 \square K_{n,t}$,*

$$\frac{(t-2)(n+t)(4n+4t+1)}{2nt-2n-2t} \text{ and } \frac{(n-2)(n+t)(4n+4t+1)}{2nt-2n-2t}$$

are both positive integers, and

$$\frac{(nt-2n)(n+t)(4n+4t+1)}{nt-n-t} \text{ and } \frac{(nt-2t)(n+t)(4n+4t+1)}{nt-n-t}$$

are both positive integer multiples of 4.

Proof. The first part is obvious. Multiplying the first two expressions by $2n$ and $2t$ respectively produces the final two expressions, which are the values of $4c$ and $4d$. $\square$

Lemma 4.2 also implies that, if $n > t$ and $C_4 \square K_{n,t}$ is distance magic, the integers from 1 to $4n + 4t$ can be partitioned into sets of size two such that the pairsums (that is, the sum of the elements in each set) take only the two values shown in the lemma, with both values actually occurring (under the assumption $n > t$, they will not be equal). It turns out that such a partition, if it exists, is unique. The following definition characterizes these partitions.

Definition 4.1. *Let $I = \{P, P+1, \ldots, Q-1, Q\}$ be a non-empty interval of integers (of length $Q - P + 1$). Suppose α and β are positive integers such that $u = \frac{Q-P+1}{\alpha+\beta}$ is an integer, and $u\alpha$ and $u\beta$ are both even. An α, β* **- partition** *of the interval is constructed as follows. First, partition I into sets A and B by letting the smallest α integers of I belong to A, the next smallest β integers belong to B, the next smallest α integers belong to A, and so on, until the largest β integers are assigned to B. The α, β-partition is the refinement of this partition into sets of size two by pairing the i-th smallest integer of A with the i-th largest integer of A for all $1 \leq i \leq \frac{1}{2}u\alpha$, and by pairing the i-th smallest integer of B with the i-th largest integer of B for all $1 \leq i \leq \frac{1}{2}u\beta$.*

Note that the conditions on α and β guarantee that no integer in I is 'left over' at either step in the construction. Also note that the subsets of A in the α, β - partition all have the same pairsum, as do the subsets of B. This kind of partition was constructed by Anholcer and Cichacz in the proof of Theorem 2.2 of [2], and the proof of the next lemma is essentially identical to what is found there.

Lemma 4.3 (Anholcer, Cichacz [2])**.** *Suppose the interval $I = \{P, P+1, \ldots, Q-1, Q\}$ of integers can be partitioned into sets of size 2, with the pairsums being L or N, and both pairsums occurring at least once. Then the partition must be an α, β-partition. Assuming $L < N$, we have $\alpha = N - P - Q$ and $\beta = P + Q - L$.*

Proof. Since the mean pairsum has to be $P + Q$, we must have $L < P + Q < N$. Each integer from P to $N - Q - 1$ is too small to be part of a pair with sum N, so the partition must include $\{P, L-P\}, \{P+1, L-P-1\}, \ldots, \{N-Q-2, L-N+Q+2\}, \{N-Q-1, L-N+Q+1\}$, all with pairsum L. Similarly, the integers from $L - P + 1$ to Q are too large to be part of a pair with sum L, so the partition must also include $\{N-L+P-1, L-P+1\}, \{N-L+P-2, L-P+2\}, \ldots, \{N-Q+1, Q-1\}, \{N-Q, Q\}$, all with pairsum N. Thus, the smallest $N-P-Q$ integers in I (from P to $N-Q-1$) belong to pairs of sum L, and the next-smallest $P+Q-L$ integers (from $N-Q$ to $N-L+P-1$) belong to pairs of sum N. Similarly, the largest $P+Q-L$ integers (from $L-P+1$ to Q) belong to pairs of sum N, and the next-largest $N-P-Q$ integers (from $L-N+Q+1$ to $L-P$) belong to pairs of sum L. Thus, the only possible values for α and β are $\alpha = N - P - Q$ and $\beta = P + Q - L$.

