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Multidesigns for the graph pair formed by the 6-cycle and 3-prism

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Abstract

Given two graphs G and H, a (G, H)-multidecomposition of K_n is a partition of the edges of K_n into copies of G and H such that at least one copy of each is used. We give necessary and sufficient conditions for the existence of $(C_6, \overline{C}_6)$-multidecomposition of K_n where C_6 denotes a cycle of length 6 and $\overline{C}_6$ denotes the complement of C_6. We also characterize the cardinalities of leaves and paddings of maximum $(C_6, \overline{C}_6)$-multipackings and minimum $(C_6, \overline{C}_6)$-multicoverings, respectively.

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1. Introduction

Let G and H be graphs. Denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. A *G-decomposition of H* is a partition of $E(H)$ into a set of edge-disjoint subgraphs of H each of which is isomorphic to G. Graph decompositions have been extensively studied. This is particularly true for the case where $H \cong K_n$, see [2] for a recent survey. A G-decomposition of K_n is sometimes referred to as a *G-design of order n*. As an extension of a graph decomposition we can permit more than one graph, up to isomorphism, to appear in the partition. A (G, H)-*multidecomposition of K_n* is a partition of $E(K_n)$ into a set of edge-disjoint subgraphs each of

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which is isomorphic to either G or H, and at least one copy of G and one copy of H are elements of the partition. When a (G, H)-multidecomposition of K_n does not exist, we would like to know how "close" we can get. More specifically, define a *(G, H)-multipacking of K_n* to be a collection of edge-disjoint subgraphs of K_n each of which is isomorphic to either G or H such that at least one copy of each is present. The set of edges in K_n that are not used as copies of either G or H in the (G, H)-multipacking is called the *leave* of the (G, H)-multipacking. Similarly, define a *(G, H)-multicovering of K_n* to be a partition of the multiset of edges formed by $E(K_n)$ where some edges may be repeated into edge-disjoint copies of G and H such that at least one copy of each is present. The multiset of repeated edges is called the *padding*. A (G, H)-multipacking is called *maximum* if its leave is of minimum cardinality, and a (G, H)-multicovering is called *minimum* if its padding is of minimum cardinality. The term *multidesign* is used to encompass multidecompositions, multipackings, and multicoverings.

A natural way to form a pair of graphs is to use a graph and its complement. To this end, we have the following definition which first appeared in [1]. Let G and H be edge-disjoint, non-isomorphic, spanning subgraphs of K_n each with no isolated vertices. We call (G, H) a *graph pair of order n* if $E(G) \cup E(H) = E(K_n)$. For example, the only graph pair of order 4 is (C_4, E_2), where E_2 denotes the graph consisting of two disjoint edges. Furthermore, there are exactly 5 graph pairs of order 5. In this paper we are interested in the graph pair formed by a 6-cycle, denoted C_6, and the complement of a 6-cycle, denoted $\overline{C}_6$.

Necessary and sufficient conditions for multidecompositions of complete graphs into all graph pairs of orders 4 and 5 were characterized in [1]. They also characterized the cardinalities of leaves and paddings of multipackings and multicoverings for the same graph pairs. We advance those results by solving the same problems for a graph pair of order 6, namely $(C_6, \overline{C}_6)$. Note that $\overline{C}_6$ is sometimes referred to as the 3-prism, but we used the former notation for brevity. We first address multidecompositions, then multipackings and multicoverings. Our main results are stated in the following three theorems.

Theorem 1.1. *The complete graph K_n admits a $(C_6, \overline{C}_6)$-multidecomposition of K_n if and only if $n \equiv 0, 1 \pmod 3$ with $n \geq 6$, except $n \in \{7, 9, 10\}$.*

Theorem 1.2. *For each $n \equiv 2 \pmod 3$ with $n \geq 8$, a maximum $(C_6, \overline{C}_6)$-multipacking of K_n has a leave of cardinality 1. Furthermore, a maximum $(C_6, \overline{C}_6)$-multipacking of K_7 has a leave of cardinality 6, and a maximum $(C_6, \overline{C}_6)$-multipacking of either K_9 or K_{10} has a leave of cardinality 3.*

Theorem 1.3. *For each $n \equiv 2 \pmod 3$ with $n \geq 8$, a minimum $(C_6, \overline{C}_6)$-multicovering of K_n has a padding of cardinality 2. Furthermore, a minimum $(C_6, \overline{C}_6)$-multicovering of K_7 has a padding of cardinality 6, and a minimum $(C_6, \overline{C}_6)$-multicoveirng of either K_9 or K_{10} has a padding of cardinality 2.*

Let G and H be vertex-disjoint graphs. The *join of G and H*, denoted $G \vee H$, is defined to be the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G), v \in V(H)\}$. We use the shorthand notation $\bigvee_{i=1}^{t} G_i$ to denote $G_1 \vee G_2 \vee \cdots \vee G_t$, and when $G_i \cong G$ for all $1 \leq i \leq t$ we write $\bigvee_{i=1}^{t} G$. For example, $K_{12} \cong \bigvee_{i=1}^{4} K_3$.

