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Total Roman domination for proper interval graphs

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Abstract

A function $f : V \to \{0, 1, 2\}$ is a total Roman dominating function (TRDF) on a graph $G = (V, E)$ if for every vertex $v \in V$ with $f(v) = 0$ there is a vertex u adjacent to v with $f(u) = 2$ and for every vertex $v \in V$ with $f(v) > 0$ there exists a vertex $u \in N_G(v)$ with $f(u) > 0$. The weight of a total Roman dominating function f on G is equal to $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a total Roman dominating function on G is called the total Roman domination number of G. In this paper, we give an algorithm to compute the total Roman domination number of a given proper interval graph $G = (V, E)$ in $\mathcal{O}(|V|)$ time.

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1. Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E. Throughout this paper, $G = (V, E)$ is a simple graph with no isolated vertices. The *open neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \in V : uv \in E\}$ and the *degree* of v is $\deg(v) = |N_G(v)|$. For any $S \subseteq V$ the *induced subgraph* $G[S]$ is the graph whose vertex set is S and whose edge set consists of all edges in E that have both endpoints in S. A graph $G = (V, E)$ is an *interval graph* if there is an one-to-one correspondence between vertices $v \in V$ and intervals I_v on the real line. A *proper interval graph* is an interval graph in which no interval properly contains another. The following is clear.

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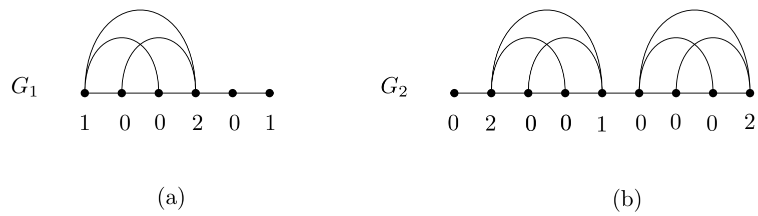


Figure 1. Illustrating (a) an 1-TRDF on G_1 and (b) a 2-TRDF on G_2.

Proposition 1.1. *Let $G = (V, E)$ be a proper interval graph. For any $S \subseteq V$, the induced subgraph $G[S]$ is a proper interval graph.*

For a graph $G = (V, E)$, a *Roman dominating function* (RDF) of G is a function $f : V \to \{0, 1, 2\}$ such that for every vertex $v \in V$ with $f(v) = 0$ there is a vertex u adjacent to v with $f(u) = 2$. Stewart [15], and ReVelle and Rosing [14] defined and discussed the concept of Roman domination. Many papers were published on the Roman domination and its several variants, see, for examples, [2, 9, 10].

Liu and Chang [11] introduced a new variant of Roman dominating functions. A RDF $f : V \to \{0, 1, 2\}$ on G is a *total Roman dominating function* (TRDF) if for every vertex $v \in V$ with $f(v) > 0$ there is a vertex $u \in N_G(v)$ with $f(u) > 0$. The *weight* of a total Roman dominating function f on G is the value $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a total Roman dominating function on G is called the *total Roman domination number* of G, denoted by $\gamma_{tR}(G)$. For further studies on total Roman domination, see, for examples, [1, 3, 4, 5].

Liu and Chang [11] showed that the decision problem related to total Roman domination number is NP-hard even when restricted to bipartite graphs and chordal graphs. Many authors proposed algorithms to compute some variants of domination on proper interval graphs, a well known subclass of chordal graphs, for example, [6, 7, 8, 13]. In this paper we propose a linear algorithm to compute the total Roman domination number of proper interval graphs.

2. Preliminaries

In this section, we introduce some notations that we will use them in our algorithm as follows. Let $G = (V, E)$ be a graph with $|V| = n$ and an ordering $(v_1, v_2, \ldots, v_n)$ of vertices of G. Let $p \in \{1, 2\}$. A function $f : V \longrightarrow \{0, 1, 2\}$ is a p-TRDF on G if f is a RDF with $f(v_n) = p$ such that for each $u \neq v_n$ with $f(u) > 0$ there is a vertex $x \in N_G(u)$ with $f(x) > 0$. See Figure 1. Let $i \in \{1, 2, \ldots, n\}$ and $j \in \{0, 1, 2\}$, let v_0 and v_{n+1} be vertices not in V and let $u, w \in V$.

- $\texttt{index}(v_i) = i$,
- $v_i^+ = v_{i+1}$,
- $v_i^- = v_{i-1}$,
- $\texttt{MAX}(i) = \begin{cases} \max\{j : v_iv_j \in E\}, & \text{if} \quad 1 \leq i < n, \\ n, & \text{if} \quad i = n, \end{cases}$

- $\texttt{MIN}(i) = \begin{cases} \min\{j : v_iv_j \in E\}, & \text{if} \quad 1 < i \leq n, \\ 1, & \text{if} \quad i = 1, \end{cases}$
- $\texttt{MAX}(v_i) = v_{\texttt{MAX}(i)}$,
- $\texttt{MIN}(v_i) = v_{\texttt{MIN}(i)}$,
- $u \leq w$ if $j \leq k$, where $u = v_j$ and $w = v_k$,
- $u < w$ if $j < k$, where $u = v_j$ and $w = v_k$,
- If $u \leq w$, then $[u, w] = \{z \in V : u \leq z \leq w\}$,
- If $u \leq w$, then $G[u, w] = G[\{z \in V : u \leq z \leq w\}]$,
- $\gamma^j(G, v_i) = \min\{w(f) : f$ is a TRDF on $G[v_1, v_i]$ with $f(v_i) = j\}$,
- $\alpha^p(G, v_i) = \min\{w(f) : f$ is a p-TRDF on $G[v_1, v_i]\}$,
- $\gamma(G, v_i) = \min\{w(f) : f$ is a TRDF on $G[v_1, v_{i-1}]\}$.

An ordering $(v_1, v_2, \ldots, v_n)$ of vertices of G is a *consecutive ordering* if $v_iv_k \in E$ for some $1 \leq i < k \leq n$ implies both $v_iv_j \in E$ and $v_jv_k \in E$ for every $i < j < k$.

Theorem 2.1 (Looges and Olariu [12])**.** *A graph G is a proper interval graph if and only if G has a consecutive ordering of its vertices.*

The following result is clear.

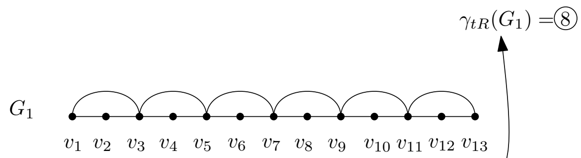
Proposition 2.1. *Let $G = (V, E)$ be a connected interval graph of order n with a consecutive ordering $(v_1, \ldots, v_n)$ of vertices of G. If $v_iv_j \in E$ for some $1 \leq i \leq j \leq n$, then the induced subgraph $G[v_i, v_j]$ is the complete graph.*

Throughout this paper, for a proper interval graph G of order n, we assume that a consecutive ordering $(v_1, \ldots, v_n)$ of vertices of G is given. If G is a disconnected proper interval graph, then clearly $\gamma_{dR}(G)$ is equal to the sum of the double Roman domination numbers of its components. So, in the following we only consider connected proper interval graphs.

