# Automorphism groups of some families of bipartite graphs

K.G. Sreekumar, K. Manilal

*Department of Mathematics, University College,*
*University of Kerala, Thiruvananthapuram, India.*

sreekumar3121@gmail.com, manilalvarkala@gmail.com

## Abstract

This paper discusses the automorphism group of a class of weakly semiregular bipartite graphs and its subclass called $WSB_{END}$ graphs. It also tries to analyse the automorphism group of the SM sum graphs and SM balancing graphs. These graphs are weakly semiregular bipartite graphs too. The SM sum graphs are particular cases of bipartite Kneser graphs. The bipartite Kneser type graphs are defined on $n$-sets for a fixed positive integer $n$. The automorphism groups of the bipartite Kneser type graphs are related to that of weakly semiregular bipartite graphs. Weakly semiregular bipartite graphs in which the neighbourhoods of the vertices in the SD part having the same degree sequence, possess non trivial automorphisms. The automorphism groups of SM sum graphs are isomorphic to the symmetric groups. The relationship between the automorphism groups of SM balancing graphs and symmetric groups are established here. It has been observed by using the well known algorithm Nauty, that the size of automorphism groups of SM balancing graphs are prodigious. Every weakly semiregular bipartite graphs with k-NSD subparts has a matching which saturates the smaller partition.

**Introduction**

A simple graph is usually denoted by $G = (V, E)$, where $V$ is the vertex set and $E$ is its edge set. The order of $G$ is the number of its vertices and size of $G$ is the number of its edges. An isomorphism from a graph $G$ to a graph $H$ is a bijection $f$ from the vertex set of $G$ to that of $H$ such that $u$ and $v$ are adjacent in $G$ if and only if $f(u)$ and $f(v)$ are adjacent in $H$, for all $u$ and $v$ in the vertex set $V(G)$. Two graphs are isomorphic when there is an isomorphism from one to the other. Moreover, when two simple graphs are isomorphic, there is a one-to-one correspondence between the vertices of the two graphs that preserves the adjacency relationship.

All the isomorphisms to the same graph itself play an important role in many applied fields of Mathematics as well as in security in information technology. An automorphism of a graph $G$ is an isomorphism with itself. The automorphism group of a graph $G$ is denoted by $Aut(G)$. The automorphisms of complete graph [10], complete bipartite graph [5] and semiregular bipartite graphs [6] were studied earlier. Here the automorphism of weakly semiregular connected bipartite graphs are studied. In the case of complete bipartite graph $K_{m,n}$, the automorphism group is isomorphic to $S_m \times S_n$, when $m \neq n$ [3]. The automorphism group of $K_{m,n}$, is isomorphic to $S_m \times S_n \times Z_2$ when $m = n$ [3]. The isomorphism and automorphism of graphs are largely used in data structure for database retrieval and in cryptography etc. In the study of graph parameters, the graph isomorphism and graph automorphisms have a big role. Many algorithms like Nauty [4], Saucy, Trace, Bliss, etc, have been introduced apart from the conventional combinatorial methods. A study of the direct product and uniqueness of automorphism groups of graphs was done by Peisert [12]. A characterisation of automorphism groups of the generalized Hamming graphs was done by Chaouche and Berrachedi [2].

The Kneser graph $K_{v:r}$ is the graph with the $r$-subsets of a fixed $v$-sets as its vertices, with two $r$-subsets adjacent if they are disjoint. SM sum graphs are related to the intrinsic connection between the powers of 2 and the natural numbers which is the basic logic of binary number system. The SM balancing graphs are associated with the balanced ternary number system which was used in the SETUN computers made in Russia. These graphs are vertex labelled graphs and are explained in the following section. The automorphism group of Kneser graph $K_{v:r}$ is isomorphic to the symmetric group $S_v$ [11].

## 1. Preliminary

In this section, we provide the basic definitions and some results from the related previous work.