For notational convenience, let (a, b, c, d, e, f) denote the copy of C_6 with vertex set $\{a, b, c, d, e, f\}$ and edge set $\{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}, \{e, f\}, \{a, f\}\}$, as seen in Figure 1. Let $[a, b, c; d, e, f]$ denote the copy of $\overline{C}_6$ with vertex set $\{a, b, c, d, e, f\}$ and edge set

$$\{\{a, b\}, \{b, c\}, \{a, c\}, \{d, e\}, \{e, f\}, \{d, f\}, \{a, d\}, \{b, e\}, \{c, f\}\}.$$

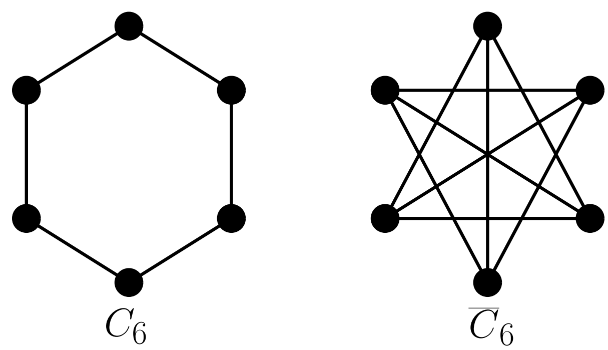

Figure 1. Labeled copies of C_6 and $\overline{C}_6$, denoted by (a, b, c, d, e, f) and $[a, e, c; d, b, f]$, respectively.

Next, we state some known results on graph decompositions that will help us prove our main result. Sotteau's theorem gives necessary and sufficient conditions for complete bipartite graphs (denoted by $K_{m,n}$ when the partite sets have cardinalities m and n) to decompose into even cycles of fixed length. Here we state the result only for cycle length 6.

Theorem 1.4 (Sotteau [5]). *A C_6-decomposition of $K_{m,n}$ exists if and only if $m \geq 4$, $n \geq 4$, m and n are both even, and 6 divides mn.*

Another celebrated result in the field of graph decompositions is that the necessary conditions for a C_k-decomposition of K_n are also sufficient. Here we state the result only for $k = 6$.

Theorem 1.5 (Šajna [4]). *Let n be a positive integer. A C_6-decomposition of K_n exists if and only if $n \equiv 1, 9 \pmod{12}$.*

The necessary and sufficient conditions for a $\overline{C}_6$-decomposition of K_n are also known, and stated in the following theorem.

Theorem 1.6 (Kang et al. [3]). *Let n be a positive integer. A $\overline{C}_6$-decomposition of K_n exists if and only if $n \equiv 1 \pmod{9}$.*

2. Multidecompositions

We first establish the necessary conditions for a $(C_6, \overline{C}_6)$-multidecomposition of K_n.

Lemma 2.1. *If a $(C_6, \overline{C}_6)$-multidecomposition of K_n exists, then*

1. $n \geq 6$, *and*
2. $n \equiv 0, 1 \pmod{3}$.

Proof. Assume that a $(C_6, \overline{C}_6)$-multidecomposition of K_n exists. It is clear that condition (1) holds. Considering that the edges of K_n are partitioned into subgraphs isomorphic to C_6 and $\overline{C}_6$, we have that there exist positive integers x and y such that $\binom{n}{2} = 6x + 9y$. Hence, 3 divides $\binom{n}{2}$, which implies $n \equiv 0, 1 \pmod 3$, and condition (2) follows. □

2.1. Small examples of multidecompositions

In this section we present various non-existence and existence results for $(C_6, \overline{C}_6)$-multidecompositions of small orders. The existence results will help with our general constructions.

2.1.1. Non-existence results

The necessary conditions for the existence of a $(C_6, \overline{C}_6)$-multidecomposition of K_n fail to be sufficient in exactly three cases, namely $n = 7, 9, 10$. We will now establish the non-existence of $(C_6, \overline{C}_6)$-multidecompositions of K_n for these cases.

