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A Study on Topological Integer Additive Set-Labeling of Graphs

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Abstract

A set-labeling of a graph G is an injective function $f : V(G) \to \mathcal{P}(X)$ such that the induced function $f^{\oplus} : E(G) \to \mathcal{P}(X) - \{\emptyset\}$ defined by $f^{\oplus}(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$, where X is a non-empty finite set and $\mathcal{P}(X)$ be its power set. A set-indexer of G is a set-labeling such that the induced function $f^{\oplus}$ is also injective. A set-indexer $f : V(G) \to \mathcal{P}(X)$ of a given graph G is called a topological set-labeling of G if $f(V(G))$ is a topology of X. An integer additive set-labeling is an injective function $f : V(G) \to \mathcal{P}(\mathbb{N}_0)$, whose associated function $f^+ : E(G) \to \mathcal{P}(\mathbb{N}_0)$ is defined by $f^+(uv) = f(u) + f(v), uv \in E(G)$, where $\mathbb{N}_0$ is the set of all non-negative integers. An integer additive set-indexer is an integer additive set-labeling such that the induced function $f^+ : E(G) \to \mathcal{P}(\mathbb{N}_0)$ is also injective. In this paper, we extend the concepts of topological set-labeling of graphs to topological integer additive set-labeling of graphs.

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1. Introduction

For all terms and definitions of graphs and graph classes, not defined specifically in this paper, we refer to [7], [8], [13] and [24] and for graph labeling concepts, we refer to [10]. For terms

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and definitions in topology, we further refer to [9], [14] and [16]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.

Research on graph labeling commenced with the introduction of β-valuations of graphs in [18]. Analogous to the number valuations of graphs, the concepts of set-assignments, set-labelings and set-indexers of graphs are introduced in [2] as follows.

Let $G(V, E)$ be a given graph. Let X be a non-empty set and $\mathcal{P}(X)$ be its power sets. Then, the set-valued function $f : V(G) \to \mathcal{P}(X)$ is called the *set-assignment* of vertices of G respectively. In a similar way, we can define a set assignment of edges of G as a function $g : E(G) \to \mathcal{P}(Y)$ and a set assignment of elements (both vertices and edges) of G as a function $h : V(G) \cup E(G) \to \mathcal{P}(Z)$, where Y and Z are non-empty sets. The term set assignment is used for set-assignment of vertices unless mentioned otherwise.

A set-assignment of a graph G is said to be a *set-labeling* or a *set-valuation* of G if it is injective. A graph with a set-labeling $f : V(G) \to \mathcal{P}(X)$ is denoted by (G, f) and is referred to as a *set-labeled graph* or a *set-valued graph*.

For a graph $G(V, E)$ and a non-empty set X of cardinality n, a *set-indexer* of G is defined as an injective set-valued function $f : V(G) \to \mathcal{P}(X)$ such that the function $f^{\oplus} : E(G) \to \mathcal{P}(X) - \{\emptyset\}$ defined by $f^{\oplus}(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$ is also injective, where $\mathcal{P}(X)$ is the set of all subsets of X and $\oplus$ is the symmetric difference of sets. A graph that admits a set-indexer is called a *set-indexed graph*. It is proved in [2] that every non-empty graph has a set-indexer.

More studies on set-labeled and set-indexed graphs have been done in [1], [2], [3] and [4]. Then, the notion of topological set-labeling of a graph is defined in [5] as follows.

Let G be a graph and let X be a non-empty set. A set-labeling $f : V(G) \to \mathcal{P}(X)$ is called a *topological set-labeling* of G if $f(V(G))$ is a topology of X. A graph G which admits a topological set-labeling is called a *topologically set-labeled graph*. More studies on topological set-labeling of different graphs have been done subsequently.

The *sumset* of two non-empty sets A and B, denoted by $A + B$, is the set defined by $A + B = \{a + b : a \in A, b \in B\}$. For every non-empty set A, we have $A + \{0\} = A$. Hence, $\{0\}$ and A are said to be the *trivial summands* of the set A. If $C = A + B$, where A and B are non-trivial summands of C, then C is said to be the *non-trivial sumset* of A and B. In this paper, by the terms sumsets and summands, we mean non-trivial sumsets and non-trivial summands respectively.

If any either A or B is countably infinite, then their sumset $A + B$ will also be a countably infinite set. Hence, all sets mentioned in this paper are finite sets.We denote the cardinality of a set A by $|A|$ and the power set of a set A by $\mathcal{P}(A)$. We also denote, by X, the finite ground set of non-negative integers that is used for set-labeling the elements of G.

Using the terminology and concepts of sumset theory, a particular type of set-labeling, called integer additive set-labeling, was introduced as follows.

Let $\mathbb{N}_0$ be the set of all non-negative integers. An *integer additive set-labeling* (IASL, in short) is an injective function $f : V(G) \to \mathcal{P}(\mathbb{N}_0)$ such that the associated function $f^+ : E(G) \to \mathcal{P}(X)$ is defined by $f^+(uv) = f(u) + f(v)$ for any two adjacent vertices u and v of G. A graph G which admits an IASL is called an IASL-graph.

