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Constructions of new integral graph families

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Abstract

We construct new families of integral graphs by considering complete products, unions and point identifications of complete graphs and complete bipartite graphs. In particular, we find a relation between arithmetic series and the integrality of complete products.

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1. Introduction

The notion of an integral graph, that is, a graph having integral eigenvalues only, was first introduced by Harary and Schwenk [5] in 1973/74. All graphs considered in this paper are simple, i.e. undirected and without loops or multiple edges. The quest of characterizing all integral graphs seems to be a challenging project. There are several infinite families of integral graphs known (cf. [5, 4, 7, 6, 1]), but they appear rarely compared to the huge number of graphs at all. Thus, there is still no satisfying answer to Harary's and Schwenk's question '*Which graphs have integral spectra?*'. Wang et. al. [6, 7] and also Hansen et. al. [4] introduced new families of integral graphs by combining common examples of integral graphs, like complete graphs or complete bipartite graphs. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs, then the *union* $G_1 \cup G_2$ of G_1 and G_2 is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The notion aG_1 is short for the a-folded union $\underbrace{G_1 \cup G_1 \cup \ldots \cup G_1}_{a}$. In the following we write $v \sim w$ for adjacent vertices

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v and w. The *direct product* $G_1 \times G_2$ is the graph with vertex set $V_1 \times V_2$ and two vertices $(v_1, v_2), (v_1', v_2') \in V_1 \times V_2$ are adjacent in $G_1 \times G_2$ if and only if $v_1 \sim v_1'$ in G_1 and $v_2 \sim v_2'$ in G_2. The graph $G_1 \Box G_2$ denotes the *cartesian product* of G_1 and G_2 and consists of vertices $V_1 \times V_2$ where two vertices $(v_1, v_2), (v_1', v_2') \in V_1 \times V_2$ are adjacent in $G_1 \Box G_2$ if and only if either $v_1 = v_1'$ and $v_2 \sim v_2'$ or $v_1 \sim v_1'$ and $v_2 = v_2'$. Note that if G_1 and G_2 are integral graphs, then the graphs $G_1 \cup G_2$, $G_1 \times G_2$ and $G_1 \Box G_2$ are integral, too, and if, in addition, G_1 and G_2 are regular, then also the respective *complement graphs* $\overline{G_1}$ and $\overline{G_2}$ are integral. A proof for this can be found in [2].

In this paper, we construct new families of integral graphs by considering the following two products:

Definition 1.1. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs, then the *complete product* $G_1 \nabla G_2$ has vertex set $V_1 \cup V_2$ and v_1 and v_2 are adjacent in $G_1 \nabla G_2$ if and only if either $v_1 \in V_1$ and $v_2 \in V_2$ or v_1 and v_2 are adjacent in G_1 or G_2.

Definition 1.2. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs and let $v_1 \in V_1$, $v_2 \in V_2$. Then the *point identification* $G_1 \bullet G_2$ arises from setting $v_1 = v_2$.

The graph obtained by applying a point identification a-times to the same graph G and the same vertex of G is denoted by $G\overset{a}{\bullet}$.

Note that these two products are not closed under integrality. However, the following lemma stated by Hansen et. al. [4] characterizes all complete products of regular graphs with integral eigenvalues:

Lemma 1.1. *For $i = 1, 2$ let G_i be regular graphs of degree r_i with n_i vertices. The complete product $G_1 \nabla G_2$ is an integral graph if and only if both, G_1 and G_2, are integral graphs and there exists $k \in \mathbb{Z}$ such that the* ***integrality condition***

$$n_1 n_2 = k(k + r_1 - r_2) \tag{1}$$

holds.

In Section 2 we construct new families of integral graphs by considering complete products satisfying Lemma 1.1. In particular, we study families of integral graphs of the form $G_1 \nabla a G_2$ and provide relations between the parameter a and some arithmetic series. In Section 3 we investigate point identifications of integral graphs such as complete graphs and complete bipartite graphs. We also find new families of integral graphs of the form $K_{m,n}\overset{a}{\bullet}$. Throughout the paper, $K_{m,n}$ denotes a complete bipartite graphs with bipartitions of order m and n, K_n denotes a complete graph of order n, and $G - v$ denotes the graph obtained by deleting the vertex v of G and all its adjacent edges.