**Definition 1.1.** *[11] For a fixed integer $n > 1$, let $\mathscr{S}_n = \{1, 2, 3, \ldots, n\}$ and $V$ be the set of all $k-$subsets and $(n - k)-$subsets of $\mathscr{S}_n$. The bipartite Kneser graph $H(n, k)$ has $V$ as its vertex set and two vertices $A, B$ are adjacent if and only if $A \subset B$ or $B \subset A$.*

A bipartite graph $G$ with bipartition $(V_1, V_2)$, is called $(q_1 + 1, q_2 + 1)-$ semiregular if the degree of the vertex $v$, $d(v) = q_i + 1$ for each $v \in V_i$, $i = 1, 2$ [6]. Furthermore $q_1 + 1$ and $q_2 + 1$ are called the degrees of $G$. We begin with the definition of some families of graphs. First let us see the definition of SM balancing graphs [8]. Consider the set $T_n = \{3^m : m$ is an integer,

$0 \leq m \leq n-1\}$ for a fixed positive integer $n \geq 2$. Let $I = \{-1, 0, 1\}$ and $x \leq \frac{1}{2}(3^n - 1)$ be any positive integer which is not a power of 3. Then $x$ can be expressed as

$$x = \sum_{j=1}^{n} \alpha_j y_j \tag{1}$$

where $\alpha_j \in I$, $y_j \in T_n$ and $y_j's$ are distinct. Each $y_j$ such that $\alpha_j \neq 0$ is called a balancing component of $x$. Consider the simple digraph $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_{\frac{1}{2}(3^n-1)}\}$ and adjacency of vertices is defined by: for any two distinct vertices $v_x$ and $v_{y_j}$, $(v_x, v_{y_j}) \in E$ if (1) holds and $\alpha_j = -1$, and $(v_{y_j}, v_x) \in E$ if (1) holds and $\alpha_j = 1$. This digraph $G$ is called the $n^{th} SMD$ Balancing Graph, denoted by $SMD(B_n)$. Its underlying undirected graph is called the $n^{th}$ SM Balancing Graph, denoted by $SM(B_n)$. Now let us see the definition of SM sum graphs [7]. For a fixed integer $n \geq 2$, consider the positive integers $p < 2^n$, that are not powers of 2, then $p = \sum_{i=1}^{n} x_i$, with $x_i = 0$ or $2^m$, $m$ is an integer, $0 \leq m \leq n-1$ and $x_i's$ are distinct. The coefficient of each $x_i's$ is 1. Each $x_i \neq 0$ is called an additive component of $p$. For a fixed integer $n \geq 2$, the simple graph $SM(\sum_n)$, called $n^{th}$ SM sum graph [9], is the graph with vertex set $\{v_1, v_2, \ldots, v_{2^n-1}\}$ and adjacency of vertices defined by: $v_i$ and $v_j$ are adjacent if either $i$ is an additive component of $j$, or $j$ is an additive component of $i$. For a fixed integer $n \geq 2$, let $T_n = \{3^m : m \text{ is an integer}, 0 \leq m \leq n-1\}$, $N_n = \{1, 2, 3, \ldots, t\}$, where $t = \frac{1}{2}(3^n - 1)$. Also, let $P_n = \{2^m : m \text{ is an integer}, 0 \leq m \leq n-1\}$, $M_n = \{1, 2, 3, \ldots, 2^n - 1\}$. Then consider $P_n^c = M_n - P_n$, $T_n^c = N_n - T_n$ throughout this paper unless otherwise specified. The Hamming weight of a string was defined as the number of 1's in the string representation using 0 and 1. Here the number of additive components is the Hamming weight of string (binary) representation of all numbers in $P_n^c$. The Hamming weight of string (binary) representation of numbers in $P_n$ is always 1. In $SM(\sum_n)$, the degree of the vertex $v_{2^n-1}$ is $n$ and $\sum_{v \in V} d(v) = 2n(2^{n-1} - 1)$ [7]. In $SM(B_n)$, the number of vertices is $\frac{1}{2}(3^n - 1)$ and $\sum_{v \in V} d(v) = 2n(3^{n-1} - 1)$ [8].

A graph $G$ is *asymmetric* if its automorphism group, $Aut(G)$ does not contain any permutation other than the identity. It has been proved by $Erd\ddot{o}s$ and $R\acute{e}nyi$, that almost all graphs are asymmetric. This means that the proportion of graphs on $n$ vertices that are asymmetric goes to one as $n \rightarrow \infty$. All regular graphs need not be non asymmetric, for example, the Frucht graph which is a 3-regular graph with 12 vertices and has no non-trivial automorphism. The problem of finding the automorphisms of a graph belongs to the class NP of computational complexity [4].