An *integer additive set-labeling* f is an integer additive set-indexer (IASI, in short) if the induced function $f^+ : E(G) \to \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is injective (see [11], [12]). A graph G which admits an IASI is called an IASI-graph.

Cardinality of the set-label of an element (a vertex or an edge) of a graph G is called the *set-indexing number* of that element. An IASL (or an IASI) is said to be a k-uniform IASL (or k-uniform IASI) if $|f^+(e)| = k \ \forall \ e \in E(G)$. The vertex set $V(G)$ is called *l-uniformly set-indexed*, if all the vertices of G have the set-indexing number l (see [21],[22]).

Motivated by the studies on topological set-labeling of graphs (see [6], [17] and [23]), we introduce the notion of topological integer additive set-labeling of graphs and study the structural properties and characteristics of the graphs which admit this type of set-labeling.

2. Topological IASL-Graphs

Note that no vertex of a given graph G has the empty set $\emptyset$ as its set-labeling with respect to a given integer additive set-labeling. Hence, in this paper, we consider only non-empty subsets of the ground set X for set-labeling the elements of G.

Analogous to topological set-labeling of graphs, we introduce the notion of topological integer additive set-labeling of certain graphs as follows.

Definition 2.1. Let G be a graph and let X be a non-empty set of non-negative integers. An integer additive set-labeling $f : V(G) \to \mathcal{P}(X) - \{\emptyset\}$ is called a *topological integer additive set-labeling* (TIASL, in short) of G if $f(V(G)) \cup \{\emptyset\}$ is a topology of X. A graph G which admits a topological integer additive set-labeling is called a *topological integer additive set-labeled graph* (in short, TIASL-graph).

The notion of a topological integer additive set-indexer of a given graph G is introduced as follows.

Definition 2.2. A topological integer additive set-labeling f is called a *topological integer additive set-indexer* (TIASI, in short) if the associated function $f^+ : E(G) \to \mathcal{P}(X)$ defined by $f^+(uv) = f(u) + f(v);\ u, v \in V(G)$, is also injective. A graph G which admits an integer additive set-graceful indexer is called an *topological integer additive set-indexed graph* (TIASI-graph, in short).

For more about topologies of finite sets, we further refer to [15], [19] and [20]. Based on the above notions, we have the following remark.

Remark 2.1. For a finite set X of non-negative integers, let the given function $f : V(G) \to \mathcal{P}(X) - \{\emptyset\}$ be an integer additive set-labeling on a graph G. Since the set-label of every edge uv is the sumset of the sets $f(u)$ and $f(v)$, it can be observed that $\{0\}$ can not be the set-label of any edge of G. More over, since f is a TIASL defined on G, X must be the set-label of some vertex, say u, of G and hence the set $\{0\}$ will be the set-label of a vertex, say v, and the vertices u and v are adjacent in G.

Let f be a topological integer additive set-indexer of a given graph G with respect to a non-empty finite ground set X. Then, $\mathcal{T} = f(V(G)) \cup \{\emptyset\}$ is a topology on X. Then, the graph G is said to be a *f-graphical realisation* (or simply *f-realisation*) of $\mathcal{T}$. The elements of the sets $f(V)$ are called *f-open sets* in G.

An interesting question that arises in this context is about the existence of an f-graphical realisation for a topology $\mathcal{T}$ of a given non-empty set X. Existence of graphical realisations for certain topologies of a given set X is established in the following theorem.

Theorem 2.1. *Let X be a non-empty finite set of non-negative integers. A topology $\mathcal{T}$ of X, consisting of the set $\{0\}$ is graphically realisable.*

Proof. Let X be a non-empty finite set of non-negative integers and let $0 \in X$. Consider a topology $\mathcal{T}$ of X consisting of the set $\{0\}$. We need to construct a graph G such that the vertices of G have the (non-empty) set-labels taken from $\mathcal{T}$ in an injective manner. Let us proceed in this direction as explained below. Take a star graph $K_{1,|\mathcal{T}|-2}$. Label its central vertex by $\{0\}$ and label the other vertices by the remaining $|\mathcal{T}| - 2$ non-empty open sets in $\mathcal{T}$. Clearly, this labeling is a TIASL defined on the graph $K_{1,|\mathcal{T}|-2}$ and hence $K_{1,|\mathcal{T}|-2}$ is a graphical realisation of $\mathcal{T}$. □

It can also be observed that if we join two vertices u and v of the above mentioned TIASL-graph $K_{1,|\mathcal{T}|-2}$ by an edge, subject to the condition that $f(u) + f(v) \subseteq X$, the resultant graph will also be a graphical realisation of $\mathcal{T}$. Hence, there may exist more than one graphical realisations for a given topology of X. In view of this fact, we have to address the questions regarding the structural properties of TIASL-graphs. Hence, we proceed to find out the structural properties of TIASL-graphs.

Proposition 2.1. *If $f : V(G) \to \mathcal{P}(X) - \{\emptyset\}$ is a TIASL of a graph G, then G must have at least one pendant vertex.*

Proof. Let f be a TIASL defined on a graph G. Then, clearly $X \in f(V)$. That is, for some vertex $v \in V(G),\ f(v) = X$. Then, by Remark 2.1, v is adjacent to a vertex whose set-label is $\{0\}$. Now we claim that, the vertex v can be adjacent to only one vertex that has the set-label $\{0\}$. This can be proved as follows.