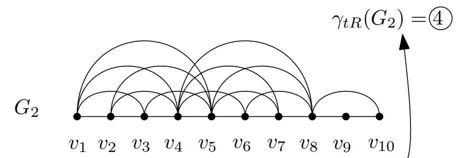
3. Total Roman domination of proper interval graphs

In this section, we propose a linear algorithm (Algorithm 3.1) that computes the total Roman domination number of a given proper interval graph. Let $G = (V, E)$ be a connected proper interval graph with $|V| = n \geq 2$ and a consecutive ordering $(v_1, v_2, \ldots, v_n)$ of vertices of G.

This algorithm uses a dynamic programming technique for computing the total Roman domination number of G. Algorithm 3.1 first initialize values $\gamma^0(G, v)$, $\gamma^1(G, v)$, $\gamma^2(G, v)$, $\alpha^1(G, v)$, $\alpha^2(G, v)$, and $\gamma(G, v)$ for each $v \in [v_1, \texttt{MAX}(v_1)]$. By Proposition 2.1, the induced subgraph $G[v_1, \texttt{MAX}(v_1)]$ is a complete graph. Then, Algorithm 3.1 using values $\gamma^0(G, v)$, $\gamma^1(G, v)$, $\gamma^2(G, v)$, $\alpha^1(G, v)$, $\alpha^2(G, v)$, and $\gamma(G, v)$ for each $v \in [v_1, v_{i-1}]$ computes values $\gamma^0(G, v_i)$, $\gamma^1(G, v_i)$,



	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}
a minimum TRDF	0	0	2	0	2	0	0	0	2	0	2	0	0
$\gamma^0(v)$	∞	∞	3	3	3	4	4	5	5	7	7	8	(8)
$\gamma^1(v)$	∞	2	3	3	3	4	5	6	6	7	7	8	9
$\gamma^2(v)$	∞	3	3	4	4	5	5	7	7	8	8	9	9
$\alpha^1(v)$	1	2	3	3	3	4	5	5	6	6	7	8	9
$\alpha^2(v)$	2	2	2	4	4	5	5	6	6	7	7	9	9
$\gamma(v)$	0	∞	2	3	3	3	4	4	5	5	7	7	8



	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
a minimum TRDF	0	0	0	2	0	0	0	2	0	0
$\gamma^0(v_i)$	∞	∞	3	3	3	3	3	3	4	(4)
$\gamma^1(v_i)$	∞	2	3	3	3	3	3	3	4	5
$\gamma^2(v_i)$	∞	3	3	3	3	4	4	4	5	5
$\alpha^1(v_i)$	1	2	3	3	3	3	3	3	4	5
$\alpha^2(v_i)$	2	2	2	2	2	4	4	4	5	5
$\gamma(v_i)$	0	∞	2	3	3	3	3	3	3	4

Figure 2. Two examples for illustrating Algorithm 3.1.

$\gamma^2(G, v_i)$, $\alpha^1(G, v_i)$, $\alpha^2(G, v_i)$, and $\gamma(G, v_i)$ and repeats this process to compute values $\gamma^0(G, v_n)$, $\gamma^1(G, v_n)$, $\gamma^2(G, v_n)$, $\alpha^1(G, v_n)$, $\alpha^2(G, v_n)$, and $\gamma(G, v_n)$. Finally, Algorithm 3.1 returns the value $\min\{\gamma^0(G, v_n), \gamma^1(G, v_n), \gamma^2(G, v_n)\}$. Examples of Algorithm 3.1 are shown in Figure 2.

To prove Algorithm 3.1 computes the total Roman domination number of proper interval graphs we need the following. Since we have $x \leftarrow x^+$ (Line 9) in each iteration of the **while** loop of Algorithm TRDN$(G, v_1, \dots, v_n)$, Algorithm TRDN$(G, v_1, \dots, v_n)$ terminates. Let $\mathtt{t}_G$ be the number of iterations of the **while** loop of Algorithm TRDN$(G, v_1, \dots, v_n)$.

Lemma 3.1. *Let $G = (V, E)$ be a connected proper interval graph with $|V| = n \geq 2$ and a consecutive ordering $(v_1, v_2, \dots, v_n)$ of vertices of G and let $i \in \{1, 2, \dots, n\}$, $j \in \{0, 1, 2\}$ and $p \in \{1, 2\}$. Then,*

- *there is a TRDF f on $G[v_1, v_i]$ with $f(v_i) = j$ and $w(f) \leq \gamma^j(v_i)$,*
- *there is a p-TRDF f on $G[v_1, v_i]$ with $f(v_i) = p$ and $w(f) \leq \alpha^p(v_i)$, and*
- *there is a TRDF f on $G[v_1, v_{i-1}]$ with $w(f) \leq \gamma(v_i)$.*

Proof. Recall that $\mathtt{t}_G$ is the number of iterations of the **while** loop of Algorithm TRDN$(G, v_1, \dots, v_n)$. The proof is by induction on $\mathtt{t}_G$. We first consider the case that the **while** loop of Algorithm TRDN$(G, v_1, \dots, v_n)$ does not hold, that is, $\mathtt{t}_G = 0$. So, we consider Lines 2-7 of Algorithm 3.1. Let $x = \mathtt{MAX}(v_1)$. Since $\mathtt{t}_G = 0$, $x \geq v_n$. Since always $\mathtt{MAX}(v_1) \leq v_n$, $\mathtt{MAX}(v_1) = v_n$, that is, G is the complete graph. In the following we first consider Lines 2-3 and then Lines 5-7 of Algorithm 3.1.

In Lines 2-3 of Algorithm 3.1, we have $\gamma^0(v_1) = \gamma^1(v_1) = \gamma^2(v_1) = \infty$, $\alpha^1(v_1) = 1$, $\alpha^2(v_1) = 2$, $\gamma(v_1) = 0$, $\gamma^0(v_2) = \gamma(v_2) = \infty$, $\gamma^1(v_2) = \alpha^1(v_2) = \alpha^2(v_2) = 2$ and $\gamma^2(v_2) = 3$. It is not difficult to verify that the lemma holds for both v_1 and v_2.