The SM sum graphs are closely related to bipartite Kneser graphs. The bipartite Kneser type-1 graph is defined as follows.

**Definition 1.2.** *Let $\mathscr{S}_n = \{1, 2, 3, \ldots, n\}$ for a fixed integer $n > 1$. Let $\phi(\mathscr{S}_n)$ be the set of all non-empty subsets of $\mathscr{S}_n$. Let $V_1$ be the set of 1- element subsets of $\mathscr{S}_n$ and $V_2 = \phi(\mathscr{S}_n) - V_1$. Define a bipartite graph with adjacency of vertices as: a vertex $A \in V_1$ is adjacent to a vertex $B \in V_2$ if and only if $A \subset B$. This graph is called a bipartite Kneser type-1 graph.*

This bipartite kneser type-1 graph has $2^n - 1$ vertices and $n(2^{n-1} - 1)$ edges for each $n \geq 2$. Also, bipartite kneser type graphs are neither vertex transitive nor edge transitive. But they are non asymmetric. The SM sum graphs are isomorphic to this bipartite kneser type graph for each $n \geq 2$.

## 2. Weakly semiregular bipartite graphs

Revisiting bipartite graphs, a simple graph $G$ is called bipartite if its vertex set $V$ can be partitioned into two disjoint sets $V_1$ and $V_2$ such that every edge in the graph connects a vertex in $V_1$ and a vertex in $V_2$ and no edge in $G$ connects either two vertices in $V_1$ or two vertices in $V_2$. We call $(V_1, V_2)$ the bipartition of $G$; $V_1$ and $V_2$ are called the parts of $G$. Here we consider only connected bipartite graphs.

A bipartite graph $G$ is semiregular of bi-degree $(k, m)$ if every vertex in one member of the bipartition has degree $k$ and every vertex in the other has degree $m$. We considered cases where one part of $G$ has vertices of equal degree.

**Definition 2.1.** *A bipartite graph $G$ with bipartition $(V_1, V_2)$, $|V_1| > 1$ and $|V_2| > 1$ is weakly semiregular if the vertices in exactly one $V_i$ have the same degree. The part of $G$ in which all vertices have the same degree is called a SD-part. The other part of $G$ is called a NSD-part.*

Let $G = (V, E)$ be a graph. The neighbourhood of $v \in V$, written $N_G(v)$ or $N(v)$, is the set of vertices adjacent to $v$.

**Definition 2.2.** *A weakly semiregular bipartite graph $G$ is called a $WSB_{END}$ graph if the vertices in the NSD-part do not have all distinct degrees and the neighbourhoods of the vertices in the SD-part have same degree sequence. For each $k \geq 1$, the set of vertices in the NSD-part of degree $k$ is called a k-NSD subpart.*

Suppose there are 3 servers, 5 computers and the maximum allowed connections to each server is 4. So we have a total of $4 \times 3 = 12$ possible connections. 5 computers need to be connected. One of the arrangements is as follows: 3 computers (having only 2 ports) to two each server and other two computers (having only 3 ports) to all the 3 servers. Only one direct connection to a server can be active at any time. This connection leads to a graph which is a $WSB_{END}$. The question of interchanging the connections without altering the connection structure raising the question of automorphism. In the cases of online examinations, these connection automorphisms may reduce the possibility of cheating. This can be solved by using a symmetric swap using an automorphism.

The automorphism relationship is an equivalence relation on the vertices of a graph. Two vertices are equivalent if there exists an automorphism taking one to the other. Like all equivalence relations, this also produces a partition of the vertex set into equivalence classes. These classes are usually called automorphism classes or orbits. Here the orbits are the vertices of each k-NSD subpart and SD part. The automorphism classes of these types of graphs are yet to be studied.

## 3. Main Results

The automorphism group is an algebraic invariant of a graph. The main results regarding the automorphisms of connected $WSB_{END}$ graphs are given in this section.