If there is any overlap at all between the interval from P to $N-L+P-1$ and the interval from $L-N+Q+1$ to Q, then the intervals must be identical to avoid a contradiction. Thus, we have an α, β - partition of the interval (with $u = 1$). If the intervals are disjoint, and the pairs already listed partition I by themselves, we have an α, β - partition with $u = 2$. Otherwise, the intervals are disjoint and there is a non-empty interval from $N-L+P$ to $L-N+Q$. This interval itself meets the conditions of the lemma, and by induction the partition of this interval is an α, β - partition, with $\alpha = N-(N-L+P)-(L-N+Q) = N-P-Q$ and $\beta = (N-L+P)+(L-N+Q)-L = P+Q-L$. Combining this partition with the pairs already listed gives an α, β - partition of the original interval. $\square$

In a distance magic labeling of $C_4 \square K_{n,t}$, we would have $P = 1$ and $Q = 4n + 4t$. Since by assumption $n > t$, the expression for $a_{i-1,j} + a_{i+1,j}$ in Lemma 4.2 must be L, and the expression for $b_{i-1,k} + b_{i+1,k}$ must be N. Thus,

$$\alpha = N - P - Q = \frac{(n-2)(n+t)(4n+4t+1)}{2nt-2n-2t} - (4n+4t+1) = \frac{(4n+4t+1)(n-t)n}{2nt-2n-2t},$$

and

$$\beta = P + Q - L = (4n+4t+1) - \frac{(t-2)(n+t)(4n+4t+1)}{2nt-2n-2t} = \frac{(4n+4t+1)(n-t)t}{2nt-2n-2t}.$$

Since

$$\alpha + \beta = \frac{(4n+4t+1)(n-t)(n+t)}{2nt-2n-2t}$$

must be a divisor of the length $4n + 4t$ of the interval I, we get another integrality condition:

Corollary 4.2. *In a distance magic labeling of $C_4 \square K_{n,t}$ with $n > t$,*

$$\frac{8nt-8n-8t}{(4n+4t+1)(n-t)}$$

is a positive integer.

We can now give a partial converse to Corollaries 4.1 and 4.2.

Theorem 4.1. *Assume n and t are positive integers with n and t both even and $n > t$. If*

$$\frac{(t-2)(n+t)(4n+4t+1)}{2nt-2n-2t} \text{ and } \frac{(n-2)(n+t)(4n+4t+1)}{2nt-2n-2t}$$

are both positive integers, and

$$\frac{8nt-8n-8t}{(4n+4t+1)(n-t)}$$

is a positive integer, then $C_4 \square K_{n,t}$ is distance magic.

Proof. Let

$$L = \frac{(t-2)(n+t)(4n+4t+1)}{2nt-2n-2t} \text{ and } N = \frac{(n-2)(n+t)(4n+4t+1)}{2nt-2n-2t},$$

and also let

$$\alpha = N-(4n+4t+1) = \frac{(4n+4t+1)(n-t)n}{2nt-2n-2t} \text{ and } \beta = 4n+4t+1-L = \frac{(4n+4t+1)(n-t)t}{2nt-2n-2t}.$$

We wish to show that there is an α, β - partition of the interval from 1 to $4n+4t$ with pairsums L and N. By assumption, L and N are positive integers, and thus α and β are as well (note that $L > 0$ implies $t \geq 4$, and thus $2nt - 2n - 2t$ is positive). Also,

$$(4n+4t)/(\alpha+\beta) = \frac{8nt-8n-8t}{(4n+4t+1)(n-t)}$$

is a positive integer that we can call u. Note that

$$u\alpha = \frac{8(nt-n-t)}{(4n+4t+1)(n-t)} \cdot \frac{(4n+4t+1)(n-t)n}{2nt-2n-2t} = 4n,$$

