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On n-connected minors of the es-splitting binary matroids

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Abstract

The es-splitting operation on an n-connected binary matroid may not yield an n-connected matroid for $(n \geq 3)$. In this paper, we show that given an n-connected binary matroid M of rank r, the resulting es-splitting binary matroid has an n-connected minor of rank-$(r+1)$ having $|E(M)|+1$ elements.

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1. Introduction

Slater [13] specified the n-line splitting operation on graphs as follows. Let G be a graph and $e = uv$ be an edge of G with deg $u \geq 2n-3$ with u adjacent to $v, x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_h$, where k and $h \geq n-2$. Let H be the graph obtained from G by replacing u by two adjacent vertices u_1 and u_2, with v adj u_1, v adj u_2, u_1 adj x_i $(1 \leq i \leq k)$, and u_2 adj y_j $(1 \leq j \leq h)$, where deg $u_1 \geq n$ and deg $u_2 \geq n$. The transition from G to H is called an *n-line splitting operation*. We also say that H is obtained from G by an n-line splitting operation. This construction is explicitly illustrated with the help of Figure 1.

Slater [13] proved that if G is n-connected and H is obtained from G by n-line-splitting operation, then H is n-connected. In fact, he characterized 4-connected graphs, in terms of the 4-line

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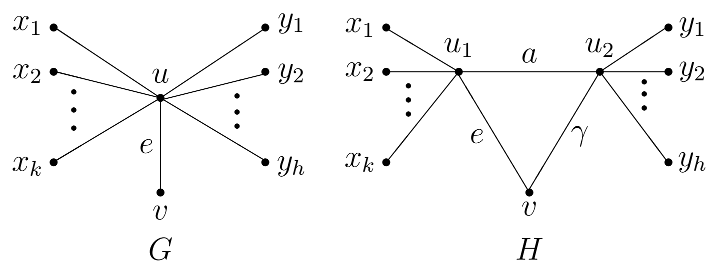

Figure 1. n-line splitting operation.

splitting operation along with some other operations. The notion of connectivity of graphs also has been studied in [6, 13] and connectivity of binary matroids has been studied in [3, 14].

Suppose G is a graph with n vertices and m edges. Let $X = \{e, x_1, x_2, \ldots, x_k\}$ be a subset of $E(G)$. The incident matrix A of G is a matrix of size $n \times m$. The row corresponding to the vertex u has 1 in the columns of $e, x_1, x_2, \ldots, x_k$, $y_1, y_2, \ldots, y_h$ and 0 in the other columns. The graph H has $(n + 1)$ vertices and $(m + 2)$ edges. The incidence matrix A' of H is a matrix of size $(n + 1) \times (m + 2)$. The row corresponding to u_2 has 1 in the columns of $y_1, y_2, \ldots, y_h, \gamma$ and 0 in the other columns, where as the row corresponding to the vertex u_1 has 1 in the columns of $e, x_1, x_2, \ldots, x_k$ and 0 in other columns. One can check that the matrix A' can be obtained from A by adjoining an extra row corresponding to the vertex u_1 to A with entries zero every where except in the columns corresponding to $e, x_1, x_2, \ldots, x_k$ where it takes the value 1. The row vector obtained by addition (mod 2) of row vectors corresponding to vertices u and u_1 will corresponds to the row vector of the vertex u_2 in A'.

Noticing the above s Azanchiler [1] extended the notion of n-line-splitting operation from graphs to binary matroids in the following way:

Definition 1. *Let M be a binary matroid on a set E and let X be a subset of E with $e \in X$. Suppose A is a matrix representation of M over GF(2). Let A^e_X be a matrix obtained from A by adjoining an extra row δ_X to A with entries zero every where except in the columns corresponding to the elements of X, where it takes the value 1 and then adjoining two columns labelled a and γ to the resulting matrix such that the column labelled a is zero everywhere except in the last row where it takes the value 1, and γ is sum of the two column vectors corresponding to the elements a and e. The vector matroid of the matrix A^e_X is denoted by M^e_X. The transition from M to M^e_X is called an es-splitting operation. We call the matroid M^e_X as es-splitting matroid.*

The following proposition characterizes the circuits of the matroid M^e_X in terms of the circuits of the matroid M.