**Proposition 3.1.** *Let $G$ be a connected $WSB_{END}$ graph. Then $G$ has non trivial automorphism group.*

*Proof.* Let $V_1$ and $V_2$ be the parts of $G$. Let $|V_1| = m > 1$ and $|V_2| = n > 1$. Let $V_1$ be the SD part having degree $p$ to each of its vertices and $V_2$ be the NSD part. Let $\{n_1, n_2, \ldots, n_j\} = \{d(x) : x \in V_2\}$ with $n_1 < n_2 < \cdots < n_j$. Let $V_2^i = \{x \in V_2 : d(x) = n_i\}$ be the $j$ k-NSD sub parts having $n_1, n_2, \ldots, n_j$ as the corresponding degrees of vertices in each k-NSD subpart. Obviously $m > n_1$ and also $mp = |V_2^1|n_1 + |V_2^2|n_2 + \cdots + |V_2^j|n_j$. Now consider a permutation on $m + n$ vertices. Since the vertices in the SD part are of same degree and each is having a neighbourhood with same degree sequence, there exist some permutations which permute among these vertices together with elements of k-NSD subpart so that it results in automorphisms. In this way these graphs have non trivial automorphisms too.

Let $Aut(G)$ be the collection of all these automorphisms of $G$. Here the trivial permutation is the identity automorphism. The collection of all permutations is closed under the operation of composition. Also, the permutations on the k-NSD subpart permute in itself in accordance with the permutation of elements of SD part. Consider two automorphisms $\alpha$ and $\beta$ having cycles on the k-NSD subpart, then there will be two cases.

**Case 1**. When $\alpha$ and $\beta$ belonging to the set of permutations containing cycles of the same orbit. In this case when the composition is taken, then $\alpha \circ \beta$ is a member of the permutation with product of cycles on the same k-NSD subpart. Therefore, $\alpha \circ \beta \in Aut(G)$.

**Case 2**. When $\alpha$ and $\beta$ belonging to the set of permutations containing cycles of the different orbits. In this case these are disjoint permutations. Therefore, $\alpha \circ \beta$ is a member of the permutation on $Aut(G)$. In both the cases $\alpha \circ \beta \in Aut(G)$. Therefore, $Aut(G)$ is closed under the operation of composition. The function composition is associative on $Aut(G)$. Now for each of these permutations, there exists an inverse permutation. This inverse permutation acts as the inverse automorphism for each of these elements of $Aut(G)$. Therefore, $G$ has a non trivial automorphism group. $\square$

The degree sequences given in this work have been already explained in [7] and [8].

**Theorem 3.2.** *The graphs* $SM(\sum_n)$ *and* $SM(B_n)$ *are* $WSB_{END}$ *graphs for all* $n > 2$.

*Proof.* Consider the graph $G = SM(\sum_n)$, $n > 2$. The graph $G$ is a bipartite graph with parts $V_1 = \{v_i : i \in P_n\}$ and $V_2 = \{v_j : j \in P_n^c\}$ where $P_n = \{2^m : m$ is an integer, $0 \leq m \leq n-1\}$. Also, $G$ has $2^n - 1$ vertices and $n(2^{n-1}-1)$ edges. All the vertices of $V_1$ are of same degree $2^{n-1}-1$. The vertices of $V_2$ are not of same degree and has a degree sequence $\{2_{(\binom{n}{2})}, 3_{(\binom{n}{3})}, \ldots, n_{(\binom{n}{n})}\}$, for $n > 2$. There are $\binom{n}{2}$ vertices of degree 2 and $\binom{n}{3}$ vertices of degree 3 and so on. Also, each vertices in $V_1$ has a neighbourhood with degree sequence $\{2_{(\binom{n-1}{1})}, 3_{(\binom{n-1}{2})}, \ldots, n_{(\binom{n-1}{n-1})}\}$. This implies that $G$ is a $WSB_{END}$ graph. Similarly for the graph $SM(B_n)$, is having $\frac{1}{2}(3^n - 1)$ number of vertices and $n(3^{n-1} - 1)$ edges. It is a bipartite graph with parts $V_3 = \{v_i : i \in T_n\}$ and $V_4 = \{v_j : j \in T_n^c\}$. The vertices in $V_3$ are of same degree $3^{n-1} - 1$ and each vertex is having a neighbourhood with same degree sequence $\{2_{(2\binom{n-1}{1})}, 3_{(2^2\binom{n-1}{2})}, \ldots, n_{(2^{n-1}\binom{n-1}{n-1})}\}$. The vertices in $V_4$ are of different degree and has a degree sequence $\{2_{(2\binom{n}{2})}, 3_{(2^2\binom{n}{3})}, \ldots, n_{(2^{n-1}\binom{n}{n})}\}$. Therefore, the graph $SM(B_n)$ is also a $WSB_{END}$ graph for each $n > 2$. Hence proved. $\square$