2. Integral complete products and airthmetic series

Lemma 1.1 provides a nice characterization of integral complete products of regular graphs. However, it is not easy to find families of such graphs satisfying Equation (1). Some examples are stated in the following theorem:

Theorem 2.1. *Every graph of any of the infinite families $K_1\nabla(2n-3)K_{n-1}$, $K_2\nabla nK_{n-2}$ and $K_m\nabla((m-1)(n-1))K_n$ is integral.*

Proof. We show that each graph of these families satisfy the respective integrality condition. Since all graphs are complete products of integral regular graphs, our statement therefore follows from Lemma 1.1.
Every graph of the first family $K_1\nabla(2n-3)K_{n-1} = K_n \overset{2n-3}{\bullet}$ satisfies the integrality condition

$$(2n-3)(n-1) = k(k+(n-2))$$

for $k = n-1 \in \mathbb{Z}$.
The graph K_2 is regular of degree one with two vertices and nK_{n-2} is regular of degree $n-3$ with $n(n-2)$ vertices. Thus, the integrality condition of $K_2\nabla nK_{n-2}$ is

$$2n(n-2) = k(k+(n-4))$$

and indeed holds for $k = n \in \mathbb{Z}$.
Similarly, we can see that the integrality condition of $K_m\nabla((m-1)(n-1))K_n$ is

$$m(m-1)(n-1)n = k(k+(m-1)-(n-1)).$$

This equation has integer solutions $k = mn-m$ and $k = n-mn$. □

This theorem motivates the question, for which $a \in \mathbb{N}$ the graph $K_m\nabla aK_n$ is integral. Some notable observations are stated in the next result, which is also a consequence of Lemma 1.1.

Theorem 2.2. *Let $a, b \in \mathbb{N}$. Furthermore, let H be an integral regular graph and G_1 and G_2 be regular graphs.*

1. *The graph $H\nabla aH$ is integral if and only if a is a square.*
2. *The graph $G_1\nabla aG_2$ is integral if and only if the graph $G_2\nabla aG_1$ is integral.*
3. *The graph $aG_1\nabla bG_2$ is integral if and only if the graph $G_1\nabla abG_2$ is integral.*

Proof. The integrality condition for $H\nabla aH$ is $a|H|^2 = k^2$. Therefore, the first statement follows from Lemma 1.1. If G_1 or G_2 is non-integral, then by Lemma 1.1 neither $G_1\nabla aG_2$ nor $G_2\nabla aG_1$ is integral. On the other hand, if both graphs, G_1 and G_2, are integral, $G_1\nabla aG_2$ and $G_2\nabla aG_1$ satisfy the same integrality condition since $k(k+(r_1-r_2)) = -k(-k+(r_2-r_1))$. This implies the second statement and the third statement follows easily from the second one. □

Besides this symmetry aspect, we also found that for fixed $n, m \in \mathbb{N}$ the parameters $a \in \mathbb{N}$, for which the graphs $K_n\nabla aK_m$ are integral, form some arithmetic series. The next theorem provides several new infinite families of integral graphs:

Theorem 2.3. *Let $p, q, r, t, u, n \in \mathbb{N}$ and*

$$s_p = \sum_{i=1}^{p} a_i \quad \textit{with} \quad a_i = 1 + (i-1)2n,$$
$$s_q = \sum_{i=1}^{q} b_i \quad \textit{with} \quad b_i = (2n-1) + (i-1)2n,$$
$$s_r = \sum_{i=1}^{r} c_i \quad \textit{with} \quad c_i = 1 + (i-1)n,$$
$$s_t = \sum_{i=1}^{t} d_i \quad \textit{with} \quad d_i = (n-1) + (i-1)n,$$
$$s_u = \sum_{i=1}^{u} e_i \quad \textit{with} \quad e_i = -5 + (i-1)24.$$

Then, every graph of any of the families $K_1\nabla s_pK_n$, $K_1\nabla s_qK_n$, $K_2\nabla s_rK_n$, $K_2\nabla s_tK_n$ and $K_3\nabla(6+s_u)K_4$ is integral.

Proof. We observe that

$$s_p = \sum_{i=1}^{p}\big(1+(i-1)2n\big) = p\frac{2+(p-1)2n}{2} = np^2 - np + p.$$

By Lemma 1.1, the graph $K_1\nabla s_pK_n$ is integral if and only if there exists $k \in \mathbb{Z}$ such that $ns_p = n^2p^2 - n^2p + np = k(k-(n-1))$ holds. Indeed, this is true for $k = pn$. The remaining cases can be proven analogously. □

In view of the latter theorem and several computer experiments we conjecture the following:

Conjecture 1. *Let $a, m, n \in \mathbb{N}$ and $G = K_m\nabla aK_n$. If G is integral, then there exists an arithmetic series $z+s_p$ for $p \in \mathbb{N}$ and $z \in \mathbb{Z}$ such that $a \in z+s_p$ and all graphs of the family $K_m\nabla(z+s_p)K_n$ are integral.*

Given integral regular graphs G_1, G_2 and H_1, H_2, the next theorem provides a relation between the integrality of $G_1\nabla aG_2$ and $H_1\nabla bH_2$ for $a, b \in \mathbb{N}$.