Proposition 1.1. *[1] Let $M(E, \mathcal{C})$ be a binary matroid together with the collection of circuits $\mathcal{C}$. Suppose $X \subseteq E$, $e \in X$ and $a, \gamma \notin E$. Then $M^e_X = (E \cup \{a, \gamma\}, \mathcal{C}')$, where $\mathcal{C}' = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \{\Delta\}$ with $\Delta = \{e, a, \gamma\}$ and*

$$\begin{aligned}
\mathcal{C}_0 &= \{C \in \mathcal{C} \mid C \textit{ contains an even number of elements of } X\ \};\\
\mathcal{C}_1 &= \textit{The set of minimal members of } \{C_1 \cup C_2 \mid C_1, C_2 \in \mathcal{C}, C_1 \cap C_2 = \phi\\
&\quad \textit{ and each of } C_1 \textit{ and } C_2 \textit{ contains an odd number of elements of}\\
&\quad X \textit{ such that } C_1 \cup C_2 \textit{ contains no member of } \mathcal{C}_0\ \};\\
\mathcal{C}_2 &= \{C \cup \{a\} \mid C \in \mathcal{C} \textit{ and } C \textit{ contains an odd number of elements of } X\};\\
\mathcal{C}_3 &= \{C \cup \{e, \gamma\} \mid C \in \mathcal{C}, e \notin C \textit{ and } \mathbf{C} \textit{ contains an odd number of elements}\\
&\quad \textit{of } \mathbf{X}\ \} \cup \{(C \setminus e) \cup \{\gamma\} \mid C \in \mathcal{C}, e \in C \textit{ and } \mathbf{C} \textit{ contains an odd number}\\
&\quad \textit{of elements of } X\ \} \cup \{(C \setminus e) \cup \{a, \gamma\} \mid C \in \mathcal{C}, e \in C \textit{ and } C \setminus e\\
&\quad \textit{contains an odd number of elements of } X\ \}.
\end{aligned}$$

Throughout this paper we assume that M is a loopless and coloopless binary matroid, $X \subset E(M)$ and M_X^e is the es-splitting matroid of M. We denote by $\mathcal{C}_{OX}$ the set of all circuits of a matroid M each of which contains an odd number of elements of the set X. The members of the set $\mathcal{C}_{OX}$ are called *OX-circuits*. On the other hand, $\mathcal{C}_{EX}$ denotes the set of all circuits of a matroid M each of which contains an even number of elements of the set X. The members of the set $\mathcal{C}_{EX}$ are called *EX-circuits*.

It is intersting to observe that $M_X^e \setminus \gamma$ and $M_X^e \setminus \{a, \gamma\}$ are isomorphic with element splitting matroid and splitting matroid of M, respectively. The main theorems of this paper, Theorem 3.1 and Theorem 3.2 are motivated by a series of earlier work on splitting operation, element splitting operation and es-splitting operation [1, 2, 4, 7, 8, 10, 11, 12, 15, 17].

The following result characterizes the rank function of the matroid M_X^e in terms of the rank function of the matroid M [4].

Lemma 1.1. *Let r and r' be the rank functions of the matroids M and M_X^e, respectively. Suppose that $A \subseteq M(E)$. Then*

1. $r'(A) = r(A) + 1$, *if A contains an OX-circuit of the matroids M;*
   $= r(A)$*; otherwise.*
2. $r'(A \cup a) = r(A) + 1$.
3. $r'(A \cup \{\gamma\}) = r(A)$, *if not A but $A \cup \{e\}$ contains an OX-circuit of M;*
   $= r(A) + 2$, *if A contains an OX-circuit of M and* $e \notin cl(A)$;
   $= r(A) + 1$, *otherwise.*
4. $r'(A \cup \{a, \gamma\}) = r(A) + 1$, *if* $e \in cl(A)$;
   $= r(A) + 2$, *if* $e \notin cl(A)$.