Let u be a vertex that is adjacent to the vertex v and let a be a non-zero element of X. Also, let l be the maximal element of X. If possible, let $a \in f(u)$. Then, the element $a + l \in f^+(uv)$ and is greater than l, which leads to a contradiction to the fact that $f^+(uv) = f(u) + f(v) \subseteq X$, as f is an IASL of G. Therefore, the vertex of G having the set-label X can be adjacent to a unique vertex that has the set-label $\{0\}$. That is, the vertex v with $f(v) = X$ is definitely a pendant vertex of G. Any TIASL-graph G has at least one pendant vertex. □

Figure 1 depicts the TIASL, say f, of a graph G, with respect to a ground set $X = \{0, 1, 2, 3, 4\}$ and a topology $\mathcal{T} = \{\emptyset, X, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$ of X, where $f(V(G)) = \mathcal{T} - \{\emptyset\}$ is the collection of the set-labels of the vertices in G.

An interesting question in this context is about the number of pendant vertices required in a TIASL-graph. Clearly, the answer to this question depends on the ground set X and the topology $\mathcal{T}$ of X we choose for labeling the vertices of G. Our next objective is to determine the minimum number of pendant vertices required in a TIASL-graph.

Proposition 2.2. *Let $f : V(G) \to \mathcal{P}(X) - \{\emptyset\}$ is a TIASL of a graph G. Then, the vertices whose set-labels containing the maximal element of the ground set X are pendant vertices which are adjacent to the vertex having the set-label $\{0\}$.*

Proof. For given ground set X of non-negative integers, let $f : V(G) \to \mathcal{P}(X) - \emptyset$ be a TIASL of G. Let l be the maximal element of the ground set X. Let v be a vertex of G whose set-label contains the element $l \in X$. Let u be an adjacent vertex of v whose set-label contains a

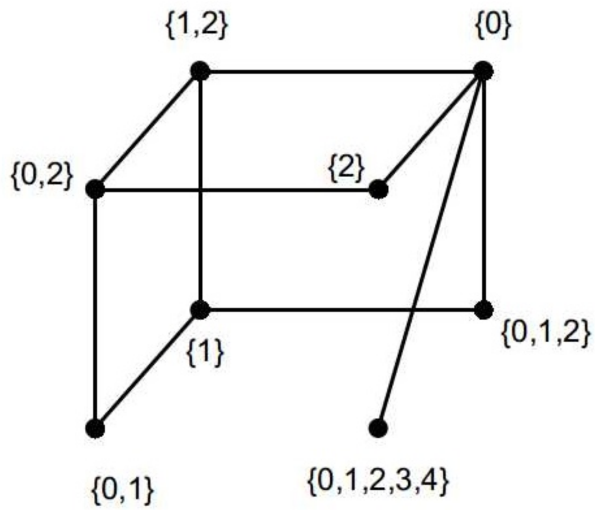

Figure 1. An example to a TIASL-graph.

non-zero element $b \in X$. Then, $b + l \notin X$, contradicting the fact that f is an IASL of G. If $l \in f(v)$ for $v \in V(G)$, then its adjacent vertices can have a set-label $\{0\}$. That is, all the vertices whose set-labels contain the maximal element of the ground set X must be adjacent to a unique vertex whose set-label is $\{0\}$. $\square$

Invoking Proposition 2.1 and Proposition 2.2, we have

Proposition 2.3. *Let X be the ground set and $\mathcal{T}$ be the topology of X which are used for set-labeling the vertices of a TIASL-graph G. Then, an element x_r in X can be an element of the set-label $f(v)$ of a vertex v of G if and only if $x_r + x_s \leq l$, where x_s is any element of the set-label of another vertex u which is adjacent to v in G and l is the maximal element in X.*

The following result is an immediate consequence of the above propositions.

Proposition 2.4. *If G has only one pendant vertex and if G admits a TIASL, then X is the only set-label of the vertices of G containing the maximal element of X.*

What is the minimum number of pendant vertices required for a graph which admits a TIASL with respect to a given topology $\mathcal{T}$ of the ground set X? The following proposition provides a solution to this question.

Proposition 2.5. *Let $\mathcal{T}$ be a given topology of the ground set X. Then,*

(i) the minimum number of pendant edges incident on a particular vertex of a TIASL-graph is equal to the number of f-open sets in $f(V(G))$ containing the maximal element of the ground set X
(ii) the minimum number of pendant vertices of a TIASL-graph G is the number of f-open sets in $\mathcal{T}$, each of which is the non-trivial summand of at most one f-open set in $\mathcal{T}$.