Here, we consider Lines 5-7 of Algorithm 3.1. Let $v_i \in [v_3, v_n]$. Recall that G is the complete graph. We have $\gamma^0(v_3) = \cdots = \gamma^0(x) = 3$ (Line 5). Function $f = \{(v_1, 2), (v_2, 1), (v_3, 0), \dots, (v_i, 0)\}$ is a TRDF on $G[v_1, v_i]$ with $f(v_i) = 0$ and $w(f) \leq \gamma^0(v_i) = 3$. We have $\gamma^1(v_3) = \cdots = \gamma^1(x) = \alpha^1(v_3) = \cdots = \alpha^1(x) = 3$ (Lines 5-6). Function $f = \{(v_1, 2), (v_2, 0), \dots, (v_{i-1}, 0),$

Algorithm 3.1: TRDN$(G, v_1, \dots, v_n)$

Input: A graph G with $|V(G)| \geq 2$ and a consecutive ordering $(v_1, \dots, v_n)$ of vertices of G.
Output: The total Roman domination number of G.

1 Compute $\mathtt{MAX}(v_1), \dots, \mathtt{MAX}(v_n), \mathtt{MIN}(v_1), \dots, \mathtt{MIN}(v_n)$;
2 $\gamma^0(v_1), \gamma^1(v_1), \gamma^2(v_1) \leftarrow \infty$; $\alpha^1(v_1) \leftarrow 1$; $\alpha^2(v_1) \leftarrow 2$; $\gamma(v_1) \leftarrow 0$;
3 $\gamma^0(v_2), \gamma(v_2) \leftarrow \infty$; $\gamma^1(v_2), \alpha^1(v_2), \alpha^2(v_2) \leftarrow 2$; $\gamma^2(v_2) \leftarrow 3$; $x \leftarrow \mathtt{MAX}(v_1)$;
4 **if** $x \geq v_3$ **then**
5     $\gamma^0(v_3), \dots, \gamma^0(x) \leftarrow 3$; $\gamma^1(v_3), \dots, \gamma^1(x) \leftarrow 3$;
6     $\gamma^2(v_3), \dots, \gamma^2(x) \leftarrow 3$; $\alpha^1(v_3), \dots, \alpha^1(x) \leftarrow 3$;
7     $\alpha^2(v_3), \dots, \alpha^2(x) \leftarrow 2$; $\gamma(v_3) \leftarrow 2$; $\gamma(v_4), \dots, \gamma(x) \leftarrow 3$;
8 **while** $x < v_n$ **do**
9     $x \leftarrow x^+$; $u \leftarrow \mathtt{MIN}(x)$; $\gamma^0(x) \leftarrow \gamma^2(u)$; $\gamma^2(x) \leftarrow \min\{\alpha^1(u), \alpha^2(u)\} + 2$;
10     $\alpha^2(x) \leftarrow \min\{\gamma(u), \alpha^1(u), \alpha^2(u)\} + 2$; $\gamma(x) \leftarrow \min\{\gamma^0(x^-), \gamma^1(x^-), \gamma^2(x^-)\}$;
11   **if** $u^+ < x$ **then**
12         $\gamma^1(x) \leftarrow \min\{\alpha^2(\mathtt{MIN}(x^-)) + 2, \alpha^2(u) + 1\}$;
13         $\alpha^1(x) \leftarrow \min\{\gamma^0(x^-), \alpha^2(u)\} + 1$;
14   **else**
15         $\gamma^1(x) \leftarrow \min\{\alpha^1(u), \alpha^2(u)\} + 1$;
16         $\alpha^1(x) \leftarrow \min\{\gamma^0(u), \alpha^1(u), \alpha^2(u)\} + 1$;
17 **return** $\min\{\gamma^0(v_n), \gamma^1(v_n), \gamma^2(v_n)\}$;

$(v_i, 1)\}$ is a TRDF on $G[v_1, v_i]$ with $f(v_i) = 1$ and $w(f) \leq \gamma^1(v_i) = 3$ and an 1-TRDF on $G[v_1, v_i]$ with $w(f) \leq \alpha^1(v_i) = 3$. We have $\gamma^2(v_3) = \cdots = \gamma^2(x) = 3$ (Line 6). Function $f = \{(v_1, 1), (v_2, 0), \dots, (v_{i-1}, 0), (v_i, 2)\}$ is a TRDF on $G[v_1, v_i]$ with $f(v_i) = 2$ and $w(f) \leq \gamma^2(v_i) = 3$. We have $\alpha^2(v_3) = \cdots = \alpha^2(x) = 2$ (Line 7). Function $f = \{(v_1, 0), \dots, (v_{i-1}, 0), (v_i, 2)\}$ is a 2-TRDF on $G[v_1, v_i]$ with $w(f) \leq \alpha^2(v_i) = 2$. We have $\gamma(v_3) = 2$ and $\gamma(v_4) = \cdots = \gamma(x) = 3$ (Line 7). Function $h = \{(v_1, 1), (v_2, 1)\}$ is a TRDF on $G[v_1, v_2]$ with $w(h) \leq \gamma(v_3) = 2$ and $f = \{(v_1, 1), (v_2, 2), (v_3, 0), \dots, (v_{i-1}, 0)\}$ is a TRDF on $G[v_1, v_{i-1}]$ with $w(f) \leq \gamma(v_i) = 3$. So, the base case of the induction holds.

Assume that the claim is true for any connected proper interval graphs H with $\mathtt{t}_H \leq m$, where $m \geq 0$. Let us consider a connected proper interval graph G with $\mathtt{t}_G = m + 1$. Assume that $(v_1, v_2, \dots, v_n)$ is a consecutive ordering of vertices of G. We have $|V(G)| \geq 3$. In the rest of the proof, we consider the last iteration of the **while** loop of Algorithm TRDN$(G, v_1, \dots, v_n)$.

Suppose $v \in V(G)$. Since the edge $\mathtt{MIN}(v)v \in E(G)$, by Proposition 2.1, the induced subgraph $G[\mathtt{MIN}(v), v]$ is the complete graph. The induced subgraph $G[v_1, v]$ is a connected proper interval graph with a consecutive ordering $(v_1, v_2, \dots, v)$. Consider Algorithm TRDN$(G[v_1, v], v_1, \dots, v)$. If $v < v_n$, then $\mathtt{t}_{G[v_1,v]} \leq m$. Since $x \leftarrow x^+$ (Line 9), $x = v_n \geq v_3$ in the last iteration of the **while** loop of Algorithm TRDN$(G, v_1, \dots, v_n)$. Assume $u = \mathtt{MIN}(v_n)$. We have $v_2 \leq u \leq v_{n-1}$.

