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Lower bounds for the algebraic connectivity of graphs with specified subgraphs

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Abstract

The second smallest eigenvalue of the Laplacian matrix of a graph G is called the algebraic connectivity and denoted by $a(G)$. We prove that

$$a(G) > \frac{\pi^2}{3}\left(p\frac{12\overline{g}(n_1,n_2,\ldots,n_p)^2-\pi^2}{4\overline{g}(n_1,n_2,\ldots,n_p)^4}+4(q-p)\frac{3\overline{g}(n_{p+1},n_{p+2},\ldots,n_q)^2-\pi^2}{\overline{g}(n_{p+1},n_{p+2},\ldots,n_q)^4}\right),$$

holds for every non-trivial graph G which contains edge-disjoint spanning subgraphs $G_1, G_2, \ldots, G_q$ such that, for $1 \leq i \leq p$, $a(G_i) \geq a(P_{n_i})$, with $n_i \geq 2$, and, for $p+1 \leq i \leq q$, $a(G_i) \geq a(C_{n_i})$, where P_{n_i} and C_{n_i} denote the path and the cycle of the corresponding order, respectively, and $\overline{g}$ denotes the geometric mean of given arguments. Among certain consequences, we emphasize the following lower bound

$$a(G) > \pi^2\frac{12(4q-3p)n^2-(16q-15p)\pi^2}{12n^4},$$

referring to G which has n ($n \geq 2$) vertices and contains p Hamiltonian paths and $q-p$ Hamiltonian cycles, such that all of them are edge-disjoint. We also discuss the quality of the obtained lower bounds.

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1. Introduction

The *Laplacian* of a graph G is the positive semidefinite matrix $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the standard adjacency matrix. Among all eigenvalues of the Laplacian of a graph, one of the most popular is the second smallest called, by Fiedler [5], the *algebraic connectivity* of a graph. The algebraic connectivity is usually denoted by $a(G)$. Its significance is due to the fact that it measures (to a certain extent) how well a graph is connected. For example, a graph G is connected if and only if $a(G) > 0$.

The number of vertices (also known as the *order*) and the number of edges of a graph G are denoted by n and m (or $m(G)$), respectively. We also use d for the diameter of a graph. A path and a cycle of order n are denoted by P_n and C_n, respectively. A graph is *Hamiltonian* if it contains a spanning subgraph which is a cycle, while every such cycle is referred to as a *Hamiltonian cycle*. Similarly, every spanning path is referred to as a *Hamiltonian path*.

There is a significantly large number of bounds for the algebraic connectivity expressed in terms of other graph invariants. One of them is a classical result of Mohar [8] stating that

$$a(G) \geq \frac{4}{dn}, \tag{1}$$

where, as said above, d is the diameter of G. Some others can be found in [1, 4, 10]. In this study we obtain a lower bound for $a(G)$ which relies on the assumption that G contains edge-disjoint spanning subgraphs such that the algebraic connectivity of each of them is not less than the algebraic connectivity of either a fixed path or a fixed cycle. This result yields the lower bound for $a(G)$ expressed in terms of orders of the longest paths or cycles contained in the corresponding spanning subgraphs. In particular, we establish a lower bound when G contains the set of edge-disjoint Hamiltonian paths and cycles.

Our contribution is reported in the forthcoming sections. Precisely, theoretical results are given in Section 2, a concluding discussion is given in Section 3, while in the Appendix we observe the existence of an upper bound for the algebraic connectivity (which is implicitly proved in [2]).

2. Results

We use the following lemma referred to Fiedler.

Lemma 2.1. [5] *Let $G_1, G_2, \ldots, G_k$ be edge-disjoint spanning subgraphs of a non-trivial signed graph G such that $m(G) = \sum_{i=1}^{k} m(G_i)$. Then*

$$a(G) \geq \sum_{i=1}^{k} a(G_i).$$

We also use the following limit point without reference:

$$\lim_{x \to 0} \left(\frac{\sum_{i=1}^{k} t_i^x}{k} \right)^{\frac{1}{x}} = \left(\prod_{i=1}^{k} t_i \right)^{\frac{1}{k}}, \tag{2}$$

for positive $t_1, t_2, \ldots, t_k$.