Using Lemma 1.1, one can obtain the following corollary.

Corollary 1.1. *Let r and r' be the rank functions of the matroids M and M_X^e, respectively. Then* $r'(M_X^e) = r(M) + 1$. □

Remark 1.1. As $\triangle = \{a, e, \gamma\} \subset M_X^e$, we have $r'(M_X^e) = r'(M_X^e \setminus \{a\}) = r'(M_X^e \setminus \{e\}) = r'(M_X^e \setminus \{\gamma\})$.

We recall that matroid M is connected if and only if for every pair of distinct elements of $E(M)$, there is a circuit containing both. The concept of n-connected matroids was introduced by W. T. Tutte [14]. If k is positive integer, the matroid M is k-separated if there is a subset $X \subset E(M)$ such that $|X| \geq k, |E \setminus X| \geq k$ and $r(X) + r(E \setminus X) - r(M) = k - 1$. Connectivity $\lambda(M)$ of M is the list positive integer j such that M is j-separated. If there is no such integer we say $\lambda(M) = \infty$. Note that $\lambda(U_{2,4}) = \infty$. The following result from [9] provides a necessary condition for a matroid to be n-connected.

Lemma 1.2. *If M is a n-connected matroid and $|E(M)| \geq 2(n-1)$ then all circuits and all cocircuits of M have at least n elements.*

Let M be an n-connected binary matroid and $X \subset E(M)$. Note that if $|X| < n$ then $X \cup \{a\}$ will be a cocircuit of M_X^e. Further, if $|X \cup \{a\}| < n$ then, by Lemma 1.2, M_X^e is not n-connected. Azanchiler [1] proved that es-splitting operation on a connected binary matroid yields a connected binary matroid. In fact, he proved the following theorem.

Theorem 1.1. *Let M be a connected binary matroid and $X \subset E(M)$ with $|X| \geq 2$. Then M_X^e is connected binary matroid.*

In the following result Dhotre, Malavadkar and Shikare [4], provided a sufficient condition for the es-splitting operation to yield a 3-connected binary matroid from a 3-connected binary matroid.

Theorem 1.2. *Let M be a 3-connected binary matroid, $X \subseteq E(M)$ and $e \in X$. Suppose that M has an OX-circuit not containing e. Then M_X^e is a 3-connected binary matroid.*

In perticular, when $X = \{x, y\}$ the es-spliting maroid is denoted by $M_{x,y}^e$. As a consequence of the above result, Dhotre, Malavadkar and Shikare [4] obtained a splitting lemma for es-splitting matroid $M_{x,y}^e$.

Corollary 1.2. (Splitting Lemma). *If M is a 3-connected binary matroid then, $M_{x,y}^e$ is a 3-connected binary matroid for any pair $\{x, y\}$ of elements of $E(M)$.*

2. 3-Connected Minors of the es-splitting Matroids.

In this section, we provide a sufficient condition for a 3-connected binary matroid M of rank r, where $M_X^e \setminus e$ and $M_X^e \setminus \gamma$ are 3-connected minors of rank $r+1$ of the matroid M_X^e.

Let M be a cycle matroid of a wheel W_5 as shown in the Figure 2, $X = \{8, 9, 10\}$ and $e = 10$. Then M_X^e is the es-splitting matroid of M. Observe that $M_X^e \setminus e$ and $M_X^e \setminus \gamma$ are 3-connected minors of M_X^e. But $M_X^e \setminus a$ is not a 3-connected minor of M_X^e.

In the following result, we provide a sufficient condition for a 3-connected binary matroid M where $M_X^e \setminus e$ is a 3-connected minor of M_X^e.