- Instruction $\gamma^0(v_n) \leftarrow \gamma^2(u)$ (Line 9 of Algorithm 3.1):

  The induction hypothesis implies that there is a TRDF h on $G[v_1, u]$ with $h(u) = 2$ and

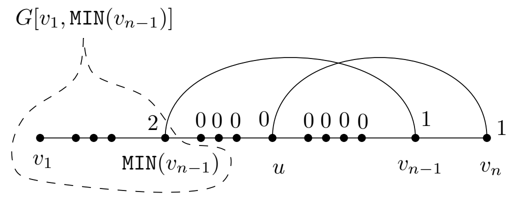


Figure 3. Illustrating a TRDF f on $G[v_1, v_n]$ with $f(v_n) = 1$ and $w(f) \leq \alpha^2(\texttt{MIN}(v_{n-1})) + 2$ by using a 2-TRDF on $G[v_1, \texttt{MIN}(v_{n-1})]$ with weight $\alpha^2(\texttt{MIN}(v_{n-1}))$; note that some edges are not drawn.

$w(h) \leq \gamma^2(u)$. Consider function $f = h \cup \{(u^+, 0), \ldots, (v_n, 0)\}$. Function f is a TRDF on $G[v_1, v_n]$ with $f(v_n) = 0$ and $w(f) = w(h) \leq \gamma^2(u) = \gamma^0(v_n)$.

- Instruction $\gamma^2(v_n) \leftarrow \min\{\alpha^1(u), \alpha^2(u)\} + 2$ (Line 9 of Algorithm 3.1):

  Let $p \in \{1, 2\}$. The induction hypothesis implies that there is a p-TRDF h_p on $G[v_1, u]$ with $w(h_p) \leq \alpha^p(u)$. Consider function $f_p = h_p \cup \{(u^+, 0), \ldots, (v_{n-1}, 0), (v_n, 2)\}$. Function f_p is a TRDF on $G[v_1, v_n]$ with $f_p(v_n) = 2$ and $w(f) = w(h_p) + 2 \leq \alpha^p(u) + 2$. So, there is a TRDF f on $G[v_1, v_n]$ with $f(v_n) = 2$ and $w(f) \leq \min\{\alpha^1(u), \alpha^2(u)\} + 2 = \gamma^2(v_n)$.

- Instruction $\alpha^2(v_n) \leftarrow \{\gamma(u), \alpha^1(u), \alpha^2(u)\} + 2$ (Line 10 of Algorithm 3.1):

  The induction hypothesis implies that there is a TRDF h on $G[v_1, u^-]$ with $w(h) \leq \gamma(u)$. Consider function $f = h \cup \{(u, 0), \ldots, (v_{n-1}, 0), (v_n, 2)\}$. Function f is a 2-TRDF on $G[v_1, v_n]$ with $w(f) = w(h) + 2 \leq \gamma(u) + 2$.

  By the proof of the previous case (Instruction $\gamma^2(v_n) \leftarrow \min\{\alpha^1(u), \alpha^2(u)\} + 2$), there is a TRDF g on $G[v_1, v_n]$ with $g(v_n) = 2$ and $w(g) \leq \min\{\alpha^1(u), \alpha^2(u)\} + 2$. Function g is a 2-TRDF on G. Hence, there is a 2-TRDF f on $G[v_1, v_n]$ with $w(f) \leq \min\{\gamma(u), \alpha^1(u), \alpha^2(u)\}$ $+2 = \alpha^2(v_n)$.

- Instruction $\gamma(v_n) \leftarrow \min\{\gamma^0(v_{n-1}), \gamma^1(v_{n-1}), \gamma^2(v_{n-1})\}$ (Line 10 of Algorithm 3.1):

  The induction hypothesis implies that there is a TRDF h_j on $G[v_1, v_{n-1}]$ with $h_j(v_{n-1}) = j$ and $w(h_j) \leq \gamma^j(v_{n-1})$, where $j \in \{0, 1, 2\}$. So, there is a TRDF f on $G[v_1, v_{n-1}]$ with $w(f) \leq \min\{\gamma^0(v_{n-1}), \gamma^1(v_{n-1}), \gamma^2(v_{n-1})\} = \gamma(v_n)$.

- Instruction $\gamma^1(v_n) \leftarrow \min\{\alpha^2(\texttt{MIN}(v_{n-1})) + 2, \alpha^2(u) + 1\}$ (Line 12 of Algorithm 3.1):

  The induction hypothesis implies that there is a 2-TRDF h on $G[v_1, u]$ with $w(h) \leq \alpha^2(u)$. Consider function $g = h \cup \{(u^+, 0), \ldots, (v_{n-1}, 0), (v_n, 1)\}$. Function g is a TRDF on $G[v_1, v_n]$ with $g(v_n) = 1$ and $w(g) = w(h) + 1 \leq \alpha^2(u) + 1$.

  The induction hypothesis implies that there is a 2-TRDF h on $G[v_1, \texttt{MIN}(v_{n-1})]$ with $w(h) \leq \alpha^2(\texttt{MIN}(v_{n-1}))$. Consider function $g = h \cup \{(\texttt{MIN}(v_{n-1})^+, 0), \ldots, (v_{n-2}, 0), (v_{n-1}, 1), (v_n, 1)\}$. See Figure 3. Function g is a TRDF on $G[v_1, v_n]$ with $g(v_n) = 1$ and $w(g) = w(h) + 2 \leq \alpha^2(\texttt{MIN}(v_{n-1})) + 2$. Hence, there is a TRDF f on $G[v_1, v_n]$ with $f(v_n) = 1$ and $w(f) \leq \min\{\alpha^2(\texttt{MIN}(v_{n-1})) + 2, \alpha^2(u) + 1\} = \gamma^1(v_n)$.

- Instruction $\alpha^1(v_n) \leftarrow \min\{\gamma^0(v_{n-1}), \alpha^2(u)\} + 1$ (Line 13 of Algorithm 3.1):

  The induction hypothesis implies that there is a TRDF h on $G[v_1, v_{n-1}]$ with $h(v_{n-1}) = 0$ and $w(h) \leq \gamma^0(v_{n-1})$. Consider function $f = h \cup \{(v_n, 1)\}$. Function f is an 1-TRDF on $G[v_1, v_n]$ with $w(f) = w(h) + 1 \leq \gamma^0(v_{n-1}) + 1 = \alpha^1(v_n)$.

  The induction hypothesis implies that there is a 2-TRDF h on $G[v_1, u]$ with $w(h) \leq \alpha^2(u)$. Consider function $f = h \cup \{(u^+, 0), \ldots, (v_{n-1}, 0), (v_n, 1)\}$. Function f is an 1-TRDF on $G[v_1, v_n]$ with $w(f) = w(h) + 1 \leq \alpha^2(u) + 1 = \alpha^1(v_n)$. Hence, there is an 1-TRDF f on $G[v_1, v_n]$ and $w(f) \leq \min\{\gamma^0(v_{n-1}), \alpha^2(u)\} + 1 = \alpha^1(v_n)$.