**Corollary 3.3.** *The graph* $G = SM(\sum_n)$, $n > 2$, *has at least one subgraph which is* $WSB_{END}$ *graph.*

**Proposition 3.4.** *Let $G$ be a $WSB_{END}$ graph and having parts $V_1$ and $V_2$, $|V_1| = p > 1$ and $|V_2| = q > 1$. Let the k-NSD subparts be $X_1, X_2, \ldots, X_m$ with $|X_i| = k_i, i = 1, 2, \ldots, m$. Then the number of automorphisms of $G$ is at most $p!(k_1! + k_2! + \cdots + k_m!)$.*

*Proof.* The graph $G$ has $m$ numbers of k-NSD subparts. It has been proved in Proposition 3.1 that $G$ has a non trivial automorphism group. Here the total number of vertices is $p + q$. Consider a permutation on $p + q$ vertices. Assume that the vertices of $V_1$ are of same degree $n$. Each vertex of $V_1$ has a neighbourhood of the same degree sequence. Therefore, we can permute the $p$ elements in the $p + q$ element permutations in $p!$ ways. And in each of these permutations again can be permuted among each of the $k_i$ vertices, $i = 1, 2, 3, \ldots, m$. Thus each of these give raise to $k_i!$ permutations, where $i = 1, 2, 3, \ldots, m$ which include automorphisms of $G$. Therefore, by using the sum and product rule of permutations, the total number of automorphisms is at most $p!(k_1! + k_2! + \cdots + k_m!)$. Hence proved. $\square$

**Corollary 3.5.** *The graph $SM(B_n)$ has less than $n!((2(\binom{n}{2}))! + (2^2\binom{n}{3})! + \cdots + (2^{n-1}\binom{n}{n})!)$ automorphisms.*

*Proof.* Let $G = SM(B_n)$, $n \geq 2$. It can be easily seen that $G$ is a $WSB_{END}$ graph. $G$ has a degree sequence $\{2_{(2\binom{n}{2})}, 3_{(2^2\binom{n}{3})}, \ldots, n_{(2^{n-1}\binom{n}{n})}, 3^{n-1} - 1_{(n)}\}$ for $n \geq 3$. The automorphisms of $G$ depends completely on the permutations of the elements of SD part but not in a unique way. So from Theorem 3.2 and Proposition 3.4, it follows that the total number of automorphisms is less than $n!((2(\binom{n}{2}))! + (2^2\binom{n}{3})! + \cdots + (2^{n-1}\binom{n}{n})!)$. $\square$

Let $X$ and $Y$ be the parts of a bipartite graph $G$. It is known that if a matching $M$ saturates $X$, then for every $S \subseteq X$ there must be at least $|S|$ vertices that have neighbours in $S$. We use $N(S)$ to denote the set of vertices having a neighbour in $S$.

**Theorem 3.6.** *Every weakly semiregular bipartite graphs with k-NSD subparts has a matching which saturates the smaller partition.*

*Proof.* Let $G$ be a weakly semiregular bipartite graph with k-NSD subparts. Let $X$ and $Y$ be the parts of $G$. Since it has k-NSD subparts, $|X| \neq |Y|$. Let $|X| < |Y|$. This implies that for all $x \in X$ and $y \in Y$, $deg\ x \geq deg\ y$. Now to prove the theorem, we use Hall's necessary and sufficient conditions. It says that 'an $X, Y$ bigraph $G$ has a matching that saturates $X$ if and only if $|N(S)| \geq |S|$, for all $S \subseteq X$'. For proving that there exists a matching that saturates the smaller partition $X$, it is enough to prove that $|N(S)| \geq |S|$, for all $S \subseteq X$. On the contrary assume that $|N(S)| < |S|$. Let $G_1$ be the sub graph induced by $S \cup N(S)$. Let $M$ be a matching in $G$ and $M$ does not saturate $X$. So we get $deg\ x < deg\ y$ and $|X| > |Y|$. This is a contradiction. Therefore, $|N(S)| < |S|$ is wrong. So we get $|N(S)| \geq |S|$. Hence proved. $\square$