Proof. Let G be a graph which admits a TIASL f with respect to a topology $\mathcal{T}$ of the ground set X.
Case (i): If an f-open set X_i contains the maximal element of X, then by Proposition 2.2, X_i can be the set-label of a pendant vertex, say v_i, which is adjacent to the vertex having set-label $\{0\}$. Hence, every f-open set containing the maximal element of X must be the set-label of a pendant vertex that is adjacent to a single vertex whose set-label is $\{0\}$. Therefore, the minimum number of pendant edges incident on a single vertex is the number of f-open sets in $\mathcal{T}$ containing the maximal element of X.
Case (ii): If an f-open set X_i is not a non-trivial summand of any f-open sets in $\mathcal{T}$, then the vertex with set-label X_i can be adjacent only to the vertex with set-label $\{0\}$. If X_i is the non-trivial summand of exactly one f-open set in $\mathcal{T}$, then the vertex v_i with the set-label X_i can be adjacent only to one vertex say v_j with set-label X_j, where $X_i + X_j \subseteq X$. If X_i is the non-trivial summand of more than one f-open sets in $\mathcal{T}$, then the vertex with set-label X_i can be adjacent to more than one vertex of G and hence v_i need not be a pendant vertex. Therefore, the minimum number of pendant vertices in G is the number of f-open sets in $\mathcal{T}$, each of which is the non-trivial summand of at most one f-open set in $\mathcal{T}$. □

Does every graph with one pendant vertex admit a TIASL? The answer to this question depends upon the choice of the ground set X. Hence, let us verify the existence of TIASL for certain standard graphs having pendant vertices by choosing a ground set X suitably. For this, first consider the following graphs.

Let G be a graph on n vertices and let P_m be a path that has no common vertex with G. We call the graph obtained by identifying one vertex of G and one end vertex of P_m an (n, m)-*ladle*.

If G is a cycle C_n, then this ladle graph is called an *(n,m)-tadpole graph* or a *dragon graph*. If $m = 1$ in a tadpole graph, then G is called an n-*pan*.

If G is a complete graph on n vertices, then the corresponding (n, m)-ladle graph is called an (n, m)-*shovel*.

Now, we proceed to discuss the admissibility of TIASL by these types of graphs. The following result establishes the admissibility of TIASL by a pan graph.

Proposition 2.6. *A pan graph admits a topological integer additive set-labeling.*

Proof. Let G be an m-pan graph. Let v be the pendant vertex and $v_1, v_2, \ldots, v_n$ be the vertices of C_n. Without loss of generality, let v_1 be the unique vertex adjacent to v in G. Label the vertices of the cycle C_n of G in such a way that we have $f(v_1) = \{0\}, f(v_i) = \{0, 1, \ldots, i-1\} : 2 \leq i \leq n$. Now, let $X = \{0, 1, 2, 3, \ldots, m\}$, where $m \geq 2n - 3$ and label the pendant vertex v by the set X. Hence, the collection of the set-labels of the vertices of G is $\mathcal{A} = \{\{0\}, \{0, 1\}, \{0, 1, 2\}, \ldots, \{0, 1, 2, \ldots, n-1\}, X\}$. Clearly, the set $\mathcal{T} = \mathcal{A} \cup \{\emptyset\}$ is a topology on X. Therefore, this labeling of G is a TIASL of G. Hence, the n-pan G admits a TIASL. □

Figure 2 illustrates the admissibility of TIASL by an n-pan with respect to the ground set $X = \{0, 1, 2, 3, \ldots, 2n-3\}$.

We now proceed to verify the admissibility of TIASL by the general tadpole graphs.

Proposition 2.7. *A tadpole graph admits a topological integer additive set-labeling.*

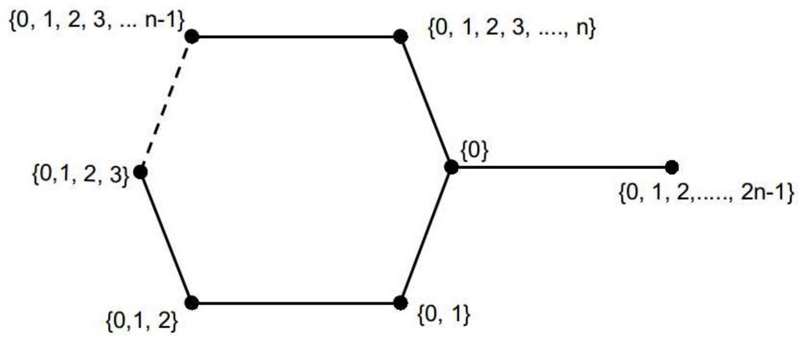

Figure 2. An n-pan graph with a TIASL defined on it.