Lemma 2.2. *A $(C_6, \overline{C}_6)$-multidecomposition of K_7 does not exist.*

Proof. Assume the existence of a $(C_6, \overline{C}_6)$-multidecomposition of K_7, call it $\mathcal{G}$. There must exist positive integers x and y such that $\binom{7}{2} = 21 = 6x + 9y$. The only solution to this equation is $(x, y) = (2, 1)$; therefore, $\mathcal{G}$ must contain exactly one copy of $\overline{C}_6$. However, upon examing the degree of each vertex contained in the single copy of $\overline{C}_6$ we see that there must exist a non-negative integer p such that $6 = 2p + 3$. This is a contradiction. Thus, a $(C_6, \overline{C}_6)$-multidecomposition of K_7 cannot exist. □

Lemma 2.3. *A $(C_6, \overline{C}_6)$-multidecomposition of K_9 does not exist.*

Proof. Assume the existence of a $(C_6, \overline{C}_6)$-multidecomposition of K_9, call it $\mathcal{G}$. There must exist positive integers x and y such that $\binom{9}{2} = 36 = 6x + 9y$. The only solution to this equation is $(x, y) = (3, 2)$; therefore, $\mathcal{G}$ must contain exactly two copies of $\overline{C}_6$.

Turning to the degrees of the vertices in K_9, we have that there must exist positive integers p and q such that $8 = 2p + 3q$. The only possibilities are $(p, q) \in \{(4, 0), (1, 2)\}$. Note that K_6 does not contain two edge-disjoint copies of $\overline{C}_6$. Since $\mathcal{G}$ contains exactly two copies of $\overline{C}_6$, there must exist at least one vertex $a \in V(K_9)$ that is contained in exactly one copy of $\overline{C}_6$. However, this contradicts the fact that vertex a must be contained in either 0 or 2 copies of $\overline{C}_6$. Thus, a $(C_6, \overline{C}_6)$-multidecomposition of K_9 cannot exist. □

Lemma 2.4. *A $(C_6, \overline{C}_6)$-multidecomposition of K_{10} does not exist.*

Proof. Assume the existence of a $(C_6, \overline{C}_6)$-multidecomposition of K_{10}, call it $\mathcal{G}$. There must exist positive integers x and y such that $\binom{10}{2} = 45 = 6x + 9y$. Thus, $(x, y) \in \{(6, 1), (3, 3)\}$; therefore, $\mathcal{G}$ must contain at least one copy of $\overline{C}_6$. However, if $\mathcal{G}$ consists of exactly one copy of $\overline{C}_6$, then the vertices of K_{10} which are not included in this copy would have odd degrees remaining after the removal of the copy of $\overline{C}_6$. Thus, the case where $(x, y) = (6, 1)$ is impossible.

Upon examining the degree of each vertex in K_{10}, we see that there must exist positive integers p and q such that $9 = 2p + 3q$. The only solutions to this equation are $(p, q) \in \{(3, 1), (0, 3)\}$. From the above argument, we know that $\mathcal{G}$ contains exactly 3 copies of $\overline{C}_6$, say A, B, and C. Let

$X = V(A) \cap V(B)$. It must be the case that $|X| \geq 2$ since K_{10} has 10 vertices. It also must be the case that $|X| \leq 5$ since K_6 does not contain two copies of $\overline{C}_6$. If $|X| \in \{2, 3\}$, then $V(C) \cap (V(A) \triangle V(B)) \neq \emptyset$, where $\triangle$ denotes the symmetric difference. This implies that there exists a vertex in $V(K_n)$ that is contained in exactly 2 copies of $\overline{C}_6$ in $\mathcal{G}$, which is a contradiction.

Observe that any set consisting of either 4 or 5 vertices in $\overline{C}_6$ must induce at least 3 or 6 edges, respectively. Furthermore, $X \subseteq V(C)$ due to the degree constraints put in place by the existence of $\mathcal{G}$. If $|X| = 4$ or $|X| = 5$, then X must induce at least 9 or at least 18 edges, respectively. This is a contradiction in either case. Thus, no $(C_6, \overline{C}_6)$-multidecomposition of K_{10} exists.

□

2.1.2. Existence results

We now present some multidecompositions of small orders that will be useful for our general recursive constructions.

Example 1. K_{13} admits a (C_6,$\overline{C}_6$)-multidecomposition.

Let $V(K_{13}) = \{1, 2, \ldots, 13\}$. The following is a $(C_6, \overline{\mathrm{C}}_6)$-multidecomposition of K_{13}.

$$\begin{aligned} &\big\{[1,2,3;7,9,8],[1,4,5;9,12,10],[3,4,6;7,11,10],[2,5,6;8,12,11]\big\} \\ &\cup\big\{(13,1,6,8,5,11),(13,2,4,7,6,12),(13,3,5,9,4,10),(13,7,12,3,9,6), \\ &\quad (13,8,10,2,7,5),(13,9,11,1,8,4),(1,10,3,11,2,12)\big\} \end{aligned}$$

Example 2. K_{15} admits a (C_6,$\overline{C}_6$) -multidecomposition.