Theorem 2.4. *Let $a \in \mathbb{N}$ and G_1, G_2 be integral regular graphs of degree r_1 and r_2, respectively, and let H_1, H_2 be integral regular graphs of degree s_1 and s_2, respectively. If*

$$s_1 - r_1 = s_2 - r_2 \quad \textit{and} \quad \frac{a|G_1||G_2|}{|H_1||H_2|} \in \mathbb{N},$$

then the graph $G_1\nabla aG_2$ is integral if and only if the graph

$$H_1\nabla\frac{a|G_1||G_2|}{|H_1||H_2|}H_2$$

is integral.

Proof. By Lemma 1.1, the graph $G_1\nabla aG_2$ is integral if and only if there is $k \in \mathbb{Z}$ with

$$a|G_1||G_2| = k(k + (r_1 - r_2)). \tag{2}$$

Since $s_1 - r_1 = s_2 - r_2$, we have that $r_1 - r_2 = s_1 - s_2$ and, therefore, Equation (2) is equivalent to

$$\frac{a|G_1||G_2|}{|H_1||H_2|}|H_1||H_2| = k(k + (s_1 - s_2)),$$

which equals the integrality condition of $H_1\nabla\frac{a|G_1||G_2|}{|H_1||H_2|}H_2$. □

Hence, the integrality of direct products $G_1\nabla aG_2$ can be verified by considering suitable complete graphs K_{r_1}, K_{r_2} in the sense that

$$G_1\nabla aG_2 \text{ is integral} \iff K_{r_1}\nabla\frac{a|G_1||G_2|}{r_1r_2}K_{r_2} \text{ is integral.}$$

The following corollary provides some explicit examples:

Corollary 2.1. *Let $a, m, n \in \mathbb{N}$.*

1. *The graph $K_m\nabla(K_n \times K_2)$ is integral if and only if $K_m\nabla 2K_n$ is integral.*
2. *The graphs*
$$\overline{K_{anm}}\nabla 2K_{n+1}, \quad \overline{K_{anm}}\nabla(K_{n+1} \times K_2)$$
and
$$\overline{K_m}\nabla(a(n+1))(K_n\square K_2)$$
are integral if and only if one of them is integral.
3. *The graph $\overline{K_a}\nabla K_{m,m}$ is integral if and only if $\overline{K_{2a}}\nabla K_m$ is integral.*
4. *The graph $K_{m,m}\nabla aK_{n,n}$ is integral if and only if $K_m\nabla 4aK_n$ is integral.*

3. Constructions of integral graphs by point identifications

We now focus on point identifications in order to construct new families of integral graphs.

Lemma 3.1. *Let G_1 and G_2 be graphs and let v be a vertex of G_1 and w be a vertex of G_2. Then the point identification $v = w$ yields the following characteristic polynomials:*

1. $\chi(G_1 \bullet G_2, x) = \chi(G_1, x)\chi(G_2 - w, x) + \chi(G_1 - v, x)\chi(G_2, x) - x\chi(G_1 - v, x)\chi(G_2 - w, x)$
2. $\chi(G_1\overset{a}{\bullet}, x) = \chi^{a-1}(G_1 - v, x)(a\chi(G_1, x) - (a-1)x\chi(G_1 - v, x))$.

Proof. A proof of the first statement can be found in the book by Cvetković et. al. [3]. For the proof of the second statement we use induction over a. For $a = 2$ we get

$$\begin{aligned}\chi(G_1 \bullet G_1, x) &= 2\chi(G_1, x)\chi(G_1 - v, x) - x\chi^2(G_1 - v, x)\\ &= \chi(G_1 - v, x)(2\chi(G_1, x)) - x\chi(G_1 - v, x).\end{aligned}$$

For the induction step we use the equality $\chi(G_1 \overset{a}{\bullet} -v, x) = \chi^a(G_1 - v, x)$ and, therefore, get

$$\begin{aligned}\chi(G_1 \overset{a+1}{\bullet}, x) &= \chi(G_1 \overset{a}{\bullet}, x)\chi(G_1 - v, x) + \chi(G_1 \overset{a}{\bullet} -v, x)\chi(G_1, x) - \\ &\quad - x\chi(G_1 \overset{a}{\bullet} -v, x)\chi(G_1 - v, x) \\ &= (\chi^{a-1}(G_1 - v, x)(a\chi(G_1, x) - (a-1)x\chi(G_1 - v, x)))\times \\ &\quad \times \chi(G_1 - v, x) + \chi^a(G_1 - v, x)\chi(G_1, x) - \\ &\quad - x\chi^a(G_1 - v, x)\chi(G_1 - v, x) \\ &= \chi^a(G_1 - v, x)((a+1)\chi(G_1, x) - ax\chi(G_1 - v, x)).\end{aligned}$$