Lemma 2.1. *Let M be a 3-connected binary matroid, $|E(M)| \geq 4$ and let $X \subset E(M)$, where $|X| \geq 3$. Suppose for $x \in E(M)$ there is an OX-circuit of M not containing x. Then $M_X^e \setminus e$ is a 3-connected binary matroid.*

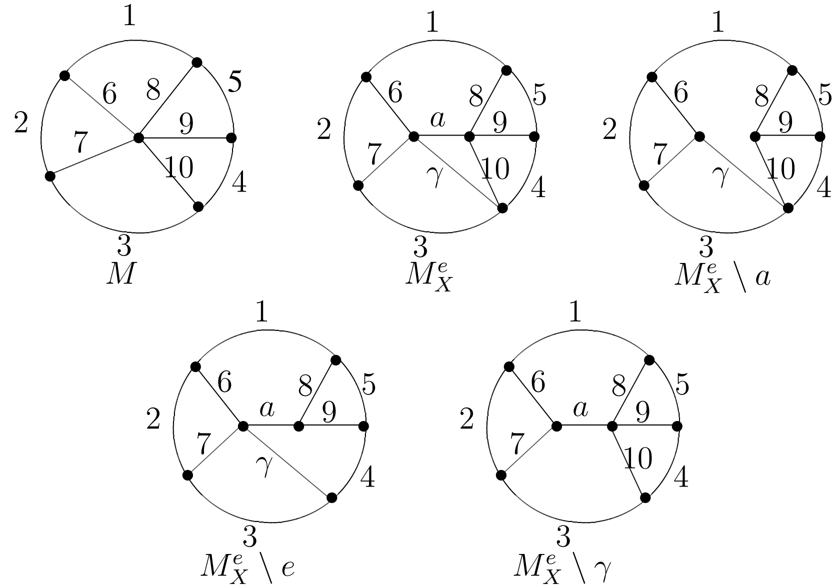

Figure 2. 3-connected minors of $M(W_5)_X^e$.

Proof. Suppose that M contains an OX-circuit C of M and $e \notin C$. Then, by Theorem 1.2, M_X^e is 3-connected. Therefore, $M_X^e \setminus e$ is at least 2-connected. It is enough to show that $M_X^e \setminus e$ has no 2-separation. On the contrary, suppose (A, B) is a 2-separation of $E(M_X^e) \setminus e$. So, $\min\{|A|, |B|\} \geq 2$ and

$$r'(A) + r'(B) - r'(M_X^e \setminus e) \leq 1. \tag{1}$$

Now one of the following two cases concerning a and γ occurs.

Case 1. $a \in A$ and $\gamma \in B$.

Let $A' = A \setminus a$ and $B' = B \setminus \gamma$.

Subcase 1.1. $|A| = 2$ or $|B| = 2$.

Suppose $A = \{a, x\}$. Then there is an OX-circuit C of M not containing x and $C \subset B'$. Thus, by Lemma 1.1 (2) and (3), $r'(A) = r(A') + 1$ and $r'(B) \geq r(B') + 1$. By inequality (1), we have $r(A') + 1 + r(B') + 1 - r(M) - 1 \leq 1$. That is, $r(A') + r(B') - r(M) \leq 0$ where $|A'|, |B'| \geq 1$. This implies (A', B') is a 1- separation of M, a contradiction. If B = 2, and suppose $B = \{\gamma, x\}$. Then by the argument similar to one as given above we get a contradiction to the 3-connectedness of M.

Subcase 1.2. $|A|, |B| > 2$.

Then, by Lemma 1.1 (2) and (3), $r'(A) = r(A') + 1$ and $r'(B) \geq r(B')$. By inequality (1), we have $r(A') + 1 + r(B') - r(M) - 1 \leq 1$. That is, $r(A') + r(B') - r(M) \leq 1$ where $|A'|, |B'| \geq 2$. This implies (A', B') is a 2- separation of M, a contradiction.