Let $V(K_{15}) = \{1, 2, \ldots, 15\}$. The following is a $(C_6, \overline{\mathrm{C}}_6)$-multidecomposition of K_{15}.

$$\begin{aligned} &\big\{[1,5,10;6,8,12],[4,8,13;9,11,15],[7,11,1;12,14,3],[10,14,4;15,2,6], \\ &\quad [13,2,7;3,5,9]\big\} \\ &\cup\big\{(1,12,11,13,5,15),(4,15,14,1,8,3),(7,3,2,4,11,6),(10,6,5,7,14,9), \\ &\quad (13,9,8,10,2,12),(1,2,11,3,6,13),(4,5,14,6,9,1),(7,8,2,9,12,4), \\ &\quad (10,11,5,12,15,7),(13,14,8,15,3,10)\big\} \end{aligned}$$

Example 3. K_{19} admits a $(C_6, \overline{C}_6)$-multidecomposition.

Let $V(K_{19}) = \{1, 2, \ldots, 19\}$. The following is a $(C_6, \overline{C}_6)$-multidecomposition of K_{19}.

$$\begin{aligned} &\big\{[2,11,14;17,4,18],[3,12,15;18,5,19],[4,13,16;19,6,11],[5,14,17;11,7,12], \\ &\quad [6,15,18;12,8,13],[7,16,19;13,9,14],[8,17,11;14,10,15],[9,18,12;15,2,16], \\ &\quad [10,19,13;16,3,17]\big\} \\ &\cup\big\{(2,12,14,3,11,1),(3,13,15,4,12,1),(4,14,16,5,13,1),(5,15,17,6,14,1), \\ &\quad (6,16,18,7,15,1),(7,17,19,8,16,1),(8,18,11,9,17,1),(9,19,12,10,18,1), \\ &\quad (10,11,13,2,19,1),(2,3,10,4,9,5),(2,6,8,7,3,4),(2,7,4,5,3,8), \\ &\quad (2,10,8,4,6,9),(3,6,10,5,7,9),(5,6,7,10,9,8)\big\} \end{aligned}$$

2.2. General constructions for multidecompositions

Lemma 2.5. *If $n \equiv 0 \pmod 6$ with $n \geq 6$, then K_n admits a $(C_6, \overline{C}_6)$-multidecomposition.*

Proof. Let $n = 6x$ for some integer $x \geq 1$. Note that $K_{6x} \cong \bigvee_{i=1}^{x} K_6$. On each copy of K_6 place a $(C_6, \overline{C}_6)$-multidecomposition of K_6. The remaining edges form edge-disjoint copies of $K_{6,6}$, which admits a C_6-decomposition by Theorem 1.4. Thus, we obtain the desired $(C_6, \overline{C}_6)$-multidecomposition of K_n. □

Lemma 2.6. *If $n \equiv 1 \pmod 6$ with $n \geq 13$, then K_n admits a $(C_6, \overline{C}_6)$-multidecomposition.*

Proof. Let $n = 6x + 1$ for some integer $x \geq 2$. The proof breaks into two cases.
Case 1: $x = 2k$ for some integer $k \geq 1$. Notice that $K_{12k+1} \cong K_1 \vee \left(\bigvee_{i=1}^{k} K_{12}\right)$. Each of the k copies of K_{13} formed by $K_1 \vee K_{12}$ admits a $(C_6, \overline{C}_6)$-multidecomposition by Example 1. The remaining edges form edge-disjoint copies of $K_{12,12}$, which admits a C_6-decomposition by Theorem 1.4. Thus, we obtain the desired $(C_6, \overline{C}_6)$-multidecomposition of K_n.
Case 2: $x = 2k + 1$ for some integer $k \geq 2$. Notice that $K_{12k+7} \cong K_1 \vee K_6 \vee \left(\bigvee_{i=1}^{k} K_{12}\right)$. The single copy of K_{19} formed by $K_1 \vee K_6 \vee K_{12}$ admits a $(C_6, \overline{C}_6)$-multidecomposition by Example 3. The remaining $k-1$ copies of K_{13} formed by $K_1 \vee K_{12}$ each admit a $(C_6, \overline{C}_6)$-multidecomposition by Example 1. The remaining edges form edge-disjoint copies of either $K_{6,12}$ or $K_{12,12}$. Both of these graphs admit C_6-decompositions by Theorem 1.4. Thus, we obtain the desired $(C_6, \overline{C}_6)$-multidecomposition of K_n. □

Lemma 2.7. *If $n \equiv 3 \pmod 6$ with $n \geq 15$, then K_n admits a $(C_6, \overline{C}_6)$ -multidecomposition.*