□

Starting with a complete graph K_n, we observe that $K_n \overset{a}{\bullet} = K_1 \nabla a K_{n-1}$. Thus, by Lemma 1.1 we can easily deduce the following corollary:

Corollary 3.1. *The graph $K_n \overset{a}{\bullet}$ is integral if and only if there exists $k \in \mathbb{Z}$ such that $a(n-1) = k(k + (n-2))$.*

These families of graphs were already studied in Section 2. Therefore, a next step is to consider graphs of the form $K_m \bullet K_n$ for $m \neq n$. But computer experiments showed that there is no integral graph of this form for $1 \leq m, n \leq 50$, and, moreover, not even an integral graph of the form $(K_l \bullet K_m) \bullet K_n$ with $1 \leq l, m, n \leq 50$. Since the search of integral graphs within these families of graphs therefore seems to be a non-promising approach, we consider point identifications of complete bipartite graphs next. In particular, we could prove the following:

Theorem 3.1. *Let $a, m, n \in \mathbb{N}$. The graph $K_{m,n} \overset{a}{\bullet}$ is integral if and only if $(m-1)n$ and $n(a+m-1)$ are squares.*

Proof. It is well-known that the characteristic polynomial χ of a complete bipartite graph $K_{m,n}$ equals

$$\chi(K_{m,n}, x) = x^{m+n-2}(x^2 - mn).$$

Thus, together with Lemma 3.1, we get that

$$\begin{aligned}\chi(K_{m,n} \overset{a}{\bullet}, x) &= \chi^{a-1}(K_{m-1,n}, x)(a\chi(K_{m,n}, x) - (a-1)x\chi(K_{m-1,n}, x)) \\ &= (x^{m+n-3}(x^2 - (m-1)n))^{a-1}\times \\ &\quad \times (ax^{m+n-2}(x^2 - mn) - (a-1)x^{m+n-2}(x^2 - (m-1)n)) \\ &= x^{a(m+n-3)+1}(x^2 - (m-1)n)^{a-1}\times \\ &\quad \times (a(x^2 - mn) - (a-1)(x^2 - (m-1)n)).\end{aligned}$$

Assuming all roots of χ to be integers, $(m-1)n$ therefore has to be a square. In particular, the last factor of χ is zero if and only if $0 = x^2 - n(a + m - 1)$. This implies the statement. □

With this theorem we again find new families of integral graphs:

Corollary 3.2. *Let $a, m, n \in \mathbb{N}$. Then, the following graphs are integral:*

1. $K_{n^2+1,n^2} \overset{(n+1)^2-n^2}{\bullet}$,
2. $K_{n+1,n} \overset{(a^2-1)n}{\bullet}$,
3. $K_{m,n} \overset{(m-1)(a^2-1)}{\bullet}$ *if* $\sqrt{(m-1)n} \in \mathbb{Z}$.

In particular, this corollary shows that for every $m, n \in \mathbb{N}$ with $\sqrt{(m-1)n} \in \mathbb{Z}$ there exist infinitely many integral graphs of the form $K_{m,n} \overset{a}{\bullet}$ for $a \in \mathbb{N}$.

References

[1] C. Adiga, B.R. Rakshith, and K.N.S. Krishna, Spectra of extended neighborhood corona and extended corona of two graphs, *Electronic Journal of Graph Theory and Applications* **4** (1) (2016), 101–110.

[2] A.E. Brouwer and W.H. Haemers, *Spectra of Graphs*. Springer-Verlag New York, 1st edition (2012).

[3] D.M. Cvetković, M. Doob, and H. Sachs, *Spectra of Graphs: Theory and Applications*. Academic Press, 1st edition (1980).

[4] P. Hansen, H. Mélot, and D. Stevanović, Integral complete split graphs, *Univerzitet u Beogradu. Publikacije Elektrotehničkog Fakulteta. Serija Matematika* **13** (2002), 89–95.

[5] F. Harary and A.J. Schwenk, *Which graphs have integral spectra?*, pages 45–51. Springer Berlin Heidelberg (1974).

[6] L. Wang, H. Broersma, C. Hoede, X. Li, and G. Still, Some families of integral graphs, *Discrete Mathematics* **308** (2008), 6383–6391.

[7] L. Wang, X. Li, and S. Zhang, Construction of integral graphs, *Applied Mathematics-A Journal of Chinese Universities Series B* **15** (2000), 239–246.