and similarly $u\beta = 4t$. Since $u\alpha$ and $u\beta$ are even, an α, β - partition of the interval from 1 to $4n+4t$ exists, with A consisting of $4n$ integers in u intervals of length α and B consisting of $4t$ integers in u intervals of length β. The pairsums can be found by adding the smallest and largest integers of A and B respectively, so these are $1+(4n+4t-\beta) = L$, and $(\alpha+1)+(4n+4t) = N$. Note that

$$2nL = \frac{(nt-2n)(n+t)(4n+4t+1)}{nt-n-t}$$

is a multiple of 4 since n is even, so let c be such that $2nL = 4c$, and similarly let d be such that $2tN = 4d$.

Suppose now that we can label the vertices of $C_4 \square K_{n,t}$ with the integers from 1 to $4n+4t$ so that (i) the α, β - partition consists of the sets $\{a_{i-1,j}, a_{i+1,j}\}$ for all i, j and $\{b_{i-1,k}, b_{i+1,k}\}$ for all i, k, (ii) the values of $c_i = \sum_{j=1}^{n} a_{i,j}$ are all equal, and (iii) the values of $d_i = \sum_{k=1}^{t} b_{i,k}$ are all equal. Then $c_i = c$ for all i since the c_i have sum $4c$, and similarly $d_i = d$ for all i. The labeling will then be distance magic, since, for all i, j, the sum of the labels of the neighbors of $v_{i,j}$ is

$$L + d = L + \frac{tN}{2} = \frac{(t-2)(n+t)(4n+4t+1)}{2nt-2n-2t} + \frac{t(n-2)(n+t)(4n+4t+1)}{4nt-4n-4t}$$

$$= \frac{(n+t)(4n+4t+1)(nt-4)}{4nt-4n-4t},$$

and, for all i, k, the sum of the labels of the neighbors of $w_{i,k}$ is

$$N + c = N + \frac{nL}{2} = \frac{(n-2)(n+t)(4n+4t+1)}{2nt-2n-2t} + \frac{n(t-2)(n+t)(4n+4t+1)}{4nt-4n-4t}$$

$$= \frac{(n+t)(4n+4t+1)(nt-4)}{4nt-4n-4t}.$$

To this end, a $u \times \alpha$ matrix $M_A = (m_{p,q})$ will be constructed that will contain the i values for all integers in A. To be more precise, if $m_{p,q} = i$ then $q + (p-1)(\alpha+\beta)$ will be $a_{i,j}$ for some j. It is sufficient for M_A to meet three conditions. First, $m_{p,q} = m_{u+1-p,\alpha+1-q} + 2$; that is, two entries symmetric over the center of the matrix must differ by $2 \pmod 4$, since these are the matrix entries corresponding to paired labels. Second, each $i \in \mathbf{Z}/4\mathbf{Z}$ must appear exactly $u\alpha/4$ times. Third, the sums $\sum_{m_{p,q}=i}[q+(p-1)(\alpha+\beta)]$ must be equal for all i, since these sums are the values of the c_i. Note that it will be sufficient to replace this third condition with the conditions that (i) $\sum_{m_{p,q}=i} q$ is the same for all i, and (ii) $\sum_{m_{p,q}=i} p$ is the same for all i. A matrix will be called u, α-*balanced* if it meets all of these conditions. It is easy to see that, if a matrix is u, α-balanced, so is its transpose. Also, if a matrix is constructed from u, α-balanced blocks, and each block is identical to a block in the position symmetric over the center of the matrix, then the entire matrix is u, α-balanced.

The following matrices are u, α-balanced:

$$M_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 3 & 2 & 1 & 0 & 2 & 3 & 0 & 1 & 1 & 0 & 3 & 2 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 & 1 & 0 & 3 & 2 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & 3 & 2 \end{pmatrix}.$$

Note that $u\alpha$ is divisible by 8, since $u\alpha = 4n$ and n is even. If $u\alpha$ is divisible by 16, the entire matrix M_A can be filled by copies of a single one of M_1, M_2, M_3, M_2^T or M_1^T.