Proof. Let G be an (n, m)-tadpole graph. Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be the vertex set of C_n and let $\{u_0, u_1, u_2, u_3, \ldots, u_m\}$ be the vertex set of P_m. Without loss of generality, let u_0 be the pendant vertex of P_m in G. Identify the vertex u_m of P_m and the vertex v_1 of the cycle to form a tadpole graph. Let us define an IASL f on G as follows. Label the vertex u_1 by the set $\{0\}$, the vertex u_2 by the set $\{0, 1\}$ and in general, the vertex u_i by the set $\{0, 1, 2, \ldots, i-1\}$, for $1 \leq i \leq m$. Therefore, the set-label of the vertex $u_m = v_1$ is $\{0, 1, 2, \ldots, m-1\}$. Now, label the remaining vertices of C_n in G as follows. Label the vertex v_2 by the set $\{0, 1, 2, \ldots, m\}$ and in general, label the vertex v_j by the set $\{0, 1, 2, \ldots, m+j-2\}$. Now, choose the set $X = \{0, 1, 2, \ldots, l\}$, where $l \geq 2(m+n) - 5$. Now, the only vertex of G that remains to be labeled is the pendant vertex. Label the vertex u_0 by the set X. Then, the collection of set-labels of G is $\mathcal{A} = \{\{0\}, \{0, 1\}, \{0, 1, 2\}, \ldots, \{0, 1, 2, \ldots, m+n-2\}, X\}$. Clearly, the set $\mathcal{T} = \mathcal{A} \cup \{\emptyset\}$ is a topology on X. Hence, this labeling is a TIASL defined on G. $\square$

Figure 3 illustrates the admissibility of TIASL by the (m, n)-tadpole graph with respect to the ground set $X = \{0, 1, 2, \ldots, 2(m+n) - 5\}$.

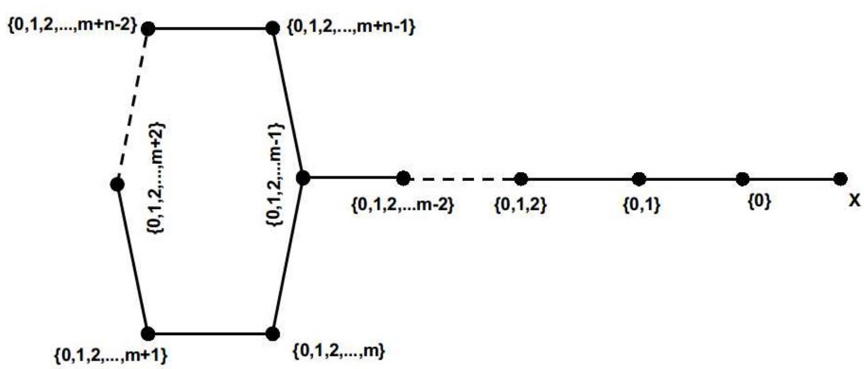

Figure 3. An (n, m)-tadpole graph with a TIASL defined on it.

We can extend the above results to the shovel graphs also. The following result establishes the admissibility of TIASL by shovel graphs by properly choosing the ground set X.

Proposition 2.8. *The (n, m)-shovel graph admits a topological integer additive set-labeling.*

Proof. Let G be an (n, m)-shovel graph. Let $\{v_0, v_1, v_2, v_3, \ldots, v_m\}$ be the vertex set of P_m and $\{v_m, v_{m+1}, v_{m+2}, \ldots, v_{m+n-1}\}$ be the vertex set of K_n in the given shovel graph G, where v_0 is the pendant vertex of P_m (and hence of G). Define an IASL f on G which assigns set-labels to the vertices of G injectively in such a way that any vertex v_i has the set-label $\{0, 1, 2, \ldots, i-1\}$, for $1 \le i \le m+n-1$. Note that, the pendant vertex v_0 remains unlabeled at the moment. It can be noted that the maximal element of the set-label $f^+(v_{m+n-2}v_{m+n-1})$ is $2(m+n)-5$. Hence, choose the set $X = \{0, 1, 2, 3, \ldots, 2(m+n)-5\}$ and label the pendant vertex v_0 by the set X itself. Therefore, $f(V(G)) = \{\{0\}, \{0, 1\}, \{0, 1, 2\}, \ldots, \{0, 1, 2, \ldots, m+n-2\}, X\}$ and $f(V(G)) \cup \{\emptyset\}$ is a topology on X. Hence, f is a TIASL on G. $\square$

Figure 4 depicts the admissibility of TIASL by an (n, m)-shovel graph with ground set $X = \{0, 1, 2, 3, \ldots, 2(m+n)-5\}$.

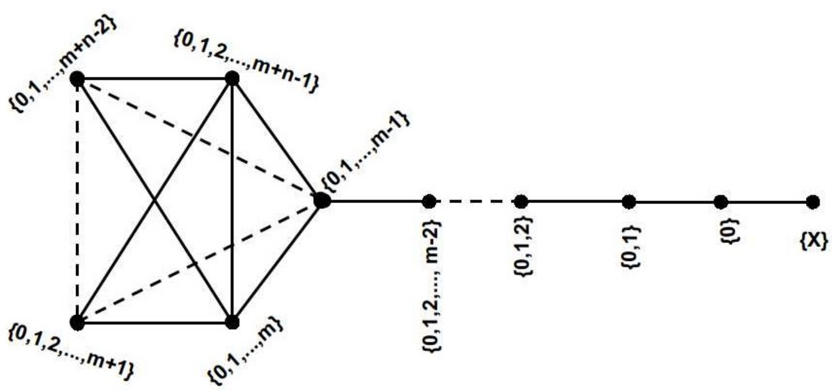

Figure 4. An (n, m)-shovel graph with a TIASL defined on it.