- Instruction $\gamma^1(v_n) \leftarrow \min\{\alpha^1(u), \alpha^2(u)\} + 1$ (Line 15 of Algorithm 3.1):

  The condition of Line 11 does not holds and so $u = v_{n-1}$. Let $p \in \{1, 2\}$. The induction hypothesis implies that there is a p-TRDF h_p on $G[v_1, v_{n-1}]$ with $w(h_p) \leq \alpha^p(v_{n-1})$. Consider function $f_p = h_p \cup \{(v_n, 1)\}$. Function f_p is a TRDF on $G[v_1, v_n]$ with $f_p(v_n) = 1$ and $w(f_p) = w(h_p) + 1 \leq \alpha^p(v_{n-1}) + 1$. So, there is a TRDF f on $G[v_1, v_n]$ with $f(v_n) = 1$ and $w(f) \leq \min\{\alpha^1(v_{n-1}), \alpha^2(v_{n-1})\} + 1 = \gamma^1(v_n)$.

- Instruction $\alpha^1(v_n) \leftarrow \min\{\gamma^0(u), \alpha^1(u), \alpha^2(u)\} + 1$ (Line 13 of Algorithm 3.1):

  Since the condition of Line 11 does not holds, we have $u = v_{n-1}$. By the correctness proof of Instruction $\alpha^1(v_n) \leftarrow \{\gamma^0(v_{n-1}), \alpha^2(u)\} + 1$ (Line 13 of Algorithm 3.1), there is an 1-TRDF g_1 on $G[v_1, v_n]$ with $w(g_1) \leq \min\{\gamma^0(v_{n-1}), \alpha^2(v_{n-1})\} + 1$.

  By the proof of the previous case, there is a TRDF g_2 on $G[v_1, v_n]$ with $g_2(v_n) = 1$ and $w(g_2) \leq \min\{\alpha^1(v_{n-1}), \alpha^2(v_{n-1})\} + 1$. Function g_2 is an 1-TRDF on $G[v_1, v_n]$. Therefore, there is an 1-TRDF f on $G[v_1, v_n]$ with $w(f) \leq \min\{\gamma^0(v_{n-1}), \alpha^1(v_{n-1}), \alpha^2(v_{n-1})\} + 1 = \alpha^1(v_n)$.

This completes the proof. □

By Lemma 3.1, we have the following result.

Corollary 3.1. *Let $G = (V, E)$ be a connected proper interval graph with $|V| = n \geq 2$ and a consecutive ordering $(v_1, v_2, \ldots, v_n)$ of vertices of G and let γ be the output of Algorithm TRDN$(G, v_1, \ldots, v_n)$. Then, there is a TRDF f on G with $w(f) \leq \gamma$.*

Lemma 3.2. *Let $G = (V, E)$ be a connected proper interval graph with $|V| = n \geq 2$ and a consecutive ordering $(v_1, v_2, \ldots, v_n)$ of vertices of G, let $i \in \{1, 2, \ldots, n\}$, $j \in \{0, 1, 2\}$ and $p \in \{1, 2\}$. Then,*

- *there is a minimum TRDF f on $G[v_1, v_i]$ with $f(v_i) = j$ such that $\gamma^j(v_i) \leq w(f)$.*
- *there is a minimum p-TRDF f on $G[v_1, v_i]$ with $f(v_i) = p$ such that $\alpha^p(v_i) \leq w(f)$, and*
- *there is a minimum TRDF f on $G[v_1, v_{i-1}]$ such that $\gamma(v_i) \leq w(f)$.*

Proof. Recall that $\mathtt{t}_G$ is the number of iterations of the **while** loop of Algorithm TRDN$(G, v_1, \ldots, v_n)$. The proof is by induction on $\mathtt{t}_G$.

Let G be a graph such that $\mathtt{t}_G = 0$. So, Algorithm TRDN$(G, v_1, \ldots, v_n)$ runs only Lines 1-7 of Algorithm 3.1. In Lines 2-3 of Algorithm 3.1, we have

- $\gamma^0(v_1), \gamma^1(v_1), \gamma^2(v_1) \leftarrow \infty$,
- $\alpha^1(v_1) \leftarrow 1$,
- $\alpha^2(v_1) \leftarrow 2$,
- $\gamma(v_1) \leftarrow 0$,
- $\gamma^0(v_2), \gamma(v_2) \leftarrow \infty$,
- $\gamma^1(v_2), \alpha^1(v_2), \alpha^2(v_2) \leftarrow 2$,
- $\gamma^2(v_2) \leftarrow 3$

It is not hard to see that $\gamma(G, v_1)$ is equal to 0, $\alpha^1(G, v_1)$ is equal to 1, all $\alpha^2(G, v_1)$, $\gamma^1(G, v_2)$, $\alpha^1(G, v_2)$ and $\alpha^2(G, v_2)$ are equal to 2, $\gamma^2(G, v_2)$ is equal to 3 and all $\gamma^0(G, v_1)$, $\gamma^1(G, v_1)$, $\gamma^2(G, v_1)$ $\gamma^0(G, v_2)$ and $\gamma(G, v_2)$ are undefined.

Let $x = \mathtt{MAX}(v_1)$ and assume that the condition of Line 4 of Algorithm 3.1 holds, that is, $x \geq v_3$. Since $\mathtt{t}_G = 0$, the condition of Line 8 of Algorithm 3.1 does not hold, that is, $x \geq v_n$. Hence, $x = v_n$, that is, $v_1 v_n \in E$ and so, by Lemma 2.1, G is the complete graph. It is easy to see that the claim holds for each $v \in [v_3, v_n]$. This proves the base case of the induction.

Assume that the result is true for any connected interval graph $H = (V, E)$ with $\mathtt{t}_H \geq m$, where $m \geq 0$. Let $G = (V, E)$ be a connected proper interval graph with $\mathtt{t}_G = m + 1$ and a consecutive ordering $(v_1, v_2, \ldots, v_n)$ of vertices of G. In the last iteration of the **while** loop of Algorithm TRDN$(G, v_1, \ldots, v_n)$, we have $x = v_n$. Let $v \in [v_1, v_{n-1}]$. The induced subgraph $G[v_1, v]$ is a connected interval graph with a consecutive ordering $(v_1, v_2, \ldots, v)$. Consider Algorithm TRDN$(G[v_1, v], v_1, \ldots, v)$. We have $\mathtt{t}_{G[v_1,v]} \leq m$. Let $u = \mathtt{MIN}(v_n)$. We deduce $u \leq v_{n-1}$.