Assume $u\alpha$ is divisible by 8 but not 16. As a first case, assume u is odd and α is divisible by 8 but not 16. If $u = 1$, then $\alpha \neq 8$, since $u\alpha = 4n = 8$ implies $n = 2$, contradicting $n > t \geq 2$. Thus, $\alpha \geq 24$. The leftmost and rightmost entries of M_A can be filled with copies of M_1 until either the central 24 or 40 positions remain. These positions can be filled with

$$0, 1, 2, 3, 3, 2, 1, 0, 0, 1, 2, 3, 3, 1, 2, 0, 2, 0, 3, 1, 3, 1, 2, 0, 2, 0, 3, 1, 1, 0, 3, 2, 2, 3, 0, 1, 1, 0, 3, 2$$

or just the central 24 values in this list as necessary. If $u \geq 3$, then the top and bottom rows of M_A can be filled with copies of M_2 until either the 3 or 5 central rows remain. These can be filled with copies of the u, α-balanced matrix

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 3 & 1 & 2 & 0 \\ 2 & 0 & 3 & 1 & 3 & 1 & 2 & 0 \\ 2 & 0 & 3 & 1 & 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 & 1 & 0 & 3 & 2 \end{pmatrix}$$

when 5 rows remain, or copies of this matrix with the top and bottom rows removed when 3 rows remain.

Next, assume u is divisible by 2 but not 4, and α is divisible by 4 but not 8. If $u = 2$, then, since $u\alpha \neq 8$, it must be that $\alpha \geq 12$. The leftmost and rightmost columns of M_A can be filled with copies of M_2 until a central 2×12 or 2×20 block remains, and this can be filled with the u, α - balanced matrix

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 2 & 0 & 2 & 0 & 0 & 1 & 2 & 3 & 3 & 1 & 3 & 1 & 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 & 3 & 1 & 3 & 1 & 1 & 0 & 3 & 2 & 2 & 0 & 2 & 0 & 1 & 0 & 3 & 2 \end{pmatrix},$$

or with its central 12 columns, as necessary. If $u \geq 6$, the top and bottom rows can be filled with copies of M_3 until only the central 6 or 10 rows remain. These can be filled with copies of the u, α-balanced matrix

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 3 & 1 & 2 & 0 \\ 2 & 0 & 3 & 1 \\ 3 & 1 & 2 & 0 \\ 2 & 0 & 3 & 1 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & 3 & 2 \end{pmatrix},$$

or copies of this matrix with the top two and bottom two rows removed.

The other cases with $u\alpha$ divisible by 8 but not 16 are just transposed versions of the cases already considered. With M_A constructed we can proceed to construct a $u \times \beta$ matrix M_B consisting of the i values of the $b_{i,k}$ in essentially the same way (this matrix can be called u, β - balanced). Note that, once we determine which subset of the integers from 1 to $4n + 4t$ will make up the values $a_{0,j}$, the j indices can be assigned arbitrarily, and then, for a particular j, $a_{2,j}$ must be the integer paired with $a_{0,j}$ in the α, β - partition. We can proceed in a similar fashion to assign the values of $a_{1,j}$ and $a_{3,j}$ for all j, the values of $b_{0,k}$ and $b_{2,k}$ for all k, and the values of $b_{1,k}$ and $b_{3,k}$ for all k, resulting in a distance magic labeling of the entire graph. □

A Maple search for examples with $n \leq 50000$ shows that $C_4 \square K_{n,t}$ is distance magic for the following values of n and t, shown with the corresponding values of u, α, and β:

n	t	u	α	β
440	344	4	440	344
756	468	2	1512	936
2514	2130	6	1676	1420
8192	7232	8	4096	3616
10074	7866	4	10074	7866
20210	18290	10	8084	7316
42072	38712	12	14024	12904

If n is odd and t is even, then the required expressions from Corollary 4.1 cannot both be multiples of 4, since the only even factors in their numerators would be $t - 2$ and t, which cannot both be multiples of 4. Thus, if n is odd and $C_4 \square K_{n,t}$ is distance magic, t must be odd as well. A Maple search found no examples with $n \leq 50000$ and n odd that met all of the integrality conditions, so it is an open question whether such examples exist and lead to distance magic labelings.