Case 2. $\{a, \gamma\} \subset A$.

Let $A' = A \setminus \{a, \gamma\}$ and $B' = B$. We have the following three subcases.

Subcase 2.1. $|A| = 2$.

Then $A = \{a, \gamma\}$, $r'(A) = 2$ and $A' = \{\phi\}$. By hypothesis, there is an OX-circuit C of M not containing e and thus, $C \subseteq B$. Then, by Lemma 1.1 (1), $r'(B) = r(B') + 1$. So, by inequality (1),

$2 + r(B') + 1 - r(M) - 1 \leq 1$. That is, $r(B') - r(M) \leq -1$ or $r(B') \leq r(M) - 1$. This is a contradiction.

Subcase 2.2. $|A| = 3$. Suppose $A = \{a, \gamma, x\}$. Then $e \notin Cl(A')$ since otherwise, $\{x, e\}$ forms a 2-circuit, which is not possible in a 3-connected matroid M. We conclude that $e \notin Cl(A')$. Consequently, by Lemma 1.1 (4), $r'(A) = r(A') + 2$. Also, by Lemma 1.1 (1), $r'(B) \geq r(B')$. Thus, by inequality (1), $r(A') + 2 + r(B') - r(M) - 1 \leq 1$. That is, $r(A') + r(B') - r(M) \leq 0$ and $|A'|, |B'| \geq 1$. This gives a 1-separation of M, a contradiction.

Subcase 2.3. $|A| > 3$.

Applying Lemma 1.1 to A and B, we get $r'(A) \geq r(A) + 1$ and $r'(B) \geq r(B)$. Then, by inequality (1), we get $r(A') + 1 + r(B') - r(M) - 1 \leq 1$. That is, $r(A') + r(B') - r(M) \leq 1$ and $|A'|, |B'| \geq 2$. This leads to a 2-separation of M; a contradiction. The above facts imply that M_X^e has no 2-separation. We conclude that $M_X^e \setminus e$ is 3-connected matroid. □

In the following lemma, we provide a sufficient condition for a 3-connected binary matroid M so that $M_X^e \setminus \gamma$ is a 3-connected minor of the es-splitting matroid M_X^e.

Lemma 2.2. *Let M be a 3-connected binary matroid, $|E(M)| \geq 4$. Let $X \subset E(M)$ with $|X| \geq 3$. Suppose for $x \in E(M)$ there is an OX-circuit of M not containing x. Then, $M_X^e \setminus \gamma$ is a 3-connected binary matroid.*

Proof. If $x = e$ then, by hypothesis, there is an OX-circuit of M not containing x. So, by Theorem 1.2, M_X^e is 3-connected and $M_X^e \setminus \gamma$ is connected. Suppose $M_X^e \setminus \gamma$ is not 3-connected and let (A, B) be a 2-separation of $E(M_X^e \setminus \gamma)$. Then min $\{|A|, |B|\} \geq 2$ and

$$r'(A) + r'(B) - r'(M_X^e \setminus \gamma) \leq 1. \tag{2}$$

Assume that $\{a\} \subset A$. Let $A' = A \setminus a$ and $B' = B$. Then, by Lemma 1.1, $r'(A) = r(A') + 1$ and $r'(B) \geq r(B')$. Now one of the following two cases occurs.

Case 1. $|A| = 2$.

Suppose $A = \{z, a\}$ and $A' = \{z\}$ where $z \in E(M)$. Then, by Lemma 1.1 (2), $r'(A) = r(A') + 1$. Now M contains an odd circuit C of M and $\{z\} \cap C = \phi$, implies $C \subseteq B$. Then, by Lemma 1.1 (1), $r'(B) = r(B') + 1$. Thus, by inequality (2), $r(A') + 1 + r(B') + 1 - r(M) - 1 \leq 1$. That is, $r(A') + r(B') - r(M) \leq 0$, and $|A'|, |B'| \geq 1$. This implies that (A', B') is a 1-separation of M, a contradiction.