## Automorphisms of $SM(\sum_n)$ and $SM(B_n)$

The automorphism of graphs is a degree preserving as well as distance preserving function. In this section, we are examining the automorphism group of SM sum graphs and SM balancing graphs. The maximum simple bipartite graph is the complete bipartite graph. If $G = K_{m,n}$ is the

complete bipartite graph, then $G$ is a subgraph of $SM(\sum_p)$ or $SM(B_p)$, where $p = max(m, n) + 1$. There is a relationship between a bipartite graph especially complete bipartite graph with the intrinsic relationship between the powers of 2 and other integers as well as powers of 3 and other integers.

**Proposition 3.7.** *Any bipartite graph is isomorphic to a subgraph of SM sum graph or SM balancing graph.*

*Proof.* Consider the graph $SM(\sum_n)$ with vertex set $V = \{v_i : 1 \leq i \leq 2^n - 1\}$ for an integer $n \geq 2$. Also, $SM(B_m)$ is a graph with the vertex set $V' = \{v_1, v_2, \ldots, v_{\frac{1}{2}(3^m-1)}\}$, where $m$ is an integer $\geq 2$. Both are bipartite graphs. Let $G$ be any bipartite graph with parts $V_1$ and $V_2$, $|V_1| = r \geq 1$ and $|V_2| = s \geq 1$. Let $\lambda = max(r, s) + 1$. From the definition of $SM(\sum_n)$ or $SM(B_m)$, it is clear that any bipartite graph $G$ is isomorphic to a subgraph of $SM(\sum_\lambda)$ or $SM(B_\lambda)$. $\square$

**Theorem 3.8.** *The graph $SM(\sum_n)$ has $n!$ automorphisms for all $n \geq 2$.*

*Proof.* Let $G = SM(\sum_n)$, $n \geq 2$. When $n = 2$, the result is obvious. Now consider the case when $n > 2$. From the results obtained earlier from Theorem 3.2, we have that $G$ is a $WSB_{END}$ graph for each $n$. Also, $G$ has a degree sequence $\{2_{(\binom{n}{2})}, 3_{(\binom{n}{3})}, \ldots, n_{(\binom{n}{n})}, (2^{n-1} - 1)_{(n)}\}$, for $n > 2$. Here $P_n$ is one of the orbits for each $n$. Since the orbits are the vertices of each k-NSD subpart and SD part, as we permute the elements of SD part, this fixes how the elements of k-NSD subpart must be permuted to give automorphisms. The automorphisms of $G$ depends completely on the permutations of the elements of SD part in a unique way. So we get that the number of automorphisms is $n!$. $\square$

**Theorem 3.9.** *The automorphism group of $SM(\sum_n)$ is isomorphic to the symmetric group $S_n$ for all $n \geq 3$.*

*Proof.* Let $G = SM(\sum_n)$. It has been observed that the graph $G$ is non asymmetric for all $n \geq 2$. By Theorem 3.8, $G$ has $n!$ automorphisms for all $n \geq 2$. Also, the automorphism group of $G$ is non abelian for all $n > 2$. Therefore, the automorphism group of $G = SM(\sum_n)$, is isomorphic to $S_n$ for all $n \geq 3$. Hence the theorem. $\square$

More precisely, we get the following Theorem 3.13 for the automorphism group of the SM balancing graphs.