Let f be a minimum TRDF on G with $f(v_n) = 0$, that is, $w(f) = \gamma^0(G, v_n)$. Since $f(v_n) = 0$, there is a vertex $v \in [u, v_{n-1}]$ with $f(v) = 2$. Since f is a TRDF, there is a vertex $v' \in N_{G[v_1,v_n]}(v)$ with $f(v') > 0$. Since $N_{G[v_1,v_n]}(v) \subseteq N_{G[v_1,v_n]}(u)$, $uv' \in E$. Assume $u \neq v$. If we replace $f(u)$ and $f(v)$ by 2 and 0, respectively, then the resulting function is a TRDF on G with weight less than or equal to $w(f)$. Hence, we may assume $f(u) = 2$. Since $N_{G[v_1,v_n]}(v) \subseteq N_{G[v_1,v_n]}(u)$ for any $v \in [u, v_n]$, if $f(v) = a > 0$, then we can replace $f(v)$ and $f(\mathtt{MIN}(u))$, respectively, by 0 and $a + b$ to obtain a new TRDF on G with weight less than or equal to $w(f)$, where $f(\mathtt{MIN}(u)) = b$ and the addition in modulo 3. So, we may assume $f(v) = 0$ for any $v \in [u^+, v_n]$. Let f' be the restriction of f to $G[v_1, u]$. Function f' is a TRDF on $G[v_1, u]$ with $f'(u) = 2$, that is, $\gamma^2(G, u) \leq w(f')$. The induction hypothesis implies that there is a minimum TRDF g on $G[v_1, u]$ with $g(u) = 2$ such that $\gamma^2(u) \leq w(g)$. We have $w(g) = \gamma^2(G, u)$. Hence, $\gamma^2(u) \leq w(g) = \gamma^2(G, u) \leq w(f') = w(f)$. In the last iteration of the **while** loop of Algorithm TRDN$(G, v_1, \ldots, v_n)$ (Line 9), we have $\gamma^0(v_n) \leftarrow \gamma^2(u)$. Therefore, $\gamma^0(v_n) \leq w(f)$.

Let f be a minimum TRDF on G with $f(v_n) = 1$, that is, $w(f) = \gamma^1(G, v_n)$. Since f is a TRDF on G and $f(v_n) = 1$, there is a vertex $v \in [u, v_{n-1}]$ with $f(v) > 0$. Since $N_{G[v_1,v_n]}(v) \subseteq N_{G[v_1,v_n]}(u)$, we can replace $f(v)$ and $f(u)$, respectively, by 0 and $a + b$ to obtain a new TRDF on G with weight less than or equal to $w(f)$, where $f(v) = a$, $f(u) = b$ and the addition in modulo 3. So, we may assume that $f(u) > 0$ and $f(v) = 0$ for any $v \in [u^+, v_{n-1}]$. We distinguish two cases depending on (i) $u < v_{n-1}$ or (ii) $u = v_{n-1}$.

(i) Let $u < v_{n-1}$.

Since $u < v_{n-1}$, the condition of Line 11 of Algorithm TRDN$(G, v_1, \ldots, v_n)$ holds in the last iteration of the **while** loop. We know $f(u) \in \{1, 2\}$. In the following we consider these cases.

- Let $f(u) = 2$.

  Let f' be the restriction of f to $G[v_1, u]$. Since $f(v) = 0$ for any $v \in [u^+, v_{n-1}]$, function f' is a 2-TRDF on $G[v_1, u]$, that is, $\alpha^2(G, u) \leq w(f')$. The induction hypothesis implies that there is a minimum 2-TRDF g on $G[v_1, u]$ such that $\alpha^2(u) \leq w(g)$. We have $w(g) = \alpha^2(G, u)$. Hence, $\alpha^2(u) \leq w(g) = \alpha^2(G, u) \leq w(f') = w(f) - 1$, that is, $\alpha^2(u) \leq w(f) - 1$.

- Let $f(u) = 1$.

  Since $f(v_{n-1}) = 0$, there is a vertex $v \in [\mathtt{MIN}(v_{n-1}), v_{n-2}]$ with $f(v) = 2$. Recall $f(x) = 0$ for any $x \in [u^+, v_{n-1}]$ and $f(u) = 1$. So, $v \in [\mathtt{MIN}(v_{n-1}), u^-]$. Since $f(v_n) = f(u) = 1$, we may assume $f(\mathtt{MIN}(v_{n-1})) = 2$ and $f(v) = 0$ for any $v \in [\mathtt{MIN}(v_{n-1})^+, u^-]$. Let f' be the restriction of f to $G[v_1, \mathtt{MIN}(v_{n-1})]$. Function f' is a 2-TRDF on $G[v_1, \mathtt{MIN}(v_{n-1})]$, that is, $\alpha^2(G, \mathtt{MIN}(v_{n-1})) \leq w(f')$. The induction hypothesis implies that there is a minimum 2-TRDF g on $G[v_1, \mathtt{MIN}(v_{n-1})]$ such that $\alpha^2(\mathtt{MIN}(v_{n-1})) \leq w(g)$. We have $w(g) = \alpha^2(G, \mathtt{MIN}(v_{n-1}))$. Hence, $\alpha^2(\mathtt{MIN}(v_{n-1})) \leq w(g) = \alpha^2(G, \mathtt{MIN}(v_{n-1})) \leq w(f') = w(f) - 2$, that is, $\alpha^2(\mathtt{MIN}(v_{n-1})) \leq w(f) - 2$.

Since $u < v_{n-1}$, in the last iteration of the **while** loop of Algorithm TRDN$(G, v_1, \ldots, v_n)$ (Line 12), we have $\gamma^1(v_n) \leftarrow \min\{\alpha^2(\mathtt{MIN}(v_{n-1})) + 2, \alpha^2(u) + 1\}$. This, together with $\alpha^2(u) \leq w(f) - 1$ and $\alpha^2(\mathtt{MIN}(v_{n-1})) \leq w(f) - 2$, implies that $\gamma^1(v_n) \leq w(f)$.

(ii) Let $u = v_{n-1}$.

Since $u = v_{n-1}$, $f(v_{n-1}) = p \in \{1, 2\}$. Let f' be the restriction of f to $G[v_1, v_{n-1}]$. Function f' is a p-TRDF on $G[v_1, v_{n-1}]$, that is, $\alpha^p(G, v_{n-1}) \leq w(f')$. The induction hypothesis implies that there is a minimum p-TRDF g on $G[v_1, v_{n-1}]$ such that $\alpha^p(v_{n-1}) \leq w(g)$. We have $w(g) = \alpha^p(G, v_{n-1})$. Hence, $\alpha^p(v_{n-1}) \leq w(g) = \alpha^p(G, v_{n-1}) \leq w(f') = w(f) - 1$, that is, $\alpha^p(v_{n-1}) \leq w(f) - 1$. In the last iteration of the **while** loop of Algorithm TRDN$(G, v_1, \ldots, v_n)$ (Line 12), we have $\gamma^1(v_n) \leftarrow \min\{\alpha^1(u), \alpha^2(u)\} + 1$. Therefore, $\gamma^1(v_n) \leq w(f)$.