Case 2. $|A| > 2$.

By (1) and (2) of Lemma 1.1, $r'(A) \geq r(A') + 1$ and $r'(B) \geq r(B')$. Then, by inequality (2), $r(A') + 1 + r(B') - r(M) - 1 \leq 1$. That is, $r(A') + r(B') - r(M) \leq 1$ and $|A'|, |B'| \geq 2$. This leads to a 2-separation of M, a contradiction. Thus, $M_X^e \setminus \gamma$ has no 2-separation. We conclude that $M_X^e \setminus \gamma$ is a 3-connected binary matroid. □

3. n-Connected Minors of the es-splitting Matroids.

In this section, we provide a sufficient condition for an n-connected binary matroid M $(n \geq 4)$ of rank r, where $M_X^e \setminus e$ and $M_X^e \setminus \gamma$ are n-connected minors of rank $r + 1$ of the es-splitting matroid M_X^e.

Let M be an n-connected binary matroid $(n \geq 4)$, $X \subseteq E(M)$ and $e \in X$. Suppose that M has an OX-circuit not containing e. Then, by Theorem 1.2, the binary matroid M_X^e is 3-connected. Note that the matroid M_X^e contains a triangle $\triangle = \{a, e, \gamma\}$. Hence, by Proposition 1.2, M_X^e is not 4-connected. We observe that for any $x \in E(M_X^e)$, M_X^e/x contains a 2-circuit or a triangle and therefore it is not 4-connected. Further, for any $x \in (E(M_X^e) - \triangle)$, the minor $M_X^e \setminus x$ contains the triangle $\triangle$ and therefore, it is not 4-connected. Thus, the possible 4-connected minors of M_X^e are $M_X^e \setminus e$ and $M_X^e \setminus \gamma$.

In the following theorem, we give a sufficient condition for an n-connected binary matroid M where $M_X^e \setminus e$ is an n-connected minor of M_X^e.

Theorem 3.1. *Let M be an n-connected binary matroid where $n \geq 4$, $|E(M)| \geq 2(n-1)$ and let $X \subset E(M)$ with $|X| \geq n$. Suppose that for any $(n-2)$-element subset S of $E(M)$ there is an OX-circuit C of M such that $S \cap C = \phi$. Then $M_X^e \setminus e$ is n-connected.*

Proof. The proof is by induction on n. First we prove the case $n = 4$. The matroid $M_X^e \setminus e$ is 3-connected by Lemma 2.1. To prove that $M_X^e \setminus e$ is 4-connected, it is enough to show that it has no 3-separation. On the contrary, suppose (A, B) forms a 3-separation of $M_X^e \setminus e$. Then
min $\{|A|, |B|\} \geq 3$ and

$$r'(A) + r'(B) - r'(M_X^e \setminus e) \leq 2. \tag{3}$$

Now one of the following two cases occurs.

Case 1. $a \in A$ and $\gamma \in B$

Subcase 1.1. $|A| = 3$ Let $A = \{a, x, y\}$. Then there is an OX-circuit C of M not containing x, y and $C \subset B'$. Thus, by Lemma 1.1 (2) and (3), $r'(A) = r(A') + 1$ and $r'(B) \geq r(B') + 1$. By inequality (1), we have $r(A') + 1 + r(B') + 1 - r(M) - 1 \leq 2$. That is, $r(A') + r(B') - r(M) \leq 1$ where $|A'|, |B'| \geq 2$. This implies (A', B') is a 2- separation of M, a contradiction.

Subcase 1.2. $|A|, |B| > 3$

Then, by Lemma 1.1 (2) and (3), $r'(A) = r(A') + 1$ and $r'(B) \geq r(B')$. By inequality (1), we have $r(A') + 1 + r(B') - r(M) - 1 \leq 2$. That is, $r(A') + r(B') - r(M) \leq 2$ where $|A'|, |B'| \geq 3$. We conclude that (A', B') is a 3- separation of M, a contradiction.