The above propositions raise the question whether the existence of a pendant vertex in a given graph G results in the admissibility of TIASL by it. The choice of X in all the above results played a major role in establishing a TIASL for G. The following is a necessary and sufficient condition for a given graph with at least one pendant vertex to admit a TIASL.

Theorem 2.2. *A graph G admits a TIASL if and only if G has at least one pendant vertices.*

Proof. Let G be a graph which admits a TIASL, say f. Then, the ground set $X \in f(V(G))$. Hence, by Proposition 2.2, the vertex with the set-label X must be a pendant vertex. More over, by Proposition 2.2, if the set-label of a vertex v_i contains the maximal element of X, then v_i is a pendant vertex. Then, G has at least one pendant vertex.

Conversely, assume that G has at least one pendant vertex. Let $V(G) = \{v_1, v_2, v_3 \ldots, v_n\}$. Without loss of generality, let v_1 be a pendant vertex of G. Now, label the vertex v_i by the set $\{0, 1, 2, 3, \ldots, i-1\}$ for $1 \le i \le n$. Then, as explained in the above results, the maximal element in all set-labels of edges of G is $2n-3$. Choose $X = \{0, 1, 2, \ldots, 2n-3\}$ and label the pendant vertex v_1 by the set X. Then, $f(V(G)) = \{\{0\}, \{0, 1\}, \{0, 1, 2\}, \ldots, \{0, 1, 2, \ldots, n-1\}, X\}$. Therefore, $f(V(G)) \cup \{\emptyset\}$ is a topology on X and hence this labeling is a TIASL of G. $\square$

Theorem 2.2 gives rise to the following result.

Theorem 2.3. *Let G be a graph with a pendant vertex v which admits a TIASL, say f, with respect to a ground set X. Let f_1 be the restriction of f to the graph $G - v$. Then, there exists a collection $\mathcal{B}$ of proper subsets of X which together with $\{\emptyset\}$ form a topology of the union of all elements of $\mathcal{B}$.*

Proof. Let G be a graph with one pendant vertex, say v and X be the ground set for labeling the vertices of G. Choose the collection $\mathcal{B}$ of proper subsets of X which contains the set $\{0\}$ and has the cardinality $n - 1$ such that the sum of the maximal elements of any two sets in it is less than or equal to the maximal element of X and the union of any two sets and the intersection of any two non-singleton sets in $\mathcal{B}$ are also in $\mathcal{B}$. Then, by Theorem 2.2, the set-labeling f under which the pendant vertex v is labeled by the set X and other vertices of G are labeled by the elements of $\mathcal{B}$ is a TIASL of G.

Let f_1 be the restriction of f to the graph $G - v$. Therefore, $\mathcal{B} = f_1(V(G - v))$. Now let $B = \bigcup_{B_i \in \mathcal{B}} B_i$ and let $\mathcal{T}' = \mathcal{B} \cup \{\emptyset\}$. Since G has only one end vertex, by Proposition 2.4, no element of $\mathcal{A}$ contains the maximal element of X. Therefore, B also does not contain the maximal element of X. Since the union of any number of sets in $\mathcal{B}$ is also in $\mathcal{B}$, the union of the elements in $\mathcal{T}'$. Then, B belongs to $\mathcal{B}$ and to $\mathcal{T}'$ and B is the maximal element of $\mathcal{T}'$. Since the intersection of any two non-singleton sets in $\mathcal{B}$ is also in $\mathcal{B}$ and $\emptyset \in \mathcal{T}'$, the finite intersection of elements in $\mathcal{T}'$ is also in $\mathcal{T}'$. The set $\mathcal{T}' = \mathcal{B} \cup \{\emptyset\}$ is a topology of the maximal set B in $\mathcal{B}$. □

Remark 2.2. If v is the only pendant vertex of a given graph G, then the collection $\mathcal{B} = f(V(G - v))$, chosen as explained in Theorem 2.3 does not induce a topological IASL on the graph $G - v$, since $f^+(uw) \neq f(u) + f(w)$, for some edge $uw \in E(G - v)$.

3. TIASLs with respect to Certain Topologies

The number of elements in the ground set X is very important in all the studies of set-labeling of graphs. Keeping this in mind, we define

Definition 3.1. The minimum cardinality of the ground set X required for a given graph to admit a topological IASL is known as the *topological set-indexing number* (topological set-indexing number) of that graph.

In this section, we discuss the existence and admissibility of topological IASLs with respect to some standard topologies like indiscrete topologies and discrete topologies.

A topology $\mathcal{T}$ is said to be an indiscrete topology of X if $\mathcal{T} = \{\emptyset, X\}$. Hence the following result is immediate.

Theorem 3.1. *A graph G admits a TIASL with respect to the indiscrete topology $\mathcal{T}$ if and only if $G \cong K_1$.*

Proof. Let v be the single vertex of the graph $G = K_1$. Let X be the ground set for set-labeling G. Let $f(v) = X$. Then $f(V) = \{X\}$ and $f(V) \cup \{\emptyset\} = \{\emptyset, X\}$, which is the indiscrete topology on X. Conversely, assume that G admits a TIASL with respect to the indiscrete topology $\mathcal{T}$ of the ground set X. Then, $f(V(G)) = \mathcal{T} - \{\emptyset\} = \{X\}$, a singleton set. Therefore, G can have only a single vertex. That is, $G \cong K_1$. □

From Proposition 3.1, we have the following result.