**Proposition 3.10.** *Let $G$ be a bipartite graph isomorphic to $K_{m,n}$ with parts $V_1$ and $V_2$, $|V_1| = m$, $|V_2| = n$, $m < n$, and $H$ be an edgeless graph with $V(H)$ as a proper subset of $V_1$. Let $|H| = r < m$. Then $Aut[(rK_1 \vee H)\dot\cup G] \cong S_r \times S_m \times S_n$, where $\dot\cup$ stands for edge disjoint union.*

*Proof.* Consider the graph $rK_1 \vee H$. It is a bipartite graph having $r$ elements in one partition. Then we have $r!$ permutations. Since the vertices of $H$ are fixed and $H$ is a proper subset of $V_1$ $\implies Aut(rK_1 \vee H) \cong S_r$, by fixing all vertices in $H$. Also, we have $Aut(K_{m,n}) \cong S_m \times S_n$. Therefore, we get $Aut[(rK_1 \vee H)\dot\cup G] \cong S_r \times S_m \times S_n$. $\square$

**Corollary 3.11.** *Let $G \cong K_{m,n}$ be the complete bipartite graph with partition $V_1 \cup V_2$, $|V_1| = m$, $|V_2| = n$, $m < n$. Let $H_i$, $1 \leq i \leq p$, be an edgeless graph with $V(H_i)$ as a proper subset of $V_1$ with $|H_i| = |H_j|$ for all $i, j$. Let $|H_i| = r < m$. Then $Aut[\dot{\bigcup}_{i=1}^{p}(rK_1 \vee H_i)\dot{\cup}G] \cong (S_r)^p \times S_m \times S_n$.*

*Proof.* Since $V(H_i)$, $1 \leq i \leq p$, are proper subsets of $V_1$ with $|H_i| = |H_j|$ for all $i, j$, $Aut[(rK_1 \vee H_i) \cong S_r$, for a fixed $i$. Therefore, $Aut[(rK_1 \vee H_1)\dot{\cup}(rK_1 \vee H_2)\dot{\cup}\ldots(rK_1 \vee H_p)\dot{\cup}G] \cong (S_r)^p \times S_m \times S_n$. $\square$

**Corollary 3.12.** $Aut[\dot{\bigcup}_{r=1}^{n-1}\dot{\bigcup}_{j=1}^{p}(rK_1 \vee H_j)\dot{\cup}G] \cong \prod_{r=1}^{n-1}(S_r)^p \times S_m \times S_n$, *where $H_j$ and $G$ are defined as in Corollary 3.11.*

For different values of $r$, $|H_j|$ may be assumed different values accordingly.

**Theorem 3.13.** *The automorphism group, $Aut(SMB_n)$ is isomorphic to $\left[ \prod_{k=2}^{n}(S_{2^{k-1}})^{\binom{n}{k}} \right] \times S_n$, for all integers $n \geq 3$.*

*Proof.* Consider $SM(B_n)$, $n \geq 3$. By Theorem 3.2, $SM(B_n)$ is a $WSB_{END}$ graph. Here $SM(B_n) \cong \left[\dot{\bigcup}_{r=1}^{n-2}\dot{\bigcup}_{j=2}^{n-1}(2^r K_1 \vee H_{\binom{n}{j}})\right]\dot{\cup}K_{2^{n-1},n}$, where $\dot{\cup}$ stands for edge disjoint union, $|H_{\binom{n}{j}}| = j$, $H_{\binom{n}{j}}$ is a collection of isolated vertices and $H_{\binom{n}{j}}$ is a proper subset of $\mathbb{T}_n$. Also, $H_{\binom{n}{j}}$ is an edgeless graph on $j$ vertices. So $r = j - 1$.

Therefore, by applying the Corollary 3.12, we get

$$Aut(SM(B_n)) \cong \left[ \prod_{k=2}^{n-1}(S_{2^{k-1}})^{\binom{n}{k}} \right] \times S_{2^{n-1}} \times S_n$$

$$\cong \left[ \prod_{k=2}^{n}(S_{2^{k-1}})^{\binom{n}{k}} \right] \times S_n.$$

$\square$

## 4. More results on automorphisms of weakly semiregular bipartite graphs

For $n = 2$, the the automorphism group of the graph $G = SM(B_n)$ is isomorphic to the dihedral group $D_4$. The automorphism groups of the graph $G = SM(B_n)$ are related to the conjugacy classes of some of the sporadic simple groups [1]. But the relation is yet to be studied furthermore. Also, it has been observed that the automorphism group of SM balancing graph is a simple group for $n = 2$.