Case 2. $\{a, \gamma\} \subset A$

Let $A' = A \setminus \{a, \gamma\}$ and $B' = B$. We have the following three subcases.

Subcase 2.1. $|A| = 3$ and $A = \{a, \gamma, x\}$, where $x \in E(M) \setminus e$

If $e \in Cl(A')$, then $\{x, e\}$ forms a 2-circuit of M. This is not possible, since M is 4-connected. Thus, $e \notin Cl(A')$ and by Lemma 1.1 (4), $r'(A) = r(A') + 2$. Also, there is an OX-circuit C of M not containing x and $C \subseteq B'$. Therefore, $r'(B) = r(B') + 1$. Consequently, by inequality (3),

$$r(A') + 2 + r(B') + 1 - r(M) - 1 \leq 2.$$

That is, $(A') + r(B') - r(M) \leq 0$ and $|A'|, |B'| \geq 1$. So M has a 1-separation; a contradiction.

Subcase 2.2. $|A| = 4$ and $A = \{a, \gamma, x, y\}$ where $x, y \in E(M) \setminus e$

If $e \in Cl(A')$, then the set $\{x, y, e\}$ itself is a 3-circuit or contains a 2-circuit of M. This is not possible, since M is 4-connected. Thus, $e \notin Cl(A')$ and, by Lemma 1.1 (4), $r'(A) = r(A') + 2$.

Now there is an OX-circuit C of M not containing x and y and $C \subseteq B'$. So, $r'(B) = r(B') + 1$. Therefore, by inequality (3),

$$r(A') + 2 + r(B') + 1 - r(M) - 1 \leq 2.$$

That is, $r(A') + r(B') - r(M) \leq 0$ and $|A'|, |B'| \geq 1$. We conclude that M has a 1-separation, a contradiction.

Subcase 2.3. $|A| > 4$

Now by (1) and (4) of Lemma 1.1, $r'(B) \geq r(B)$ and $r'(A) \geq r(A) + 1$. By inequality (3), we get

$$r(A') + 1 + r(B') - r(M) - 1 \leq 2.$$

That is, $r(A') + r(B') - r(M) \leq 2$ and $|A'|, |B'| \geq 3$. This leads to a 3-separation of M, a contradiction.

Thus, M^e_X has no 3-separation. We conclude that $M^e_X \setminus e$ is 4-connected.

Now we assume that the result is true for $k \geq 4$ and prove that the result is true for $k + 1$.

Let M be a $(k+1)$-connected binary matroid and M^e_X be the es-splitting matroid of M and any $(k-1)$-element subset S of $E(M)$ there is an OX-circuit C of M such that $S \cap C = \phi$. Note that $M^e_X \setminus e$ is a k-connected minor by induction hypothesis. Thus, it is enough to show that $M^e_X \setminus e$ has no k-separation.

On the contrary, suppose $M^e_X \setminus e$ is not (k+1)-connected. Let (A, B) be a k-separation of $E(M^e_X \setminus e)$. Then, min $\{|A|, |B|\} \geq k$, and

$$r'(A) + r'(B) - r'(M^e_X \setminus a) \leq k - 1. \tag{4}$$

Now one of the following two cases occurs.

Case 1. $a \in A$ and $\gamma \in B$

Let $A' = A \setminus a$ and $B' = B \setminus \gamma$. Then, by (2) and (3) of Lemma 1.1, $r'(A) = r(A') + 1$ and $r'(B) \geq r(B') + 1$. By inequality (4), we have $r(A') + 1 + r(B') + 1 - r(M) - 1 \leq k - 1$. That is, $r(A') + r(B') - r(M) \leq k - 2$, where $|A'|, |B'| \geq k$. Thus, (A', B') is a k- separation of M and this is a contradiction.