Proof. Let $n = 6x + 3$ for some integer $x \geq 2$. The proof breaks into two cases.
Case 1: $x = 2k$ for some integer $k \geq 1$. Notice that $K_{12k+3} \cong K_1 \vee K_{14} \vee \left(\bigvee_{i=1}^{k-1} K_{12}\right)$. The remainder of the proof is similar to the proof of Case 1 of Lemma 2.6 where the ingredients required are C_6-decompositions of $K_{12,12}$, and $K_{12,14}$, as well as $(C_6, \overline{C}_6)$-multidecompositions of K_{13} and K_{15}.
Case 2: $x = 2k + 1$ for some integer $k \geq 1$. Notice that $K_{12k+9} \cong K_1 \vee K_8 \vee \left(\bigvee_{i=1}^{k} K_{12}\right)$. The remainder of the proof is similar to the proof of Case 2 of Lemma 2.6 where the ingredients required are C_6-decompositions of K_9 (which exists by Theorem 1.5), $K_{8,12}$, and $K_{12,12}$, as well as a $(C_6, \overline{C}_6)$-multidecomposition of K_{13}.

□

Lemma 2.8. *If $n \equiv 4 \pmod 6$ with $n \geq 16$, then K_n admits a $(C_6, \overline{C}_6)$ -multidecomposition.*

Proof. Let $n = 6x + 4$ where $x \geq 2$ is an integer. Note that $K_{6x+4} \cong K_{10} \vee \left(\bigvee_{i=1}^{x-1} K_6\right)$. The remainder of the proof is similar to the proof of Case 2 of Lemma 2.6 where the ingredients required are C_6-decompositions of $K_{6,6}$ and $K_{6,10}$, a $\overline{C}_6$-decomposition of K_{10} (which exists by Theorem 1.6), as well as a $(C_6,\overline{C}_6)$-multidecomposition of K_6.

□

Combining Lemmas 2.5, 2.6, 2.7, and 2.8, we have proven Theorem 1.1.

3. Maximum Multipackings

Now we turn our attention to $(C_6, \overline{C}_6)$-multipackings in the cases where $(C_6, \overline{C}_6)$- multidecompositions do not exist.

3.1. Small examples of maximum multipackings

Example 4. A maximum $(C_6, \overline{C}_6)$-multipacking of K_7 has a leave of cardinality 6.

Note that the number of edges used in a $(C_6, \overline{C}_6)$-multipacking of any graph must be a multiple of 3, since $\gcd(6, 9) = 3$. Since no $(C_6, \overline{C}_6)$-multidecomposition of K_7 exists the next possibility is a leave of cardinality 3. However, the equation $18 = 6x + 9y$ has no positive integer solutions. Thus, the minimum possible cardinality of a leave is 6. Let $V(K_7) = \{1, ..., 7\}$. The following is a $(C_6, \overline{C}_6)$-multipacking of K_7, with leave $\{\{1, 7\}, \{2, 7\}, \{3, 7\}, \{4, 7\}, \{5, 7\}, \{6, 7\}\}$.

$$\{[1, 3, 5; 4, 6, 2], (1, 2, 3, 4, 5, 6)\}$$

Example 5. A maximum $(C_6, \overline{C}_6)$-multipacking of K_8 has a leave of cardinality 1.

Let $V(K_8) = \{1, ..., 8\}$. The following is a $(C_6, \overline{C}_6)$-multipacking of K_8, with leave $\{3, 6\}$.

$$\{[2, 5, 7; 4, 1, 8], (1, 2, 3, 4, 5, 6), (1, 3, 5, 8, 6, 7), (3, 8, 2, 6, 4, 7)\}$$

Example 6. A maximum $(C_6, \overline{C}_6)$-multipacking of K_9 has a leave of cardinality 3.

Let $V(K_9) = \{1, ..., 9\}$. The following is a $(C_6, \overline{C}_6)$-multipacking of K_9, with leave $\{\{2, 4\}, \{2, 9\}, \{4, 9\}\}$.

$$\{[1, 2, 3; 6, 5, 4], [1, 4, 7; 9, 8, 3], [2, 6, 8; 7, 9, 5], (1, 5, 3, 6, 7, 8)\}$$

Example 7. A maximum $(C_6, \overline{C}_6)$-multipacking of K_{10} has a leave of cardinality 3.

A $(C_6, \overline{C}_6)$-multipacking of K_{10} with a leave of cardinality 3 can be obtained by starting with a $\overline{C}_6$-decomposition of K_{10}. Then remove three vertex-disjoint edges from one copy of $\overline{C}_6$, forming a C_6. This gives us the desired $(C_6, \overline{C}_6)$-multipacking of K_{10} where the three removed edges form the leave.