Theorem 2.1. *Assume that a graph G with n ($n \geq 2$) vertices contains edge-disjoint spanning subgraphs $G_1, G_2, \dots, G_q$ such that for $1 \leq i \leq p$ it holds $a(G_i) \geq a(P_{n_i})$ with $n_i \geq 2$ and for $p+1 \leq i \leq q$ it holds $a(G_i) \geq a(C_{n_i})$. Then*

$$a(G) > \frac{\pi^2}{3}\left(p\frac{12\overline{g}(n_1, n_2, \dots, n_p)^2 - \pi^2}{4\overline{g}(n_1, n_2, \dots, n_p)^4} + 4(q-p)\frac{3\overline{g}(n_{p+1}, n_{p+2}, \dots, n_q)^2 - \pi^2}{\overline{g}(n_{p+1}, n_{p+2}, \dots, n_q)^4}\right), \quad (3)$$

where $\overline{g}$ denotes the geometric mean of given arguments.

Proof. By Lemma 2.1, $a(G) \geq \sum_{i=1}^{q} a(G_i)$, i.e., $a(G) \geq \sum_{i=1}^{p} a(P_{n_i}) + \sum_{i=p+1}^{q} a(C_{n_i})$. It holds $a(P_{n_i}) = 2\big(1 - \cos(\frac{\pi}{n_i})\big)$ and $a(C_{n_i}) = 2\big(1 - \cos(\frac{2\pi}{n_i})\big)$; see, for example, [1].

Using the Taylor series, we get

$$a(P_{n_i}) > 2\left(1 - 1 + \frac{\pi^2}{2n_i^2} - \frac{\pi^4}{24n_i^4}\right) = \frac{\pi^2}{12n_i^4}(12n_i^2 - \pi^2)$$

and

$$a(C_{n_i}) > 2\left(1 - 1 + \frac{4\pi^2}{2n_i^2} - \frac{16\pi^4}{24n_i^4}\right) = \frac{4\pi^2}{3n_i^4}(3n_i^2 - \pi^2)$$

that gives

$$a(G) > \frac{\pi^2}{3}\left(\frac{1}{4}\sum_{i=1}^{p}\frac{12n_i^2 - \pi^2}{n_i^4} + 4\sum_{i=p+1}^{q}\frac{3n_i^2 - \pi^2}{n_i^4}\right). \quad (4)$$

We consider the first sum of (4). For $\alpha \geq 2$, we define the function

$$f_\alpha(x) = \frac{12x^\alpha - \pi^2}{x^{2\alpha}}.$$

It holds $f''_\alpha(x) = \frac{2a}{x^{2(\alpha+1)}}\big(6(\alpha+1)x^\alpha - \pi^2(2\alpha+1)\big)$, and so, for $x \geq 2$, f_α is convex. Using the Jensen's inequality, we get

$$\sum_{i=1}^{p}\frac{12n_i^2 - \pi^2}{n_i^4} \geq pf_\alpha\left(\frac{\sum_{i=1}^{p} n_i^{2/\alpha}}{p}\right) = p\frac{12\left(\frac{\sum_{i=1}^{p} n_i^{2/\alpha}}{p}\right)^\alpha - \pi^2}{\left(\frac{\sum_{i=1}^{p} n_i^{2/\alpha}}{p}\right)^{2\alpha}}.$$

If $\alpha \to \infty$, by (2), we have

$$\sum_{i=1}^{p}\frac{12n_i^2 - \pi^2}{n_i^4} \geq p\frac{12\overline{g}(n_1, n_2, \dots, n_p)^2 - \pi^2}{\overline{g}(n_1, n_2, \dots, n_p)^4}. \quad (5)$$

The second sum of (4) is considered in a similar way. For $\alpha \geq 3$, we define the function

$$h_\alpha(x) = \frac{3x^\alpha - \pi^2}{x^{2\alpha}},$$

which is convex for $x \geq 3$ (as $h''_\alpha(x) = \frac{\alpha}{x^{2(\alpha+1)}}\big(3(\alpha+1)x^\alpha - 2\pi^2(2\alpha+1)\big)$). This leads to

$$\sum_{i=p+1}^{q} \frac{3n_i^2 - \pi^2}{n_i^4} \geq (q-p)h_\alpha\left(\frac{\sum_{i=p+1}^{q} n_i^{2/\alpha}}{q-p}\right) = (q-p)\frac{3\left(\frac{\sum_{i=p+1}^{q} n_i^{2/\alpha}}{(q-p)}\right)^\alpha - \pi^2}{\left(\frac{\sum_{i=p+1}^{q} n_i^{2/\alpha}}{q-p}\right)^{2\alpha}}.$$

Letting $\alpha \to \infty$, we get

$$\sum_{i=p+1}^{q} \frac{3n_i^2 - \pi^2}{n_i^4} \geq (q-p)\frac{3\overline{g}(n_{p+1}, n_{p+2}, \dots, n_q)^2 - \pi^2}{\overline{g}(n_{p+1}, n_{p+2}, \dots, n_q)^4}. \tag{6}$$

The inequality (4), in conjunction with (5) and (6), gives (3). □

Here are some consequences.