Case 2. $\{a, \gamma\} \subset A$

Let $A' = A \setminus \{a, \gamma\}$ and $B' = B$. We have the following two subcases.

Subcase 2.1. $|A| = 4$ and $A = \{a, \gamma, x, y\}$ where $x, y \in E(M) \setminus e$

If $e \in Cl(A')$ then the set $\{x, y, e\}$ itself is a 3-circuit or contains a 2-circuit of M. This is not possible, since M is 4-connected. If $e \notin Cl(A')$ then, by Lemma 1.1 (4), $r'(A) = r(A') + 2$. Since there is an OX-circuit C of M not containing x and y, $C \subseteq B'$. So $r'(B) = r(B') + 1$. Consequently, by inequality (4),

$$r(A') + 2 + r(B') + 1 - r(M) - 1 \leq k - 1.$$

That is, $r(A') + r(B') - r(M) \leq k - 2$ and $|A'|, |B'| \geq k$. This implies that M has a k-separation, a contradiction.

Subcase 2.1. $|A| > 4$

Now by (1) and (4) of Lemma 1.1, $r'(B) \geq r(B)$ and $r'(A) \geq r(A) + 1$. By inequality (4), we get

$r(A')+1+r(B')-r(M)-1 \leq k-1$. That is, $r(A')+r(B')-r(M) \leq k-2$ and $|A'|, |B'| \geq k$. This leads to a k-separation of M, a contradiction.

Thus, M_X^e has no k-separation. We conclude that $M_X^e \setminus e$ is $k+1$-connected. We conclude that, by principle of mathematical induction, the result is true for all $n \geq 4$. □

In the following theorem, we give a sufficient condition for an n-connected binary matroid M so that $M_X^e \setminus \gamma$ is an n-connected minor of M_X^e.

Theorem 3.2. *Let M be an n-connected binary matroid with $n \geq 4$, $|E(M)| \geq 2(n-1)$ and let $X \subset E(M)$, where $|X| \geq n$. Suppose that for any $(n-2)$-element subset S of $E(M)$ there is an OX-circuit C of M such that $S \cap C = \phi$. Then $M_X^e \setminus \gamma$ is n-connected.*

The proof follows by the arguments similar to one as given for the proof of Theorem 3.1.

Thus, we proved that given an n-connected binary matroid M of rank r, $M_X^e \setminus e$ and $M_X^e \setminus \gamma$ are the n-connected minors of rank $(r+1)$ of the es-splitting matroid M_X^e. In other words, we provide a procedure to obtain n-connected matroids of rank $(r+1)$ from an n-connected matroid of rank r. The matroids also have the property that each of them has exactly one additional element than M. We illustrate Theorems 3.1 and 3.2 with the help of the following example.

Example 1. Let matrix M be a cycle matroid of a complete bipartite graph $K_{4,4}$ shown in Figure 3. M is 4-connected matroid. Let $X = \{1, 2, 5, 6\}$. Observe that there is an OX-circuit in M avoiding every pair of elements $\{x, y\}$. Let A be the matrix representation of the cycle matroid M over $GF(2)$ where

$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Let $X = \{1, 2, 5, 6\}$ and $10 = e$. Then representation of es-splitting matroid M_X^e over the field $GF(2)$ is given by the matrix

$$A_X^e = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & a & \gamma \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

Note that, by Theorem 1.2, the es-splitting matroid M_X^e is 3-connected. But if $A = \{a, e, \gamma\}$ and $B = E(M_X^e) \setminus A$, then $r'(A) + r'(B) - r'(M_X^e) = 2 + r'(B) - 8 \leq 2$. Thus (A, B) is a 3-separation of M_X^e and hence M_X^e is not 4-connected. Further, it is easy to verify that $M_X^e \setminus e$ and $M_X^e \setminus \gamma$ are 4-connected minors of the es-splitting matroid M_X^e.

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