Let f be a minimum TRDF on G with $f(v_n) = 2$, that is, $w(f) = \gamma^2(G, v_n)$. Since f is a TRDF on G and $f(v_n) = 2$, there is a vertex $v \in [u, v_{n-1}]$ with $f(v) > 0$. We may assume $f(u) = p \in \{1, 2\}$ and $f(v) = 0$ for any $v \in [u^+, v_{n-1}]$. Let f' be the restriction of f to $G[v_1, u]$. Function f' is a p-TRDF on $G[v_1, u]$, that is, $\alpha^p(G, u) \leq w(f')$. The induction hypothesis implies that there is a minimum p-TRDF g on $G[v_1, u]$ such that $\alpha^p(u) \leq w(g)$. We have $w(g) = \alpha^p(G, v_{n-1})$. Hence, $\alpha^p(u) \leq w(g) = \alpha^p(G, v_{n-1}) \leq w(f') = w(f) - 2$, that is, $\alpha^p(u) \leq w(f) - 2$. In the last iteration of the **while** loop of Algorithm TRDN$(G, v_1, \ldots, v_n)$ (Line 9), we have $\gamma^2(v_n) \leftarrow \min\{\alpha^1(u), \alpha^2(u)\} + 2$. Therefore, $\gamma^2(v_n) \leq w(f)$.

Assume $j \in \{0, 1, 2\}$ and let f be a minimum TRDF on $G[v_1, v_{n-1}]$, that is, $w(f) = \gamma(G, v_n)$. Clearly, $f(v_{n-1}) \in \{0, 1, 2\}$, that is, $w(f) = \min\{\gamma^0(G, v_{n-1}), \gamma^1(G, v_{n-1}), \gamma^2(G, v_{n-1})\}$. The induction hypothesis implies $\gamma^j(v_{n-1}) \leq \gamma^j(G, v_{n-1})$. Thus, $\min\{\gamma^0(v_{n-1}), \gamma^1(v_{n-1}), \gamma^2(v_{n-1})\} \leq \min\{\gamma^0(G, v_{n-1}), \gamma^1(G, v_{n-1}), \gamma^2(G, v_{n-1})\}$. In the last iteration of the **while** loop of Algorithm TRDN$(G, v_1, \ldots, v_n)$ (Line 10), we have $\gamma(v_n) \leftarrow \min\{\gamma^0(v_{n-1}), \gamma^1(v_{n-1}), \gamma^2(v_{n-1})\}$. So, $\gamma(v_n) \leq w(f)$.

Let f be a minimum 1-TRDF on G, that is, $w(f) = \alpha^1(G, v_n)$. We distinguish two cases depending on (i) $u < v_{n-1}$ or (ii) $u = v_{n-1}$.

(i) Let $u < v_{n-1}$.

Assume $f(v) = a > 0$ for some $v \in [\mathtt{MIN}(v_{n-1})^+, v_{n-1}]$. Since $N_{G[v_1,v_{n-1}]}(v) \subseteq N_{G[v_1,v_{n-1}]}(\mathtt{MIN}(v_{n-1}))$, we can replace $f(\mathtt{MIN}(v_{n-1}))$ and $f(v)$ by $a + b$ and 0, respectively, to obtain a new 1-TRDF on G with weight less than or equal to $w(f)$, where $f(\mathtt{MIN}(v_{n-1})) = b$ and the addition in module 3. So, we may assume $f(v) = 0$ for any $v \in [\mathtt{MIN}(v_{n-1})^+, v_{n-1}]$. Since $f(v_{n-1}) = 0$ and f is a 1-TRDF on G, $f(\mathtt{MIN}(v_{n-1})) = 2$. Since $v_{n-1} < v_n$, $\mathtt{MIN}(v_{n-1}) \leq u$. So, either $\mathtt{MIN}(v_{n-1}) < u$ or $\mathtt{MIN}(v_{n-1}) = u$. In the following we consider these cases.

- Assume $\mathtt{MIN}(v_{n-1}) < u$.
  Let f' be the restriction of f to $G[v_1, v_{n-1}]$. Function f' is a TRDF on $G[v_1, v_{n-1}]$ with $f'(v_{n-1}) = 0$, that is, $\gamma^0(G, v_{n-1}) \leq w(f')$. The induction hypothesis implies that there is a minimum TRDF g on $G[v_1, v_{n-1}]$ with $g(v_{n-1}) = 0$ such that $\gamma^0(v_{n-1}) \leq w(g)$. We have $w(g) = \gamma^0(G, v_{n-1})$. Hence, $\gamma^0(v_{n-1}) \leq w(g) = \gamma^0(G, v_{n-1}) \leq w(f') = w(f) - 1$, that is, $\gamma^0(v_{n-1}) \leq w(f) - 1$. Since $u < v_{n-1}$ and $\mathtt{MIN}(v_{n-1}) < u$, in the last iteration of the **while** loop of Algorithm TRDN$(G, v_1, \ldots, v_n)$ (Line 13), we have $\alpha^1(v_n) \leftarrow \{\gamma^0(v_{n-1}), \alpha^2(u)\} + 1$. Therefore, $\alpha^1(v_n) \leq w(f)$.
- Assume $\mathtt{MIN}(v_{n-1}) = u$.
  Let f' be the restriction of f to $G[v_1, u]$. Since $f(v_n) = 1$, function f' is a 2-TRDF on $G[v_1, u]$, that is, $\alpha^2(G, u) \leq w(f')$. The induction hypothesis implies that there is a minimum 2-TRDF g on $G[v_1, u]$ such that $\alpha^2(u) \leq w(g)$. We have $w(g) = \alpha^2(G, u)$. Hence, $\alpha^2(u) \leq w(g) = \alpha^2(G, u) \leq w(f') = w(f) - 1$, that is, $\alpha^2(u) \leq w(f) - 1$. Since $u < v_{n-1}$ and $\mathtt{MIN}(v_{n-1}) = u$, in the last iteration of the **while** loop of Algorithm TRDN$(G, v_1, \ldots, v_n)$ (Line 13), we have $\alpha^1(v_n) \leftarrow \{\gamma^0(v_{n-1}), \alpha^2(u)\} + 1$. Therefore, $\alpha^1(v_n) \leq w(f)$.

(ii) Let $u = v_{n-1}$.

Let f' be the restriction of f to $G[v_1, v_{n-1}]$. Since $u = v_{n-1}$ and f is a 1-TRDF on G, $f(v_{n-1}) \in \{0, 1, 2\}$. In the following we consider these cases.