Proposition 3.1. *The topological set-indexing number of K_1 is 1.*

Another basic topology of a set X is the *Sierpenski's topology*. If X is a two point set, say $X = \{0, 1\}$, then the topology $\mathcal{T}_1 = \{\emptyset, \{0\}, X\}$ and $\mathcal{T}_2 = \{\emptyset, \{0\}, X\}$ are the Sierpenski's topologies. The following result establishes the conditions required for a graph to admit a TIASL with respect to the Sierpenski's topology.

Theorem 3.2. *A graph G admits a TIASL with respect to the Sierpenski's topology if and only if $G \cong K_2$.*

Proof. Let G be the given graph, with vertex set V, which admits a TIASL with respect to the Sierpenski's topology. Let a two point set $X = \{0, 1\}$ be the ground set used for set-labeling the graph G. Then, $f(V) = \{\{0\}, X\}$. Therefore, G can have exactly two vertices. That is, $G \cong K_2$.

Conversely, assume that $G \cong K_2$. Let u and v be the two vertices of G. Choose a two point set X as the ground set to label the vertices of G. Label the vertex u by X. Then by Proposition 2.1, v must have the set-label $\{0\}$. Then $f(V(G)) = \{\{0\}, X\}$. Then, $f(v(G)) \cup \{\emptyset\}$ is a topology on X, which is a Sierpenski's topology of X. Therefore, $G \cong K_2$ admits a TIASL with respect to the Sierpenski's topology. □

From the above result, we observe the following.

Observation 3.1. *The only Sierpenski's topology of the two point set $X = \{0, 1\}$ that induces a TIASL on the graph K_2 is $\mathcal{T} = \{\emptyset, \{0\}, X\}$.*

In view of Proposition 3.2, we claim that for any ground set X containing two or more elements, one of which is 0, induces a TIASL on K_2. Therefore, the following result is immediate.

Proposition 3.2. *The topological set-indexing number of K_2 is 2.*

The following results are the immediate consequences of 2.2.

Proposition 3.3. *For $n \geq 3$, no complete graph K_n admits a TIASL.*

Proof. The proof follows from Theorem 2.2 and from the fact that a complete graph on more than two vertices does not have any pendant vertex. □

Proposition 3.4. *For $m, n \geq 2$, no complete bipartite graph $K_{m,n}$ admits a TIASL.*

Proof. The proof is immediate from the fact that a complete bipartite graph has no pendant vertices. □

Corollary 3.1. *A path graph P_m admits a TIASL.*

Proof. Every path graph P_m has two pendant vertices and hence satisfy the condition mentioned in Theorem 2.2. Hence P_m admits a TIASL. $\square$

Proposition 3.5. *Every tree admits a TIASL.*

Proof. Since every tree G has at least two pendant vertices, by Theorem 2.1, G admits a TIASL. $\square$

Proposition 3.6. *No cycle graph C_n admits a TIASL.*

Proof. A cycle does not have any pendant vertex. Then, the proof follows immediately by Theorem 2.2. $\square$

In view of the above results, we arrive at the following inference.

Proposition 3.7. *For $k \geq 2$, no k-connected graph admits a TIASL with respect to a ground set X.*

Proof. No biconnected graph G can have pendant vertices. Hence, by Theorem 3.3, G can not admit a TIASL. $\square$

We have already discussed the admissibility of a TIASL by a graph with respect to the indiscrete topology of the ground set X. In this context, it is natural to ask whether a given graph admits the TIASL with respect to the discrete topology of a given set X. The following theorem establishes the condition required for G to admit a TIASL with respect to the discrete topology of X.

Theorem 3.3. *A graph G, on n vertices, admits a TIASL with respect to the discrete topology of the ground set X if and only if G has at least $2^{|X|-1}$ pendant vertices which are adjacent to a single vertex of G.*

Proof. Let $|X| = m$. Let the graph G admits a TIASL f with respect to the discrete topology $\mathcal{T}$ of X. Therefore, $f(V(G)) = \mathcal{P}(X) - \{\emptyset\}$. Then, $|f(V(G))| = 2^{|X|} - 1$. Now, let l be the maximal element in X. The number of subsets of X containing l is 2^{m-1}. Since f is a TIASL with respect to the discrete topology, all these sets containing l must also be the set-labels of some vertices of G. By Proposition 2.2, all these vertices must be adjacent to the vertex whose set-label is $\{0\}$. By Proposition 2.3, no two vertices whose set-labels contain l can be adjacent among themselves or to any other vertex which has a set-label with non-zero elements. Therefore, G has 2^{m-1} pendant vertices which are adjacent to a single vertex whose set-label is $\{0\}$.