**Proposition 4.1.** *The graph $SM(\sum_n)$ has no perfect matching and but have a non trivial biclique for all $n \geq 2$.*

*Proof.* Let $X$ and $Y$ be the parts of $G = SM(\sum_n)$. Here $|X| \neq |Y|$. Then $G$ has no perfect matching but a biclique by Proposition 3.7. $\square$

Also, we can observe that a $WSB_{END}$ graph having odd number of edges is non-Eulerian.

**Definition 4.2.** *A semiregular bipartite graph $G$ is called with equal parity if each vertex in either of the parts have a neighbourhood with same degree sequence.*

**Theorem 4.3.** *A semiregular bipartite graph with equal parity is non asymmetric.*

*Proof.* Let $G$ be a $(k, m)$ semiregular bipartite graph with parts $X$ and $Y$. Since it has same parity, it will have non trivial automorphisms. Therefore, it is non asymmetric. $\quad\square$

**Lemma 4.4.** *[9] The graph $SM(\sum_n)$ is isomorphic to an edge induced sub graph of $SM(B_n)$.*

**Theorem 4.5.** *Suppose $G = SM(\sum_n)$, $n \geq 2$ be an $n^{th}$ SM sum graph. Then the number of isomorphisms of $G = SM(\sum_n)$ to edge induced subgraph of $SM(B_n)$ is $n! \times n!$.*

*Proof.* The proof follows from Lemma 4.4 and above Theorem 3.8. $\quad\square$

**Example 4.6.** *The automorphism group of $SM(\sum_n)$, $n = 3$ is isomorphic to $S_3$.*

When $n = 3$, the graph $G = SM(\sum_n)$ has 7 vertices and 9 edges. From Theorem 3.2, we have that $G$ is a $WSB_{END}$ graph. Therefore, $G$ has parts $V_1 = \{v_1, v_2, v_4\}$ and $V_2 = \{v_3, v_5, v_6, v_7\}$. Also, the vertices in $V_1$ are of degree 3 each and the vertices of $V_2$ are having degree 2 each except $v_7$. The vertex $v_7$ has degree 3 and is adjacent to vertices of degree 3. The permutations $(v_1, v_2)(v_5, v_6), (v_1, v_4)(v_3, v_6), (v_2, v_4)(v_3, v_5)$ produce 3 automorphisms. Furthermore if we fix any permutation of the vertices of $\{v_1, v_2, v_4\}$, this fixes how $\{v_3, v_5, v_6\}$ must be permuted to give an automorphism. So we get two more automorphisms which adds to a total of 6 automorphisms including the trivial automorphism. On the other hand, no automorphisms can result from swapping the vertex from the first bipartite set and second bipartite set because unless such a swap is done in its entirety, the adjacency will be lost. A swap can be done in entirety only if $|V_1| = |V_2|$ which is not the case here as $G$ is not a complete bipartite graph as well. Therefore, finally we can see that $Aut(G)$ is isomorphic to $S_3$. The orbits are $v_1, v_2, v_4$; $v_3, v_5, v_6$ and $v_7$.

A graph $G$ is vertex-transitive [13] if for every vertex pair $u, v \in V(G)$ there is an automorphism that maps $u$ to $v$. A graph $G$ is edge transitive if for every edge pair $(d, e) \in E(G)$, there is an automorphism that maps $d$ to $e$. Also, it is observed that $SM(\sum_n)$ or $SM(B_n)$ are neither bi-regular, nor edge transitive nor vertex transitive. The graph $SM(\sum_n)$ contains vertex transitive subgraphs for all $n > 2$ because it contains biclique for each $n > 2$.

## 5. Conclusion

The automorphisms given in this paper are worthwhile as the non trivial symmetries of graphs are concerned. We have proved that the $WSB_{END}$ graphs are having a non trivial automorphism group. Also as $n$ increases, the number of automorphisms of SM sum graphs as well as SM balancing graphs is increasing. The number of automorphisms was calculated for each $n$ by using the Nauty algorithm. The automorphism group of the SM family of graphs are noteworthy as almost all other graphs are asymmetric. Also, the automorphism classes of the SM family of graphs are yet to be studied in detail. The automorphism classes of SM family of graphs may lead to a decomposition of these graphs. In which all cases the binary number system or balanced ternary number system are being used, in those cases these automorphisms will make significant effects. Further scope of edge automorphisms of these weakly semiregular bipartite graphs is to be examined.

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