Corollary 2.1. *Under the assumptions of Theorem* 2.1*, we have*

$$a(G) > \frac{\pi^2}{3}\left(p\frac{12\overline{a}(n_1, n_2, \dots, n_p)^2 - \pi^2}{4\overline{a}(n_1, n_2, \dots, n_p)^4} + 4(q-p)\frac{3\overline{a}(n_{p+1}, n_{p+2}, \dots, n_q)^2 - \pi^2}{\overline{a}(n_{p+1}, n_{p+2}, \dots, n_q)^4}\right), \tag{7}$$

where $\overline{a}$ denotes the arithmetic mean of given arguments.

Proof. The function $\frac{12x^2-\pi^2}{4x^2}$ decreases for $x \geq 2$, and so

$$\frac{12\overline{g}(n_1, n_2, \dots, n_p)^2 - \pi^2}{4\overline{g}(n_1, n_2, \dots, n_p)^4} \geq \frac{12\overline{a}(n_1, n_2, \dots, n_p)^2 - \pi^2}{4\overline{a}(n_1, n_2, \dots, n_p)^4}.$$

Similarly, as $\frac{3x^2-\pi^2}{x^2}$ decreases for $x \geq 3$, we have

$$\frac{3\overline{g}(n_{p+1}, n_{p+2}, \dots, n_q)^2 - \pi^2}{\overline{g}(n_{p+1}, n_{p+2}, \dots, n_q)^4} \geq \frac{3\overline{a}(n_{p+1}, n_{p+2}, \dots, n_q)^2 - \pi^2}{\overline{a}(n_{p+1}, n_{p+2}, \dots, n_q)^4},$$

and the proof follows. □

Corollary 2.2. *Under the assumptions of Theorem* 2.1*, we have*

$$a(G) > q\pi^2\frac{12\overline{g}(n_1, n_2, \dots, n_q)^2 - \pi^2}{12\overline{g}(n_1, n_2, \dots, n_q)^4} \geq q\pi^2\frac{12\overline{a}(n_1, n_2, \dots, n_q)^2 - \pi^2}{12\overline{a}(n_1, n_2, \dots, n_q)^4}, \tag{8}$$

where $\overline{g}$ and $\overline{a}$ denote the geometric mean and the arithmetic mean of given arguments, respectively.

Proof. In the notation of Theorem 2.1, since $a(C_{n_i}) > a(P_{n_i})$, we have $a(G_i) \geq a(P_{n_i})$, for $1 \leq i \leq q$. The first inequality follows by setting $p = q$ in (3), and then the second follows by the previous corollary. □

We proceed with the following particular result.

Theorem 2.2. *If a non-trivial graph G contains p Hamiltonian paths and $q-p$ Hamiltonian cycles, such that all of them are edge disjoint, then*

$$a(G) > \pi^2 \frac{12(4q-3p)n^2 - (16q-15p)\pi^2}{12n^4}. \tag{9}$$

Proof. Obviously, G contains edge-disjoint spanning subgraphs $G_1, G_2, \dots, G_q$ such that the first p of them contain a Hamiltonian path and the remaining ones contain a Hamiltonian cycle. By Lemma 2.1, the algebraic connectivity of G_i is at least the algebraic connectivity of its spanning subgraph, i.e., all the assumptions of Theorem 2.1 are satisfied (with $n_i = n$, for $1 \leq i \leq q$). By (3), we compute

$$a(G) > \frac{\pi^2}{3}\left(p\frac{12n^2-\pi^2}{4n^4} + 4(q-p)\frac{3n^2-\pi^2}{n^4}\right),$$

giving the desired inequality. □

Since, for a connected graph G, we have $a(G) \geq 2\epsilon(1-\cos\frac{\pi}{n})$ (see [5]), where ϵ denotes the edge connectivity of G, it follows that Theorem 2.1 can be applied to any connected non-trivial graph with itself in the role of the unique spanning subgraph. Here is another criterion concerning graphs with small diameter.