Example 8. A maximum $(C_6, \overline{C}_6)$-multipacking of K_{11} has a leave of cardinality 1.

Let $V(K_{11}) = \{1, ..., 11\}$. The following is a $(C_6, \overline{C}_6)$-multipacking of K_{11}, with leave $\{1, 2\}$.

$$\{[1, 7, 10; 9, 6, 3], [1, 5, 6; 4, 10, 2], [2, 5, 7; 11, 8, 4], [1, 3, 11; 8, 2, 9]\}$$
$$\cup\{(3, 4, 9, 10, 6, 8), (4, 5, 9, 7, 11, 6), (3, 5, 11, 10, 8, 7)\}$$

Example 9. A maximum $(C_6, \overline{C}_6)$-multipacking of K_{17} has a leave of cardinality 1.

Let $V(K_{17}) = \{1, ..., 17\}$. The following is a $(C_6, \overline{C}_6)$-multipacking of K_{17}, with leave $\{1, 10\}$.

$$\big\{[2,3,5;7,8,1],[3,6,4;9,8,10],[2,4,9;6,5,7]\big\}$$
$$\cup\big\{(2,12,5,10,11,14),(2,10,17,4,13,11),(4,7,13,14,5,15),(4,11,15,8,16,12),$$
$$(1,15,14,16,5,17),(3,12,11,17,6,15),(1,2,16,7,14,4),(2,13,5,8,14,17),$$
$$(7,15,10,13,9,17),(1,13,6,9,11,16),(1,9,12,7,3,11),(3,10,12,8,4,16),$$
$$(3,13,16,6,12,14),(2,8,13,17,12,15),(6,10,16,15,9,14),(5,9,16,17,8,11),$$
$$(1,3,17,15,13,12),(1,6,11,7,10,14)\big\}$$

3.2. General Constructions of maximum multipackings

Lemma 3.1. *If $n \equiv 2 \pmod 6$ with $n \geq 14$, then K_n admits a $(C_6, \overline{C}_6)$-multipacking with leave cardinality 1.*

Proof. Let $n = 6x + 2$ for some integer $x \geq 2$. Notice that $K_{6x+2} \cong K_2 \vee \big(\bigvee_{i=1}^{x} K_6\big)$. Let $\{u, v\} = V(K_2)$. Each of the x copies of K_8 formed by $K_2 \vee K_6$ admit a $(C_6, \overline{C}_6)$-multipacking with leave cardinality 1 by Example 5. Note that we can always choose the leave edge to be $\{u, v\}$ in each of these multipackings. The remaining edges form edge disjoint copies of $K_{6,6}$, each of which admits a C_6-decomposition by Theorem 1.4. Thus, we obtain the desired $(C_6, \overline{C}_6)$-multipacking of K_n.

□

Lemma 3.2. *If $n \equiv 5 \pmod 6$ with $n \geq 11$, then K_n admits a $(C_6, \overline{C}_6)$-multipacking with leave cardinality 1.*

Proof. Let $n = 6x + 5$ for some integer $x \geq 1$.

Case 1: $x = 2k$ for some integer $k \geq 1$. Notice that $K_{12k+5} \cong K_1 \vee K_{16} \vee \big(\bigvee_{i=1}^{k-1} K_{12}\big)$. Each of the $k-1$ copies of K_{13} formed by $K_1 \vee K_{12}$ admit a $(C_6, \overline{C}_6)$-multidecomposition by Example 1. The copy of K_{17} formed by $K_1 \vee K_{16}$ admits a $(C_6, \overline{C}_6)$-multipacking with leave of cardinality 1 by Example 9. The remaining edges form edge disjoint copies of $K_{12,12}$ or $K_{12,16}$, each of which admits a C_6-decomposition by Theorem 1.4. Thus, we obtain the desired $(C_6, \overline{C}_6)$-multipacking of K_n.

Case 2: $x = 2k + 1$ for some integer $k \geq 1$. Notice that $K_{12k+11} \cong K_1 \vee K_{10} \vee \big(\bigvee_{i=1}^{k} K_{12}\big)$. On each of the k copies of K_{13} formed by $K_1 \vee K_{12}$ admit a $(C_6, \overline{C}_6)$-multidecomposition by Example 1. The copy of K_{11} formed by $K_1 \vee K_{10}$ admits a $(C_6, \overline{C}_6)$-multipacking with leave of cardinality 1 by Example 8. The remaining edges form edge disjoint copies of $K_{12,12}$ or $K_{10,12}$, each of which admits a C_6-decomposition by Theorem 1.4. Thus, we obtain the desired $(C_6, \overline{C}_6)$-multipacking of K_n. □

Combining Lemmas 3.1 and 3.2 along with Examples 4, 6, and 7 we have proven Theorem 1.2.