- Let $f(v_{n-1}) = 0$
  Function f' is a TRDF on $G[v_1, v_{n-1}]$ with $f(v_{n-1}) = 0$, that is, $\gamma^0(G, v_{n-1}) \leq w(f')$. The induction hypothesis implies that there is a minimum TRDF g on $G[v_1, v_{n-1}]$ with $g(v_{n-1}) = 0$ such that $\gamma^0(v_{n-1}) \leq w(g)$. We have $w(g) = \gamma^0(G, v_{n-1})$. Hence, $\gamma^0(v_{n-1}) \leq w(g) = \gamma^0(G, v_{n-1}) \leq w(f') = w(f) - 1$, that is, $\gamma^0(v_{n-1}) \leq w(f) - 1$.

- Let $f(v_{n-1}) = p \in \{1, 2\}$. Since $f(v_n) = 1$, function f' is a p-TRDF on $G[v_1, v_{n-1}]$, that is, $\alpha^p(G, v_{n-1}) \leq w(f')$. The induction hypothesis implies that there is a minimum p-TRDF g on $G[v_1, v_{n-1}]$ such that $\alpha^p(v_{n-1}) \leq w(g)$. We have $w(g) = \alpha^p(G, v_{n-1})$. Hence, $\alpha^p(v_{n-1}) \leq w(g) = \alpha^p(G, v_{n-1}) \leq w(f') = w(f) - 1$, that is, $\alpha^p(v_{n-1}) \leq w(f) - 1$.

Since $u = v_{n-1}$, in the last iteration of the **while** loop of Algorithm TRDN$(G, v_1, \dots, v_n)$ (Line 16), we have $\alpha^1(v_n) \leftarrow \min\{\gamma^0(u), \alpha^1(u), \alpha^2(u)\} + 1$. This, together with $\gamma^0(v_{n-1}) \leq w(f) - 1$ and $\alpha^p(v_{n-1}) \leq w(f) - 1$, implies that $\alpha^1(v_n) \leq w(f)$.

Let f be a minimum 2-TRDF on G, that is, $w(f) = \alpha^2(G, v_n)$. If $f(v) = a > 0$ for some $v \in [u^+, v_{n-1}]$, then since $N_{G[v_1,v_n]}(v) \subseteq N_{G[v_1,v_n]}(u)$, we can replace $f(u)$ and $f(v)$ by $a + b$ and 0, respectively, to obtain a new 2-TRDF on G with weight less than or equal to $w(f)$, where $f(u) = b$ and the addition in module 3. So, we may assume $f(v) = 0$ for any $v \in [u^+, v_{n-1}]$. Let f' and f'' be the restrictions of f to $G[v_1, u]$ and $G[v_1, u^-]$, respectively. Clearly, $f(u) \in \{0, 1, 2\}$. In the following we consider these cases.

- Let $f(u) = 0$.

  Function f'' is a TRDF on $G[v_1, u^-]$, that is, $\gamma(G, u) \leq w(f'')$. The induction hypothesis implies that there is a minimum TRDF g on $G[v_1, u^-]$ with $g(u) = 0$ such that $\gamma(u) \leq w(g)$. We have $w(g) = \gamma(G, u)$. Hence, $\gamma(u) \leq w(g) = \gamma(G, u) \leq w(f'') = w(f) - 2$, that is, $\gamma(u) \leq w(f) - 2$.

- Let $f(u) = p \in \{1, 2\}$. Function f' is a p-TRDF on $G[v_1, u]$, that is, $\alpha^p(G, u) \leq w(f')$. The induction hypothesis implies that there is a minimum p-TRDF g on $G[v_1, u]$ such that $\alpha^p(u) \leq w(g)$. We have $w(g) = \alpha^p(G, u)$. Hence, $\alpha^p(u) \leq w(g) = \alpha^p(G, u) \leq w(f') = w(f) - 2$, that is, $\alpha^p(u) \leq w(f) - 2$.

In the last iteration of the **while** loop of Algorithm TRDN$(G, v_1, \dots, v_n)$ (Line 10), we have $\alpha^2(v_n) \leftarrow \min\{\gamma(u), \alpha^1(u), \alpha^2(u)\} + 2$. This, together with $\gamma(u) \leq w(f) - 2$ and $\alpha^p(u) \leq w(f) - 2$, implies that $\alpha^2(v_n) \leq w(f)$. This completes the proof. □

By Lemma 3.2, we have the following result.

Corollary 3.2. *Let $G = (V, E)$ be a connected proper interval graph with $|V| = n \geq 2$ and a consecutive ordering $(v_1, v_2, \dots, v_n)$ of vertices of G and let γ be the output of Algorithm TRDN$(G, v_1, \dots, v_n)$. Then, there is a minimum TRDF f on G with $\gamma \leq w(f)$.*

Theorem 3.1. *Let $G = (V, E)$ be a connected proper interval graph with $|V| = n \geq 2$ and a consecutive ordering $(v_1, v_2, \dots, v_n)$ of vertices of G. Algorithm TRDN$(G, v_1, \dots, v_n)$ computes the total Roman domination number of G in $\mathcal{O}(n)$ time.*

Proof. Let γ be the output of Algorithm TRDN$(G, v_1, \dots, v_n)$. By Corollaries 3.1 and 3.2, we have $\gamma = \gamma_{tR}(G)$. In the following we consider the time complexity of Algorithm TRDN$(G, v_1, \dots, v_n)$. By (Algorithm 2 of) [6], we can compute all values $\texttt{MAX}(v_1), \dots, \texttt{MAX}(v_n)$ in $\mathcal{O}(n)$ time. Clearly,

$(v_n, v_{n-1}, \ldots, v_2, v_1)$ is a consecutive ordering of vertices of G. Also, we can compute all values $\texttt{MIN}(v_1), \texttt{MIN}(v_2), \ldots, \texttt{MIN}(v_n)$ in $\mathcal{O}(n)$ time. It suffices by (Algorithm 2 of) [6] to compute all values $\texttt{MAX}(v_n), \texttt{MAX}(v_{n-1}), \ldots, \texttt{MAX}(v_2), \texttt{MAX}(v_1)$ for G with consecutive ordering $(v_n, v_{n-1}, \ldots, v_2, v_1)$. So, the running time of Line 1 of Algorithm TRDN$(G, v_1, \ldots, v_n)$ is $\mathcal{O}(n)$. Since we know $\texttt{MAX}(v_i)$ and $\texttt{MIN}(v_i)$ for all $i \in \{1, 2, \ldots, n\}$, the running time of Lines 2-7 of Algorithm TRDN$(G, v_1, \ldots, v_n)$ is $\mathcal{O}(n)$ and each iteration of Algorithm TRDN$(G, v_1, \ldots, v_n)$ (Lines 9-16) is $\mathcal{O}(1)$. So, the running time of Algorithm TRDN$(G, v_1, \ldots, v_n)$ is $\mathcal{O}(n)$. This completes the proof. □

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