Theorem 2.3. *If a connected graph G with n ($n \geq 2$) vertices and diameter d contains a path P_k (resp. a cycle C_k) such that $4k^2 \geq dn\pi^2$ (resp. $k^2 \geq dn\pi^2$), then $a(G) > a(P_k)$ (resp. $a(G) > a(C_k)$).*

Proof. We use the inequality (1). Considering the existence of a path P_k, we get

$$a(G) \geq \frac{4}{dn} \geq \frac{4}{\frac{4k^2}{\pi^2}} = \frac{\pi^2}{k^2} = 2\left(1-1+\frac{\pi^2}{2k^2}\right) > 2\left(1-\cos\frac{\pi}{k}\right).$$

The existence of a cycle satisfying the assumption of the theorem is considered in the same way. □

3. Remarks

The bound (3) and its consequences (7)–(9) are always non-trivial, in the sense that they are never negative. An easy consequence of (9) is the following lower bound

$$a(G) > 4q\pi^2\frac{3n^2-\pi^2}{3n^4}, \tag{10}$$

where q stands for the number of edge-disjoint Hamiltonian cycles. In general, the bound (10) is incomparable with (1), but it gives a better estimate whenever

$$q \geq \frac{3n^3}{d\pi^2(n^2-\pi^2)}. \tag{11}$$

In particular, this occurs for every Hamiltonian graph with $d \geq \frac{3n^3}{\pi^2(3n^2-\pi^2)}$, as then the right hand side of (11) is at most 1; this lower bound for d is asymptotically n/π^2.

Example 1. Consider the graph G obtained by inserting an edge between every pair of vertices at distance 2 of a cycle C_{2k+1}, for $k \geq 2$. Obviously, G has exactly 2 edge-disjoint Hamiltonian cycles, and thus due to (10) we have $a(G) > 8\pi^2 \frac{3(2k+1)^2-\pi^2}{3(2k+1)^4}$. Say, for $k = 4$, we get $2.12 \approx a(G) > 0.94$.

As the right hand side of (10) increases with the number of edge-disjoint Hamiltonian cycles, it would be natural to consider it in conjunction with a lower bound for the number of such cycles. In this context, we recall that Nash-Williams proved that the assumptions of the well-known Dirac's theorem guarantee the existence of many edge-disjoint Hamiltonian cycles. Precisely, every graph with n vertices and minimum vertex degree at least $n/2$ contains at least $\lfloor 5n/224 \rfloor$ edge-disjoint Hamiltonian cycles [9]. It is conjectured in the same reference that every r-regular graph with at most $2r$ vertices contains $r/2$ Hamiltonian cycles. This conjecture is still open; an approximate version stating that every r-regular graph with n $(14 \leq n \leq 2r+1)$ vertices contains $\lfloor (3r-n+1)/6 \rfloor$ edge-disjoint Hamiltonian cycles is proved by Jackson [7]. For some asymptotic results, we refer to Christofides, Kühn and Osthus [3]. Particular constructions of arbitrarily large graphs with a specified number of Hamiltonian cycles can be found in Haythorpe's [6].

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Appendix

We recall an interesting upper bound, obtained by Bollobás and Nikiforov [2], for the sum of the $k-1$ least eigenvalues of a Hermitian matrix. Namely, if $N_1 \sqcup N_2 \cdots \sqcup N_k$ is a partition of a Hermitian matrix $M = (m_{ij})$ with eigenvalues $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$, then

$$\sum_{p=n-k+2}^{n} \nu_p \leq \sum_{p=1}^{k} \frac{1}{|N_p|} \sum_{(i,j)\,:\,i,j \in N_p} m_{ij} - \frac{1}{n}\,\mathrm{sum}(M), \tag{12}$$

where $\mathrm{sum}(M)$ denotes the sum of the entries of M.

By considering the Laplacian matrix of a graph in the role of M and inserting $k = 3$ in (12), we get

$$a(G) \leq \sum_{p=1}^{3} \frac{c(N_p)}{|N_p|}, \tag{13}$$

where, clearly $N_1 \sqcup N_2 \sqcup N_3$ is a vertex set partition, while $c(N_p)$ denotes the *cut* of N_p, i.e., the number of edges with exactly one end in N_p. Indeed, if $L = (l_{ij})$ is the Laplacian matrix, then $\sum_{(i,j)\,:\,i,j \in N_p} l_{ij} = c(N_p)$ and $\mathrm{sum}(L) = 0$, so we get (13). This upper bound can be used to estimate the algebraic connectivity of graphs with given tripartition of a vertex set. For example, if G contains at least two cut-edges, then we have

$$a(G) \leq \frac{1}{|N_1|} + \frac{2}{|N_2|} + \frac{1}{|N_3|},$$

where cut-edges are located between N_1 and N_2, and N_2 and N_3.