Using a different approach it is possible to show that $C_4 \square K_{n,t}$ is distance magic for infinitely many pairs (n, t) with $n > t$. For a particular value of u, the values of α and β must satisfy a quadratic diophantine equation, and this equation possibly has an infinite family of solutions in positive integers.

Theorem 4.2. *Let $\alpha_1 = 0$ and $\beta_1 = 7$. Let the sequences $\{\alpha_i\}$ and $\{\beta_i\}$ be defined recursively by $\alpha_{i+1} = 5\alpha_i + 2\beta_i - 6$ and $\beta_{i+1} = 2\alpha_i + \beta_i + 1$. Then $\alpha_{i+1} > \alpha_i$ and $\beta_{i+1} > \beta_i$ for all i, and $C_4 \square K_{n,t}$ is distance magic for all $i \geq 1$ when $n = \frac{1}{4}\alpha_{8i+2}$ and $t = \frac{1}{4}\beta_{8i+2}$.*

Proof. If $\alpha_i \geq 0$ and $\beta_i \geq 7$, then $\alpha_{i+1} = 5\alpha_i + 2\beta_i - 6 > 5\alpha_i + 8 > \alpha_i$, and $\beta_{i+1} = 2\alpha_i + \beta_i + 1 > \beta_i$, so both sequences are increasing by induction.

In a distance magic labeling of $C_4 \square K_{n,t}$ with $u = 1$, each $v_{i,j}$ would be adjacent to a pair of v-vertices with sum $\alpha + 1$, and to a set of w-vertices with label sum equal to 1/4 of the sum of the integers from $\alpha + 1$ to $\alpha + \beta$. Meanwhile, each $w_{i,k}$ must be adjacent to a pair of w-vertices with sum $2\alpha + \beta + 1$, and to a set of v-vertices with label sum equal to 1/4 of the sum of the integers from 1 to α. Thus, the equation

$$\alpha + 1 + \frac{\beta(2\alpha + \beta + 1)}{8} = 2\alpha + \beta + 1 + \frac{\alpha(\alpha + 1)}{8}$$

must be satisfied, and it is equivalent to

$$\alpha^2 - 2\alpha\beta - \beta^2 + 9\alpha + 7\beta = 0.$$

Methods for solving two-variable quadratic diophantine equations are due to Euler and Lagrange [4]. The site [1] was used to find the sequences stated in the theorem. It is straightforward to show by induction that the terms of the sequence of pairs (α_i, β_i) satisfy the quadratic equation and that $\alpha_i > \beta_i > 0$ for $i \geq 3$. It can also be checked that this sequence is cyclic with a period of 8 when taken modulo 8, and $\alpha_i \equiv \beta_i \equiv 0 \pmod 8$ when $i \equiv 2 \pmod 8$. Now suppose $n = \frac{1}{4}\alpha_{8i+2}$ and $t = \frac{1}{4}\beta_{8i+2}$ for some $i \geq 1$. Then n and t are both even and $n > t \geq 2$. Using a u, α - balanced matrix of dimension $1 \times 4n$ and a u, β-balanced matrix of dimension $1 \times 4t$, a labeling can be constructed that is guaranteed by the quadratic equation to be distance magic. □

More families of distance magic graphs can be found by taking other values of u and solving the corresponding quadratic diophantine equations.

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