Conversely, let G be a graph with $n = 2^{|X|} - 1$ vertices such that at least $2^{|X|-1}$ of them are pendant vertices incident on a single vertex of G. Label these pendant vertices by the $2^{|X|-1}$ subsets of X containing the maximal element l of X. Label remaining vertices of G by the remaining $2^{|X|-1} - 1$ subsets of X which do not contain the element l, in such a way that the sum of the maximal elements of the set-labels of two adjacent vertices is less than or equal to l. This labeling is clearly a TIASL on G. That is, G admits a TIASL with respect to the discrete topology of X. $\square$

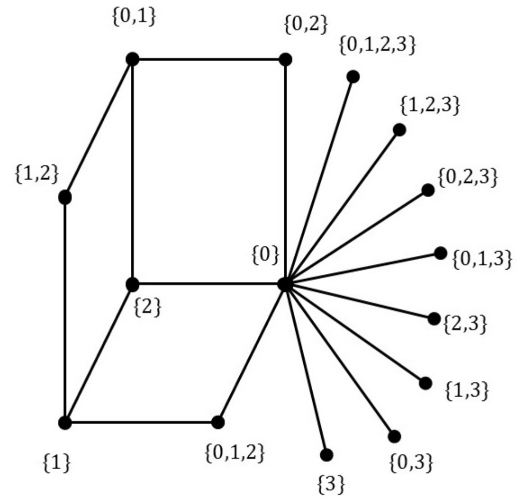

Figure 5. *A TIASL of graph with respect to the discrete topology of X.*

Figure 5 depicts the existence of a TIASL with respect to the discrete topology of a ground set $X = \{0, 1, 2, 3\}$ for a graph G.

Since the necessary and sufficient condition for a graph to admit a TIASL with respect to the discrete topology of ground set X is that G has at least $2^{|X|-1}$ pendant vertices that incident at a single vertex of G, no paths $P_n;\ n \geq 3$, cycles, complete graphs and complete bipartite graphs can have a TIASL with respect to discrete topology of X.
Theorem 2.2 gives rise to the following results also.

Corollary 3.2. *A graph on even number of vertices does not admit a TIASL with respect to the discrete topology of the ground set X.*

Proof. If a graph on n vertices admits a TIASL with respect to the discrete topology of the ground set X, then by Theorem 3.3, $n = 2^{|X|-1}$, which can never be an even integer. Therefore, G on even number of vertices does not admit a TIASL with respect to the discrete topology of X. □

Corollary 3.3. *A star graph $K_{1,r}$ admits a TIASL with respect to the discrete topology of the ground set X, if and only if $r = 2^{|X|} - 2$.*

Proof. First assume that the star graph $G = K_{1,r}$ admits a TIASL f with respect to the discrete topology of the ground set X. Then, $f(V(G)) = \mathcal{P}(X) - \{\emptyset\}$. That is, $|f(V(G))| = 2^{|X|} - 1$. Hence, G must have $2^{|X|} - 1$ vertices. That is, $r + 1 = 2^{|X|} - 1$. Therefore, $r = 2^{|X|} - 2$.

Conversely, consider a star graph $G = K_{1,r}$, where $r = 2^n - 2$ for some positive integer n. Choose a set X with cardinality n, which consists of the element 0. Note that the number of non-empty subsets of X is $2^n - 1$. Define a set-labeling f of G which assigns $\{0\}$ to the central vertex of G and the other non-empty subsets of X to the pendant vertices of G. Clearly, this labeling is an IASL of G. Also, $f(V(G)) = \mathcal{P}(X) - \{\emptyset\}$. Therefore, f is a TIASL of G with respect to the discrete topology of X. □

Figure 6 illustrates the existence of a TIASL with respect to the discrete topology of the ground set X for a star graph.

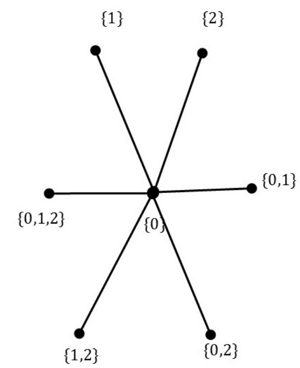

Figure 6.

Figure 7. A Star grph with a TIASL with respect to the discrete topology of X.

4. Conclusion

In this paper, we have discussed the concepts and properties of topological integer additive set-indexed graphs analogous to those of topological IASI-graphs and have done a characterisation based on this labeling.

We note that the admissibility of topological integer additive set-indexers by the given graphs depends also upon the number and nature of the elements in X and the topology $\mathcal{T}$ of X concerned. Hence, choosing a ground set X is very important in the process of checking whether a given graph admits a TIASL-graph.

Certain problems in this area are still open. Some of the areas which seem to be promising for further studies are listed below.

Problem 1. Characterise different graph classes which admit topological integer additive set-labelings.

Problem 2. Estimate the topological set-indexing number of different graphs and graph classes which admit topological integer additive set-labelings.

Problem 3. Verify the existence of topological integer additive set-labelings for different graph operations and graph products.

Problem 4. Establish the necessary and sufficient condition for a graph to admit topological integer additive set-indexer.

Problem 5. Characterise the graphs and graph classes which admit TIASI.

The integer additive set-indexers under which the vertices of a given graph are labeled by different standard sequences of non negative integers, are also worth studying. All these facts highlight a wide scope for further studies in this area.

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