4. Minimum Multicoverings

Now we turn our attention to minimum $(C_6, \overline{C}_6)$-multicoverings in the cases where $(C_6, \overline{C}_6)$-multidecompositions do not exist.

4.1. Small examples of minimum multicoverings

Example 10. A minimum $(C_6, \overline{C}_6)$-multicovering of K_7 has a padding of cardinality 6.

We first rule out the possibility of a minimum $(C_6, \overline{C}_6)$-multicovering of K_7 with a padding of cardinality 3. The only positive integer solution to the equation $24 = 6x + 9y$ is $(x, y) = (1, 2)$. In such a covering there would be one vertex left out of one of the copies of $\overline{C}_6$. It would be impossible to use all edges at this vertex with the remaining copies of C_6 and $\overline{C}_6$. Thus, the best possible cardinality of a padding is 6. Let $V(K_7) = \{1, ..., 7\}$. The following is a minimum $(C_6, \overline{C}_6)$-multicovering of K_7, with padding of $\{\{1, 2\}, \{1, 5\}, \{1, 6\}, \{3, 6\}, \{4, 5\}, \{5, 6\}\}$.

$$\{[1, 2, 3; 6, 5, 4], (1, 4, 7, 6, 3, 5), (1, 6, 2, 4, 5, 7), (1, 2, 7, 3, 6, 5)\}$$

Example 11. A minimum $(C_6, \overline{C}_6)$-multicovering of K_8 has a padding of cardinality 2.

Let $V(K_8) = \{1, ..., 8\}$. The following is a minimum $(C_6, \overline{C}_6)$-multicovering of K_8, with padding of $\{\{1, 8\}, \{3, 5\}\}$.

$$\{[1, 2, 8; 4, 3, 5], [1, 5, 6; 3, 7, 8], (1, 7, 2, 6, 4, 8), (2, 4, 7, 6, 3, 5)\}$$

Example 12. A minimum $(C_6, \overline{C}_6)$-multicovering of K_9 has a padding of cardinality 3.

A $(C_6, \overline{C}_6)$-multicovering of K_9 with a padding of cardinality 3 can be obtained by starting with a C_6-decomposition of K_9 which exists by Theorem 1.5. One of the copies of C_6 contained in this decomposition can be transformed into a copy of $\overline{C}_6$ by carefully adding 3 edges. This gives us the desired $(C_6, \overline{C}_6)$-multicovering of K_9 where the three added edges form the padding.

Example 13. A minimum $(C_6, \overline{C}_6)$-multicovering of K_{10} has a padding of cardinality 3.

A $(C_6, \overline{C}_6)$-multicovering of K_{10} with a padding of cardinality 3 can be obtained by starting with a $\overline{C}_6$-decomposition of K_{10}. One copy of $\overline{C}_6$ can be transformed into two copies of C_6 by carefully adding three edges. This gives us the desired $(C_6, \overline{C}_6)$-multicovering of K_{10} where the three added edges form the padding.

Example 14. A minimum $(C_6, \overline{C}_6)$-multicovering of K_{11} has a padding of cardinality 2.

Let $V(K_{11}) = \{1, ..., 11\}$. The following is a minimum $(C_6, \overline{C}_6)$-multicovering of K_{11}, with padding of $\{\{3, 4\}, \{8, 11\}\}$.

$$\begin{aligned}&\{[1, 2, 11; 6, 5, 7], [1, 3, 5; 10, 2, 9], [4, 6, 10; 7, 9, 8]\}\\ &\quad\cup\big\{(3, 4, 5, 8, 11, 6), (1, 8, 2, 7, 3, 9), (2, 4, 9, 11, 8, 6), (1, 4, 3, 11, 10, 7),\\ &\qquad(3, 8, 4, 11, 5, 10)\big\}\end{aligned}$$

Example 15. A minimum $(C_6, \overline{C}_6)$-multicovering of K_{17} has a padding of cardinality 2.

Let $V(K_{17}) = \{1, ..., 17\}$. Apply Theorem 1.5 and let $\mathcal{B}_1$ be a C_6-decomposition on the copy of K_9 formed by the subgraph induced by the vertices $\{9, \ldots, 17\}$. Apply Theorem 1.4 and let $\mathcal{B}_2$ be a C_6-decomposition of the copy of $K_{6,8}$ formed by the subgraph of K_{17} with vertex bipartition

(A, B) where $A = \{1, \ldots, 8\}$ and $B = \{12, \ldots, 17\}$. The following is a minimum $(C_6, \overline{C}_6)$-multicovering of K_{17}, with padding of $\{\{3,5\},\{7,8\}\}$.

$$\begin{aligned}&\mathcal{B}_1 \cup \mathcal{B}_2 \cup \{[1,2,3;6,5,4],[1,4,8;7,2,6]\}\\ &\cup\big\{(1,5,7,8,3,9),(1,10,3,7,4,11),(2,8,7,11,6,9)\big\}\\ &\cup\big\{(5,11,8,9,7,10),(3,5,9,4,10,6),(2,11,3,5,8,10)\big\}\end{aligned}$$

4.2. General constructions of minimum multicoverings

Lemma 4.1. *If $n \equiv 2 \pmod 6$ with $n \geq 8$, then K_n admits a minimum $(C_6, \overline{C}_6)$-multicovering with a padding of cardinality 2.*

Proof. Let $n = 6x + 2$ for some integer $x \geq 1$. Notice that $K_{6x+2} \cong K_8 \vee \left(\bigvee_{i=1}^{x-1} K_6\right)$. Each of the $x - 1$ copies of K_6 admit a $(C_6, \overline{C}_6)$-multidecomposition by Lemma 2.5. The copy of K_8 admits a $(C_6, \overline{C}_6)$-multicovering with a padding of cardinality 2 by Example 11. The remaining edges form edge disjoint copies of $K_{6,6}$ or $K_{6,8}$, each of which admit a C_6-decomposition by Theorem 1.4. Thus, we obtain the desired $(C_6, \overline{C}_6)$-multicovering of K_n. □

Lemma 4.2. *If $n \equiv 5 \pmod 6$ with $n \geq 11$, then K_n admits a minimum $(C_6, \overline{C}_6)$- multicovering with a padding of cardinality 2.*

Proof. Let $n = 6x + 5$ for some integer $x \geq 1$. The proof breaks into two cases.
Case 1: $x = 2k$ for some integer $k \geq 1$. Notice that $K_{12k+5} \cong K_1 \vee K_4 \vee \left(\bigvee_{i=1}^{k} K_{12}\right)$. One copy of K_{17} is formed by $K_1 \vee K_4 \vee K_{12}$, and admits a $(C_6, \overline{C}_6)$-multicovering with a padding of cardinality 2 by Example 15. The $k - 1$ copies of K_{13} formed by $K_1 \vee K_{12}$ admit a $(C_6, \overline{C}_6)$-multidecomposition by Example 1. The remaining edges form edge disjoint copies of $K_{12,12}$ or $K_{4,12}$, each of which admits a C_6-decomposition by Theorem 1.4. Thus, we obtain the desired $(C_6, \overline{C}_6)$-multicovering of K_n.
Case 2: $x = 2k + 1$ for some integer $k \geq 1$. Notice that $K_{12k+11} \cong K_1 \vee K_4 \vee K_6 \vee \left(\bigvee_{i=1}^{k} K_{12}\right)$. One copy of K_{11} is formed by $K_1 \vee K_4 \vee K_6$, and admits a $(C_6, \overline{C}_6)$-multicovering with a padding of cardinality 2 by Example 14. The k copies of K_{13} formed by $K_1 \vee K_{12}$ admit a $(C_6, \overline{C}_6)$-multidecomposition by Example 1. The remaining edges form edge disjoint copies of $K_{12,12}$, $K_{4,12}$, or $K_{6,12}$, each of which admits a C_6-decomposition by Theorem 1.4. Thus, we obtain the desired $(C_6, \overline{C}_6)$-multicovering of K_n. □

Combining Lemmas 4.1 and 4.2, we have proven Theorem 1.3.

5. Conclusion

The cardinalities of the leaves of maximum $(C_6, \overline{C}_6)$-multipackings and paddings of minimum $(C_6, \overline{C}_6)$-multicoverings of K_n have been characterized. However, the achievable structures of these leaves and paddings are still yet to be characterized. This leads to the following open question.

Open Problem 1. *For each positive integer n, characterize all possible graphs (multigraphs) which are leaves (paddings) of a $(C_6, \overline{C}_6)$-multipacking (multicovering) of K_n.*

Furthermore, it would be of interest to know when a $(C_6, \overline{C}_6)$-multidecomposition of K_n exists with p copies of C_6 and q copies of $\overline{C}_6$ where (p, q) is any solution to the equation $6p + 9q = \binom{n}{2}$. This leads to the following open problem.

Open Problem 2. *Let p, q and n be positive integers for which $6p + 9q = \binom{n}{2}$. Determine whether a $(C_6, \overline{C}_6)$-multidecomposition of K_n exists with p copies of C_6 and q copies of $\overline{C}_6$.*

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